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Maximum Effective Hole Mathematical Model and Exact Solution for Commingled Reservoir*

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Abstract The maximum effective hole-diameter mathematical model describing the flow of slightly compressible fluid through a commingled reservoir was solved rigorously with consideration of wellbore storage and different skin factors. The exact solutions for wellbore pressure and the production rate obtained from layer j for a well production at a constant rate from a radial drainage area with infinite and constant pressure and no flow outer boundary condition were expressed in terms of ordinary Bessel functions. These solutions were computed numerically by the Crump's numerical inversion method and the behavior of systems was studied as a function of various reservoir parameters. The model was compared with the real wellbore radii model. The new model is numerically stable when the skin factor is positive and negative, but the real wellbore radii model is numerically stable only when the skin factor is positive.

Keywords well-testing, mathematical model, effective hole diameter, layered reservoir

1 INTRODUCTION

Real reservoirs normally consist of many layers with different permeabilities. The behavior of pressure transient for layered reservoir has been studied in detail for two types of systems: one is layers which are separated by impermeable barriers (called commingled reservoir), the other is layers which communicate in the reservoir (called crossflow reservoir).

The earliest rigorous study of pressure behavior of commingled layered reservoirs with an arbitrary number of layers was performed by Lefkovits *et al.*[1] in 1961. Their work served as the basis for much of the work that followed. Tariq *et al.*[2] extended the study of the commingled system by considering the effect of wellbore storage and skin factors. Kucuk *et al.*[3] developed a new testing method in a two-layer commingled reservoir. Their multilayer testing technique consists of a number of sequential flow rates with a production logging tool that simultaneously measures the wellbore pressure and flow rate at the top of each layer. They developed two different techniques to estimate layer parameters. The first technique was a logarithmic convolution method, estimating the approximate values of parameters, and the second method was a nonlinear least-squares estimation, improving the first estimations. The estimation of reservoir parameters from an observed well response is an inverse problem that requires matching the observed response to a model that is a function of the unknown parameters. In general, a nonlinear parameter estimation procedure will minimize the sum of the squares of the difference between observations and the reservoir model response

$$E(\boldsymbol{\beta})=\sum_{i=1}^{n}[\Delta p_{\mathrm{m}}(\boldsymbol{\beta},t_i)-\Delta p_{\mathrm{o}}(t_i)]^2 \qquad (1)$$

where $E(\boldsymbol{\beta})$ is the objective function, $\Delta p_{\mathrm{m}}(\boldsymbol{\beta},t_i)$ is the model pressure response, which is a function of $\boldsymbol{\beta}$, $\boldsymbol{\beta}$ is a parameter vector, $\Delta p_{\mathrm{o}}(t_i)$ is the observed pressure, and n is the number of measured data point. For a detailed view of nonlinear parameter estimation, see Refs.[3, 4].

Unfortunately, the exact solution for wellbore pressure is not convergent when the skin factors are negative[5]. It is impossible to estimate reservoir parameters by using nonlinear least-squares method. In this paper, a new model of commingled reservoir is developed by the application of the maximum effective hole-diameter concept.

2 MAXIMUM EFFECTIVE HOLE DIAMETER

The improper drilling and wellbore completion technology causes serious damage around the well, especially in the near wellbore zone. Later in the life of a well, the production, injection and stimulation can also cause damage around the well that will reduce the

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production of the well. Usually, the boundary condition at the well is given by the following equation which accounts for skin factor[5]

$$p_{\mathrm{wD}} = p_{\mathrm{D}} - S\left(\frac{\partial p_{\mathrm{D}}}{\partial r_{\mathrm{D}}}\right)_{r_{\mathrm{D}}=1} \tag{2}$$

As Agarwal *et al.*[5] pointed out, the exact solution for wellbore pressure is not convergent when the skin factors are negative. They described the skin effect in terms of an effective wellbore radius. The rate of single layer which accounts for skin factor can be written as

$$q = \frac{2\pi kh(p_{\mathrm{i}} - p_{\mathrm{wf}})}{\mu[\ln(r_{\mathrm{e}}/r_{\mathrm{w}}) + S]} \tag{3}$$

the denominator can be written as

$$\ln\left(\frac{r_{\mathrm{e}}}{r_{\mathrm{w}}}\right) + S = \ln\left(\frac{r_{\mathrm{e}}}{r_{\mathrm{w}}\mathrm{e}^{-S}}\right) = \ln\left(\frac{r_{\mathrm{e}}}{r_{\mathrm{we}}}\right) \tag{4}$$

The relationship of wellbore radius and skin effect is defined as effective wellbore radius

$$r_{\mathrm{we}} = r_{\mathrm{w}}\mathrm{e}^{-S} \tag{5}$$

Similar to single layer reservoir, the rate of the jth layer is

$$q_j = \frac{2\pi kh(p_{\mathrm{i}} - p_{\mathrm{wf}j})}{\mu[\ln(r_{\mathrm{e}}/r_{\mathrm{w}}) + S_j]} \tag{6}$$

We introduce a new variable $S_{\min}$. It is the minimum skin factor in an oilfield, and is negative. The denominator can be written as

$$\begin{aligned}\ln\left(\frac{r_{\mathrm{e}}}{r_{\mathrm{w}}}\right) + S_j &= \ln\left(\frac{r_{\mathrm{e}}}{r_{\mathrm{w}}\mathrm{e}^{-S_{\min}}}\right) + S_j - S_{\min} \\ &= \ln\left(\frac{r_{\mathrm{e}}}{r_{\mathrm{we}}}\right) + S'_j \end{aligned} \tag{7}$$

the maximum effective hole-diameter can be defined as follows

$$r_{\mathrm{we}} = r_{\mathrm{w}}\mathrm{e}^{-S_{\min}} \tag{8}$$

The skin factor of layer j can be written as

$$S'_j = S_j - S_{\min} \tag{9}$$

Eq. (6) can be written as follows

$$q_j = \frac{2\pi kh(p_{\mathrm{i}} - p_{\mathrm{wf}})}{\mu[\ln r_{\mathrm{e}}/r_{\mathrm{we}}) + S'_j]} \tag{10}$$

Eq. (10) in dimensionless form is

$$p_{\mathrm{wD}} = p_{\mathrm{D}} - S'_j\left(\frac{\partial p_{j\mathrm{D}}}{\partial r_{\mathrm{D}}}\right)_{r_{\mathrm{D}}=1} \tag{11}$$

3 MODEL DESCRIPTION

The reservoir consists of n layers. There is no crossflow between layers. Each layer of the reservoir system is assumed to be homogeneous, isotropic and filled with a single phase fluid. The reservoir is assumed to be horizontal and cylindrical, enclosed at the top and bottom. The initial pressure is assumed to be the same in each layer and the production rate is constant.

The governing equation in dimensionless form is

$$\gamma_j \nabla^2 p_{j\mathrm{D}} = \frac{\omega_j}{C_{\mathrm{D}}\mathrm{e}^{2S_{\min}}} \frac{\partial p_{j\mathrm{D}}}{\partial(t_{\mathrm{D}}/C_{\mathrm{D}})} \tag{12}$$

with initial condition

$$p_{j\mathrm{D}}(r_{\mathrm{D}}, 0) = 0 \tag{13}$$

Infinite outer boundary condition

$$p_{j\mathrm{D}}(r_{\mathrm{D}} \to \infty, t_{\mathrm{D}}) = 0 \tag{14}$$

No-flow outer boundary condition

$$\left(\frac{\partial p_{j\mathrm{D}}}{\partial r_{\mathrm{D}}}\right)_{r_{\mathrm{D}}=r_{\mathrm{eD}}} = 0 \tag{15}$$

Constant pressure outer boundary condition

$$p_{j\mathrm{D}}(r_{\mathrm{eD}}, t_{\mathrm{D}}) = 0 \tag{16}$$

and wellbore boundary conditions

$$p_{\mathrm{wD}} = p_{j\mathrm{D}} - S'_j\left(\frac{\partial p_{j\mathrm{D}}}{\partial r_{\mathrm{D}}}\right)_{r_{\mathrm{D}}=1} \tag{17}$$

$$\sum_{j=1}^{N} \gamma_j \left(\frac{\partial p_{j\mathrm{D}}}{\partial r_{\mathrm{D}}}\right)_{r_{\mathrm{D}}=1} = -1 + \frac{\partial p_{\mathrm{wD}}}{\partial(t_{\mathrm{D}}/C_{\mathrm{D}})} \tag{18}$$

where

$$j : j\text{th layer}, 1, 2, \cdots, N$$

$$p_{j\mathrm{D}} = \frac{2\pi \sum kh}{qB\mu}(p_{\mathrm{i}} - p_j)$$

$$r_{\mathrm{D}} = \frac{r}{r_{\mathrm{w}}\mathrm{e}^{-S_{\min}}}$$

$$t_{\mathrm{D}} = \frac{\sum kh}{\sum(\phi h C_{\mathrm{t}})\mu r_{\mathrm{w}}^2}$$

$$\gamma_j = \frac{(kh)_j}{\sum kh}$$

$$C_{\mathrm{D}} = \frac{C}{2\pi \sum(\phi h C_{\mathrm{t}}) r_{\mathrm{w}}^2}$$

$$\omega_j = \frac{(\phi h C_{\mathrm{t}})_j}{\sum(\phi h C_{\mathrm{t}})}$$

4 DERIVATION OF SOLUTION FOR PRESSURE AND RATE

Equations(12)—(18)are transformed into Laplace domain with respect to (t_D/C_D)

$$\gamma_j \nabla^2 \overline{p}_{jD} = \frac{\omega_j s}{C_D e^{2S_{\min}}} \overline{p}_{jD} \tag{19}$$

$$\overline{p}_{jD}(r_D, 0) = 0 \tag{20}$$

$$\overline{p}_{jD}(r_D \to \infty, s) = 0 \tag{21}$$

$$\left(\frac{\partial \overline{p}_{jD}}{\partial r_D}\right)_{r_D = r_{eD}} = 0 \tag{22}$$

$$\overline{p}_{jD}(r_{eD}, s) = 0 \tag{23}$$

$$\overline{p}_{wD} = \overline{p}_{jD} - S'_j \left(\frac{\partial \overline{p}_{jD}}{\partial r_D}\right)_{r_D=1} \tag{24}$$

$$\sum_{j=1}^{N} \gamma_j \left(\frac{\partial \overline{p}_{jD}}{\partial r_D}\right)_{r_D=1} = -\frac{1}{s} + s\overline{p}_{wD} \tag{25}$$

The solutions for this system are the modified Bessel functions K_0 and I_0. The dimensionless wellbore pressure can be written as follows[6]

$$\overline{p}_{jD} = A_j[K_0(\sigma_j r_D) + \partial_j I_0(\sigma_j r_D)] \tag{26}$$

where σ_j is a function of ω_j, C_D, $S_{\min}$, γ_j and Laplace space variable s. Substitution of Eq. (26) into Eq. (19) gives

$$\sigma_j = \sqrt{\omega_j s / C_D e^{2S_{\min}} \gamma_j} \tag{27}$$

To satisfy the outer boundary condition, for the infinite reservoir the coefficient ∂_j must be zero. For the no-flow outer boundary condition

$$\partial_j = K_1(\sigma_j r_{eD}) / I_1(\sigma_j r_{eD}) \tag{28}$$

For the constant pressure outer boundary condition

$$\partial_j = -K_0(\sigma_j r_{eD}) / I_0(\sigma_j r_{eD}) \tag{29}$$

By differentiation of Eq. (26) with respect to r_D, we obtain

$$\left(\frac{\partial \overline{p}_{jD}}{\partial r_D}\right)_{r_D=1} = -\sigma_j A_j [K_1(\sigma_j) - \partial_j I_1(\sigma_j)] \tag{30}$$

Substitution of Eq. (30) into Eq. (24)results in

$$A_j = \frac{\overline{p}_{wD}}{X_1 + Y_1} \tag{31}$$

where

$$X_1 = K_0(\sigma_j) + \partial_j K_1(\sigma_j)$$
$$Y_1 = S'_j \sigma_j [K_1(\sigma_j) - \partial_j I_1(\sigma_j)]$$

Substitution of Eqs. (30) and (31) into Eq. (25) gives

$$\overline{p}_{wD} = \frac{1}{s\left[s + \sum_{j=1}^{N} \frac{\gamma_j}{S'_j + X_1/\sigma_j Y_2}\right]} \tag{32}$$

where

$$Y_2 = K_1(\sigma_j) - \partial_j I_1(\sigma_j)$$

The solution for the production rate from layer j is

$$\overline{q}_{jD} = \frac{\gamma_j \overline{p}_{wD}}{S'_j + X_1/\sigma_j Y_2} \tag{33}$$

If $S_{\min}$ is equal to zero, and S'_j is equal to S_j, the solutions of our model are identical to the solutions of real wellbore radii model[2].

5 INVERSION OF LAPLACE TRANSFORM

For numerical inversion of the Laplace transform of petroleum engineering problems Stehfest's algorithm[7] is probably the most common. Stehfest's algorithm is simple and easy to use, but it is not convergent when the curve is steep. Crump's method[8,9] is also available to solve the problem of transient flow of slightly compressible fluid in porous media. Numerical inversion solution of Crump method is accordant well with the exact solution. In this paper, these solutions are computed numerically by Crump numerical inversion method.

For a two-layer infinite commingled reservoir, Eq. (32) becomes

$$\overline{p}_{wD} = \frac{1}{s\left[s + \sum_{j=1}^{2} \frac{\gamma_j}{S'_j + K_0(\sigma_j)/\sigma_j K_1(\sigma_j)}\right]} \tag{34}$$

The maximum effective hole-diameter mathematical model is compared with the real wellbore radii model (Table 1). The new model is numerically stable whether the skin factor is positive or negative. The real wellbore radii model is numerically stable when the skin is positive. The parameters of the reservoir and fluid are as follows

$\omega_j = 0.01, 0.99, \gamma_j = 0.99, 0.01, S_{\min} = -2, S_j = 2, -2,$
$C_D = 1000.$

6 A COMPARISION OF TWO LAYER COMMINGLED RESERVOIR AND CROSSFLOW RESERVOIR

Figures 1 and 2 are the pressure curves and pressure derivative curves respectively of two layer commingled reservoir and crossflow reservoir[10]. In the early period, the pressure response of commingled system is identical to that of crossflow system. In the latter period, the pressure response reaches a semi-log straight line behavior similar to that of the single layer homogeneous system. The value of derivative curve is about 0.5.

Table 1 A comparison of results obtained by maximum effective hole-diameter mathematical model and real wellbore radii model

$\lg(t_D/C_D)$	Real wellbore radii p_{wD}	Maximum effective hole-diameter p_{wD}
−1	0.09925	0.09865
−0.8	0.15675	0.15547
−0.6	0.24664	0.24436
−0.4	0.38769	0.38257
−0.2	0.60612	0.59545
0	0.94123	0.91877
0.2	1.44764	1.39946
0.4	2.198	2.09156
0.6	3.26687	3.04141
0.8	25.9538	4.29253
1	39660	4.88091
1.2	4.778×10^7	6.53938
1.4	1.31×10^{10}	8.21733
1.6	4.491×10^9	9.02202
1.8	269	9.53815
2	10.14949	9.90935
2.2	10.22234	10.22155
2.4	10.50537	10.50614
2.6	10.77471	10.77525
2.8	11.03434	11.03469
3	11.28752	11.28776
3.2	11.53626	11.53641
3.4	11.78179	11.78188
3.6	12.02496	12.02505
3.8	12.26635	12.26644
4	12.50637	12.50639
4.2	12.74525	12.74528
4.4	12.98326	12.98326
4.6	13.22046	13.22048
4.8	13.45707	13.45706
5	13.69313	13.69314

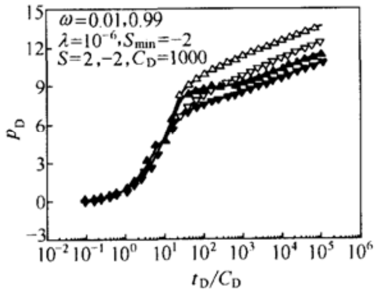

Figure 1 Pressure response of two layer commingled reservoir and crossflow reservoir
▽ no crossflow, $\gamma_1 = 0.99$; △ no crossflow, $\gamma_1 = 0.9$;
▼ crossflow, $\gamma_1 = 0.99$; ▲ crossflow, $\gamma_1 = 0.9$

7 CONCLUSIONS

(1)The maximum effective hole-diameter mathematical model describing the flow of slightly compressible fluid through a commingled reservoir is solved rigorously. The new model is numerically stable whether the skin factor is positive or negative.

(2) In the early stage, pressure response of commingled system is identical to that of crossflow system. At a later stage, the pressure response reaches a semi-log straight-line behavior similar to that of the homogeneous single layer system. The value of derivative curve is about 0.5.

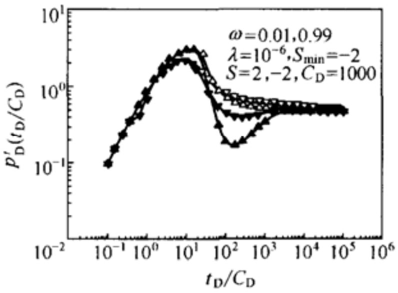

Figure 2 Pressure derivative response of two layer commingled reservoir and crossflow reservoir
▽ no crossflow, $\gamma_1 = 0.99$; △ no crossflow, $\gamma_1 = 0.9$;
▼ crossflow, $\gamma_1 = 0.99$; ▲ crossflow, $\gamma_1 = 0.9$

(3) The new model can be used for nonlinear least-squares parameter estimation with the methods of Kucuk *et al.*[3]. Because the curves explicitly include skin factor and other reservoir parameters, the result is more accurate than the real wellbore radii model.

NOMENCLATURE

A_j	coefficient of solution
B	formation volume factor
B_j	coefficient of solution
C	wellbore storage constant, $m^3 \cdot MPa^{-1}$
C_D	dimensionless wellbore storage constant
C_t	total compressibility, MPa^{-1}
h_j	jth reservoir height, m
I_0, I_1	modified Bessel function
j	layer number
k_j	jth permeability, μm^2
K_0, K_1	modified Bessel function
p_D, p_{wD}	dimensionless wellbore pressure
p_i	initial formation pressure, MPa
p_j	jth pressure, MPa
p_{jD}	jth dimensionless pressure
p_w	wellbore pressure, MPa
p_{wf}	wellbore flow pressure, MPa
p_{wfj}	jth wellbore flow pressure, MPa
p'_D	dimensionless pressure derivative
$\bar{p}_{jD}$	jth dimensionless pressure in Laplace space
q_D	dimensionless total rate
q_j	jth rate, $m^3 \cdot d^{-1}$
q_{jD}	jth dimensionless rate
$\bar{q}_{jD}$	jth dimensionless rate in Laplace space

r	radial distance, m
r_D	dimensionless radius
r_e	reservoir radius, m
r_{eD}	dimensionless reservoir radii
r_w	wellbore radius, m
r_{we}	maximum effective hole-diameter, m
S	skin factor of single layer reservoir
S_j	jth skin factor
S_{min}	minimum skin factor in an oilfield
s	Laplace space variable
t	production time, h
t_D	dimensionless time
γ_j	dimensionless productivity of layer j
μ	viscosity, mPa·s
ϕ	porosity
σ_j	eigenvalue
ω_j	dimensionless storativity of layer j

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