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Unknown Input Extended Kalman Filter and Applications in Nonlinear Fault Diagnosis*

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Abstract Unknown input observer is one of the most famous strategies for robust fault diagnosis of linear systems, but studies on nonlinear cases are not sufficient. On the other hand, the extended Kalman filter (EKF) is well-known in nonlinear estimation, and its convergence as an observer of nonlinear deterministic system has been derived recently. By combining the EKF and the unknown input Kalman filter, we propose a robust nonlinear estimator called unknown input EKF (UIEKF) and prove its convergence as a nonlinear robust observer under some mild conditions using linear matrix inequality (LMI). Simulation of a three-tank system "DTS200", a benchmark in process control, demonstrates the robustness and effectiveness of the UIEKF as an observer for nonlinear systems with uncertainty, and the fault diagnosis based on the UIEKF is found successful.

Keywords extended Kalman filter, fault diagnosis, unknown input, convergence analysis, linear matrix inequality

1 INTRODUCTION

With increasing demand on safety in industrial processes, fault detection and isolation (FDI) has received much more attention in the past three decades, in which one important strategy is based on analytical models. The FDI approaches based on analytical models include observers, parity space, Kalman filter, parameter estimation, and so on[1]. Due to the universal existence of nonlinearity and model uncertainty in practice, robust FDI of nonlinear systems is of great significance. Since model uncertainties are unexpected dynamics of the system as well as faults, they constitute a source of false alarms which corrupt the performance of the FDI system.

The robustness of FDI systems is an important research topic, in which the unknown input observer (UIO) scheme is very famous[2]. The basic idea is to design a fault diagnosis observer decoupled from the unknown disturbances. However, UIO for nonlinear systems has not been studied sufficiently. In the surveys of Frank *et al.*, they summarized some results of linear and nonlinear UIO[3,4]. Yu and Shields extended the classical linear UIO to bilinear systems and polynomial systems, respectively[5,6].

For nonlinear systems the extended Kalman filter (EKF) is very famous and has been widely used as an estimator[7]. The convergence of the EKF used as an observer for nonlinear deterministic discrete-time systems was discussed by Boutayeb *et al.* and Guo *et al.*[8–10]. In these papers, proper selection of $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ based on the convergence analysis are also discussed, which would improve the convergence capability of the EKF. On the other hand, unbiased minimum-variance linear state estimator for linear stochastic systems with unknown inputs was firstly investigated by Kitanidis[11], which was called unknown input Kalman filter (UIKF). It was also analyzed by Darouach *et al.*[12] and Keller *et al.*[13] later.

Combining the results of the UIKF with the EKF, we propose a new nonlinear robust estimator for a class of nonlinear systems with structural uncertainty in this paper, which is called unknown input EKF (UIEKF). The convergence of the UIEKF as an observer for deterministic systems is analyzed theoretically using Lyapunov method. Simulation results illustrate that the UIEKF is robust to unknown disturbances and is exponentially convergent as a nonlinear observer. The UIEKF is also applied to the fault diagnosis of three-tank system "DTS200", a benchmark in process control, and the results are satisfactory.

2 BRIEF REVIEW OF THE UIKF

The UIKF was first proposed by Kitanidis[11]. Consider the linear discrete-time stochastic system with unknown disturbance as follows:

$$\begin{cases} \boldsymbol{x}_{k+1} = \boldsymbol{F}_k\boldsymbol{x}_k + \boldsymbol{B}_k\boldsymbol{u}_k + \boldsymbol{E}_k\boldsymbol{d}_k + \boldsymbol{w}_k \\ \boldsymbol{y}_{k+1} = \boldsymbol{H}_{k+1}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1} \end{cases} \tag{1}$$

where $\boldsymbol{x}_k \in \boldsymbol{R}^n$ is the state vector; $\boldsymbol{y}_k \in \boldsymbol{R}^m$ is the observation vector; $\boldsymbol{u}_k \in \boldsymbol{R}^r$ is the known input vector; $\boldsymbol{d}_k \in \boldsymbol{R}^q$ is the unknown input vector which represents unknown disturbances or model uncertainties; the sys-

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tem and measurement noises ($\boldsymbol{w}_k$ and $\boldsymbol{v}_{k+1}$) are zero mean uncorrelated random Gaussian sequences with variance matrix $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$, respectively. $\boldsymbol{F}_k$, $\boldsymbol{B}_k$, $\boldsymbol{E}_k$ and $\boldsymbol{H}_{k+1}$ are known matrices with proper dimensions. The present aim is to design a minimum variance filter as (2):

$$\hat{\boldsymbol{x}}_{k+1|k+1}=\boldsymbol{F}_k\hat{\boldsymbol{x}}_{k|k}+\boldsymbol{B}_k\boldsymbol{u}_k+\boldsymbol{L}_{k+1}(\boldsymbol{y}_{k+1}-\boldsymbol{H}_{k+1}\hat{\boldsymbol{x}}_{k+1|k}) \tag{2}$$

In the UIKF, the gain matrix $\boldsymbol{L}_{k+1}$ should satisfy the decoupling condition which is the same as that in general unknown input observer[2]:

$$\boldsymbol{L}_{k+L}\boldsymbol{H}_{k+1}\boldsymbol{E}_k=\boldsymbol{E}_k \tag{3}$$

It has been proved[2] that Eq. (3) has a solution if and only if

$$\operatorname{rank}(\boldsymbol{H}_{k+1}\boldsymbol{E}_k)=\operatorname{rank}(\boldsymbol{E}_k)=q \tag{4}$$

Condition (4) means that the disturbances can be decoupled only if the dimension of disturbances is no larger than that of measurements. Based on the disturbance decoupling condition Eq. (3), the optimal filter gain $\boldsymbol{L}_{k+1}$ is obtained by solving an optimization problem with constraints using Lagrange method in Ref. [11].

Keller *et al.*[13] derived a brief formulation of Kitanidis's result, which demonstrated the relation between the UIKF and the classical Kalman filter that only the filter gain and the filtered state estimation variance are modified from the original Kalman filter. Via the results of the optimal state estimation of singular systems, Darouach *et al.* proposed another equivalent form of the UIKF[12].

3 UIEKF AND ITS CONVERGENCE

3.1 Algorithm of UIEKF

In this section we will extend the UIKF to nonlinear case as the EKF does. Our algorithm is based on Keller et al.'s result[13]. Consider the following nonlinear discrete-time system with unknown disturbance,

$$\begin{cases}\boldsymbol{x}_{k+1}=\boldsymbol{f}(\boldsymbol{x}_k,\boldsymbol{u}_k)+\boldsymbol{E}(\boldsymbol{x}_k)\boldsymbol{d}_k\\ \boldsymbol{y}_{k+1}=\boldsymbol{h}(\boldsymbol{x}_{k+1},\boldsymbol{u}_{k+1})\end{cases} \tag{5}$$

where $\boldsymbol{f}$, $\boldsymbol{h}$ and $\boldsymbol{E}$ are assumed to be smooth and are known. For simplicity we only investigate deterministic cases in this paper. Notice that the disturbance distribution matrix $\boldsymbol{R}(\boldsymbol{x}_k)$ in system (5) is assumed to be related to current state $\boldsymbol{x}_k$, which is a more general case and is different from that in system (1).

Then like the EKF, we extend the UIKF to system (5) as follows,

$$\hat{\boldsymbol{x}}_{k+1|k}=\boldsymbol{f}(\hat{x}_{k|k},\boldsymbol{u}_k) \tag{6}$$

$$\boldsymbol{P}_{k+1|k}=\boldsymbol{F}_k\boldsymbol{P}_{k|k}\boldsymbol{F}_k^{\mathrm{T}}+\boldsymbol{Q}_k \tag{7}$$

$$\hat{\boldsymbol{x}}_{k+1|k+1}=\hat{\boldsymbol{x}}_{k+1|k}+\boldsymbol{L}_{k+1}[\boldsymbol{y}_{k+1}-\boldsymbol{h}(\hat{\boldsymbol{x}}_{k+1|k},\boldsymbol{u}_{k+1})] \tag{8}$$

$$\begin{aligned}\boldsymbol{P}_{k+1|k+1}=&(\mathbf{I}-\boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})\boldsymbol{P}_{k+1|k}+\\ &\boldsymbol{\eta}_{k+1}\boldsymbol{\Pi}_{k+1}\boldsymbol{V}_{k+1}\boldsymbol{\Pi}_{k+1}^{\mathrm{T}}\boldsymbol{\eta}_{k+1}^{\mathrm{T}}\end{aligned} \tag{9}$$

with

$$\boldsymbol{L}_{k+1}=\boldsymbol{K}_{k+1}+\boldsymbol{\eta}_{k+1}\boldsymbol{\Pi}_{k+1} \tag{10}$$

$$\boldsymbol{K}_{k+1}=\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^{\mathrm{T}}\boldsymbol{V}_{k+1}^{-1} \tag{11}$$

$$\boldsymbol{\eta}_{k+1}=(\boldsymbol{I}-\boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})\hat{\boldsymbol{E}}_k \tag{12}$$

$$\begin{aligned}\boldsymbol{\Pi}_{k+1}=&[(\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k)^{\mathrm{T}}\boldsymbol{V}_{k+1}^{-1}(\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k)]^{-1}\cdot\\ &(\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k)^{\mathrm{T}}\boldsymbol{V}_{k+1}^{-1}\end{aligned} \tag{13}$$

$$\boldsymbol{V}_{k+1}=\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k}\boldsymbol{H}_{k+1}^{\mathrm{T}}+\boldsymbol{R}_{k+1} \tag{14}$$

where

$$\boldsymbol{F}_k=\partial\boldsymbol{f}/\partial\boldsymbol{x}|_{(\hat{x}_{k|k},u_k)} \tag{15}$$

$$\boldsymbol{H}_{k+1}=\partial\boldsymbol{h}/\partial\boldsymbol{x}|_{(\hat{x}_{k+1|k},u_{k+1})} \tag{16}$$

$$\hat{\boldsymbol{E}}_k=\boldsymbol{E}(\hat{\boldsymbol{x}}_{k|k}) \tag{17}$$

Compared to the linear case, the UIEKF algorithm is mainly different from the UIKF in the calculation of the matrices $\boldsymbol{F}_k$, $\boldsymbol{H}_{k+1}$ as that in EKF, and $\boldsymbol{E}(\boldsymbol{x}_k)$ is substituted by its estimation $\hat{\boldsymbol{E}}_k$. Only deterministic cases are considered in this paper, then the matrices $\boldsymbol{Q}_k\geqslant 0$ and $\boldsymbol{R}_{k+1}>0$ can be arbitrarily chosen. The choice of $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ will be discussed in the end of next subsection.

3.2 Convergence of UIEKF

The convergence of the nonlinear estimator is a difficult and important problem, such as that of the EKF which is analyzed recently in Refs. [8—10], though the EKF has been proposed and applied successfully in practice for about 40 years. For briefness, only the convergence conditions of UIEKF and some discussion is given in this subsection, and the detailed proof is in the Appendix.

First we define the state error vectors of $\widetilde{\boldsymbol{x}}_{k+1|k+1}$, $\widetilde{\boldsymbol{x}}_{k+1|k}$, and the residual $\boldsymbol{\gamma}_{k+1}$, respectively, by

$$\widetilde{\boldsymbol{x}}_{k+1|k+1}=\boldsymbol{x}_{k+1}-\hat{\boldsymbol{x}}_{k+1|k+1} \tag{18}$$

$$\widetilde{\boldsymbol{x}}_{k+1|k}=\boldsymbol{x}_{k+1}-\hat{x}_{k+1|k} \tag{19}$$

$$\gamma_{k+1}=\boldsymbol{y}_{k+1}-\boldsymbol{h}(\hat{x}_{k+1|k},\boldsymbol{u}_{k+1}) \tag{20}$$

and a candidate Lyapunov function V_{k+1} as follows,

$$V_{k+1}=\widetilde{\boldsymbol{x}}_{k+1|k+1}^{\mathrm{T}}\boldsymbol{P}_{k+1|k+1}^{-1}\widetilde{\boldsymbol{x}}_{k+1|k+1} \tag{21}$$

Similar to the technology used by Boutayeb *et al.* in Refs. [8, 9], $\boldsymbol{\beta}_k=\operatorname{diag}\{\beta_{k,k}\cdots\beta_{n,k}\}$ and $\boldsymbol{\alpha}_{k+1}=$

$\mathrm{diag}\{\alpha_{1,k+1}\cdots\alpha_{m,k+1}\}$, two unknown diagonal matrices, are introduced to compensate the errors due to nonlinearity and unknown disturbances such that,

$$\widetilde{\boldsymbol{x}}_{k+1|k}=\boldsymbol{\beta}_k\boldsymbol{F}_k\widetilde{x}_{k|k}+\hat{\boldsymbol{E}}_k\boldsymbol{d}_k=\overline{\boldsymbol{x}}'_{k+1|k}+\hat{\boldsymbol{E}}_k\boldsymbol{d}_k \quad (22)$$

$$\begin{aligned}\gamma_{k+1}&=\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k\widetilde{x}_{k|k}+\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k\boldsymbol{d}_k\\&=\gamma'_{k+1}+\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k\boldsymbol{d}_k\end{aligned} \quad (23)$$

where

$$\widetilde{\boldsymbol{x}}'_{k+1|k}:=\boldsymbol{\beta}_k\boldsymbol{F}_k\widetilde{x}_{k|k} \quad (24)$$

$$\gamma'_{k+1}:=\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k\widetilde{\boldsymbol{x}}_{k|k} \quad (25)$$

It is obvious that $\boldsymbol{\beta}_k$ and $\boldsymbol{\alpha}_{k+1}$ are both equal to the identity matrix in linear cases. The decoupling condition (3) is satisfied as $\boldsymbol{L}_{k+1}\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k=\hat{\boldsymbol{E}}_k$, Then we can obtain following relations:

$$\begin{aligned}\widetilde{x}_{k+1|k+1}=&\boldsymbol{x}_{k+1}-\hat{\boldsymbol{x}}_{k+1|k+1}\\=&\boldsymbol{x}_{k+1}-(\hat{\boldsymbol{x}}_{k+1|k}+\boldsymbol{L}_{k+1}\gamma_{k+1})\\=&\widetilde{\boldsymbol{x}}_{k+1|k}-\boldsymbol{L}_{k+1}\gamma_{k+1}\\=&\widetilde{\boldsymbol{x}}'_{k+1|k}-\boldsymbol{L}_{k+1}\gamma'_{k+1}+\\&(\hat{\boldsymbol{E}}_k-\boldsymbol{L}_{k+1}\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k)\boldsymbol{d}_k\\=&\widetilde{\boldsymbol{x}}'_{k+1|k}-\boldsymbol{k}_{k+1}\gamma'_{k+1}\end{aligned} \quad (26)$$

Some lemmas are needed for the convergence analysis.

Lemma 1. The inverse of $\boldsymbol{P}_{k+1|k+1}$ and the relation of $\boldsymbol{L}_{k+1}$ with $\boldsymbol{P}_{k+1|k-1}$ in the UIEKF (6)—(17) are given below[12][13],

$$\begin{aligned}\boldsymbol{P}^{-1}_{k+1|k+1}=&\boldsymbol{P}^{-1}_{k-1|k}+\boldsymbol{H}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{H}_{k+1}-\\&\boldsymbol{P}^{-1}_{k-1|k}\hat{\boldsymbol{E}}_k(\hat{\boldsymbol{E}}_k^{\mathrm{T}}\boldsymbol{P}^{-1}_{k+1|k}\hat{\boldsymbol{E}}_k)^{-1}\hat{\boldsymbol{E}}_k^{\mathrm{T}}\boldsymbol{P}^{-1}_{k+1|k}\end{aligned} \quad (27)$$

$$\boldsymbol{L}_{k+1}=\boldsymbol{P}_{k+1|k+1}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{R}^{-1}_{k+1} \quad (28)$$

Lemma 2. Given any symmetric positive definite matrix $\boldsymbol{P}\in\boldsymbol{R}^{n\times n}$ and any full column rank matrix $\boldsymbol{E}\in\boldsymbol{R}^{n\times q}(q\leqslant n)$. Let $\boldsymbol{S}\in\boldsymbol{R}^{(n-1)\times n}$ is any full row rank matrix which satisfies $\boldsymbol{SE}=0$. Then,

$$\boldsymbol{M}:=\boldsymbol{P}-\boldsymbol{PE}(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{PE})^{-1}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}=\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{SP}^{-1}\boldsymbol{S}^{\mathrm{T}})^{-1}\boldsymbol{S} \quad (29)$$

Lemma 3. In the algorithm of the UIEKF, we have the following equivalent relations,

$$\boldsymbol{P}^{-1}_{k+1|k+1}=\boldsymbol{S}^{\mathrm{T}}_k(\boldsymbol{S}_k\boldsymbol{P}_{k+1|k}\boldsymbol{S}^{\mathrm{T}}_k)^{-1}\boldsymbol{S}_k+\boldsymbol{H}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{H}_{k+1} \quad (30)$$

$$\boldsymbol{V}^{-1}_{k+1}=\boldsymbol{R}^{-1}_{k+1}(\boldsymbol{I}-\boldsymbol{H}_{k+1}\boldsymbol{K}_{k+1}) \quad (31)$$

$$\begin{aligned}\boldsymbol{C}_{k+1}:=&\boldsymbol{R}^{-1}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k+1}\boldsymbol{H}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}-\boldsymbol{R}^{-1}_{k+1}\\=&-\boldsymbol{T}^{\mathrm{T}}_{k+1}(\boldsymbol{T}_{k+1}\boldsymbol{V}_{k+1}\boldsymbol{T}^{\mathrm{T}}_{k+1})^{-1}\boldsymbol{T}_{k+1}\end{aligned} \quad (32)$$

where $\boldsymbol{S}_k\in\boldsymbol{R}^{(n-q)\times n}$ and $\boldsymbol{T}_{k+1}\in\boldsymbol{R}^{(m-q)\times m}$ are both arbitrary full row rank matrix which satisfies $\boldsymbol{S}_k\hat{\boldsymbol{E}}_k=0$ and $\boldsymbol{T}_{k+1}\boldsymbol{H}_{k+1}\hat{\boldsymbol{E}}_k=0$, respectively.

Definition 1.[9] Let $\boldsymbol{f}_{\boldsymbol{u}_k}(\cdot):=\boldsymbol{f}(\cdot,\boldsymbol{u}_k)$ and $\boldsymbol{h}_{\boldsymbol{u}_k}(\cdot):=\boldsymbol{h}(\cdot,\boldsymbol{u}_k)$. Then the nonlinear system (5) is N-locally uniformly rank observable, if there exists an integer $N\geqslant 1$ such that,

$$\mathrm{rank}\frac{\partial}{\partial\boldsymbol{x}}\left.\begin{pmatrix}\boldsymbol{h}_{\boldsymbol{u}_k}(\boldsymbol{x})\\\boldsymbol{h}_{\boldsymbol{u}_{k+1}}\circ\boldsymbol{f}_{\boldsymbol{u}_k}(\boldsymbol{x})\\\vdots\\\boldsymbol{h}_{\boldsymbol{u}_{k+N-1}}\circ\boldsymbol{f}_{\boldsymbol{u}_{k+N-2}}\circ\cdots\circ\boldsymbol{f}_{\boldsymbol{u}_k}(\boldsymbol{x})\end{pmatrix}\right|_{\boldsymbol{x}=\boldsymbol{x}_k}=n \quad (33)$$

for all $\boldsymbol{x}_k\in\boldsymbol{K}$ and N-tuple of controls $(\boldsymbol{u}_k,\cdots,\boldsymbol{u}_{k+N-1})\in\boldsymbol{U}(\boldsymbol{K}$ and $\boldsymbol{U}$ are two compact subsets of $\boldsymbol{R}^n$ and $(\boldsymbol{R}^r)^N$, respectively).

Assumption 1. The nonlinear system (5) is N-locally uniformly rank observable. Thus $\boldsymbol{P}_{k|k}$ is uniformly bounded[9,10], *i.e.* $\forall k>0$, there exists constants $0<\mu_1$, $\mu_2<\infty$ such that $\|\boldsymbol{P}_{k|k}\|\leqslant\mu_1$ and $\|\boldsymbol{P}^{-1}_{k|k}\|\leqslant\mu_2$, which also ensure the Lyapunov candidate function (21) to be positive.

The next theorem is the main result in the convergence analysis.

Theorem 1. Under Assumption 1, a sufficient condition to ensure the exponential convergence of the UIEKF is that $\boldsymbol{\beta}_k$ and $\boldsymbol{\alpha}_{k+1}$ satisfy the following LMI:

$$\begin{bmatrix}(1-\zeta)\boldsymbol{P}^{-1}_{k|k} & \boldsymbol{F}^{\mathrm{T}}_k\boldsymbol{\beta}_k\boldsymbol{S}^{\mathrm{T}}_k & (\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k-\boldsymbol{H}_{k+1}\boldsymbol{\beta}_k\boldsymbol{F}_k)^{\mathrm{T}}\\ \boldsymbol{S}_k\boldsymbol{\beta}_k\boldsymbol{F}_k & \boldsymbol{S}_k(\boldsymbol{F}_k\boldsymbol{P}_{k|k}\boldsymbol{F}^{\mathrm{T}}_k+\boldsymbol{Q}_k)\boldsymbol{S}^{\mathrm{T}}_k & \boldsymbol{0}\\ \boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k-\boldsymbol{H}_{k+1}\boldsymbol{\beta}_k\boldsymbol{F}_k & \boldsymbol{0} & \boldsymbol{R}_{k+1}\end{bmatrix}\geqslant 0 \quad (34)$$

where $\boldsymbol{S}_k\in\boldsymbol{R}^{(n-q)\times n}$ is any full row rank matrix which satisfies $\boldsymbol{S}_k\hat{\boldsymbol{E}}_k=0$, and ζ is a constant which satisfies $0<\zeta<1$.

Remark 1. Introduce $\boldsymbol{\delta}_{k+1}=\mathrm{diag}\{\delta_{1,k+1}\cdots\delta_{m,k+1}\}$ such that,

$$(\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k-\boldsymbol{H}_{k+1}\boldsymbol{\beta}_k\boldsymbol{F}_k)\widetilde{\boldsymbol{x}}'_{k|k}:=\boldsymbol{\delta}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k\widetilde{\boldsymbol{x}}'_{k|k} \quad (35)$$

Then LMI (34) can be rewritten as,

$$\begin{bmatrix} (1-\zeta)\boldsymbol{P}_{k|k}^{-1} & \boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{\beta}_k\boldsymbol{S}_k^{\mathrm{T}} & \boldsymbol{F}_k^{\mathrm{T}}\boldsymbol{H}_{k+1}^{\mathrm{T}}\boldsymbol{\delta}_{k+1} \\ \boldsymbol{S}_k\boldsymbol{\beta}_k\boldsymbol{F}_k & \boldsymbol{S}_k(\boldsymbol{F}_k\boldsymbol{P}_{k|k}\boldsymbol{F}_k^{\mathrm{T}}+\boldsymbol{Q}_k)\boldsymbol{S}_k^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{\delta}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k & \boldsymbol{0} & \boldsymbol{R}_{k+1} \end{bmatrix} \geqslant 0 \quad (36)$$

which is more explicit.

Remark 2. The LMI (34)/(36) give the sufficient conditions of exponential convergence of the UIEKF, but $\boldsymbol{\beta}_k$ and $\boldsymbol{\alpha}_{k+1}$ (or $\boldsymbol{\beta}_k$ and $\boldsymbol{\delta}_{k+1}$) that are introduced to prove convergence cannot be known in practice. However, $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ are free parameters that can be arbitrarily chosen, which would help enlarging feasible domain of the LMIs. It is obvious that larger $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ make LMI (34)/(36) easier to be satisfied, i.e. the UIEKF is more likely to be convergent. Moreover, large $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ will make $\gamma_{k+1}^{\prime\mathrm{T}}\boldsymbol{C}_{k+1}\gamma_{k+1}^{\prime}$ (see Appendix) to be more negative, which also makes the UIEKF easier to be convergent. However, Boutayeb *et al.* and Guo *et al.* both pointed out that if $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ is too large, it leads to a very slow convergence rate of EKF, which will also appear in the UIEKF. Summarily, we could choose a proper larger $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ to ensure the convergence of the UIKEF, or similar to the adaptive choice of $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ in Refs. [8—10].

4 APPLICATIONS IN NONLINEAR FAULT DIAGNOSIS

Now we apply the UIEKF to the fault detection and isolation (FDI) of DTS200 three-tank system produced by Amira company in German, which is a benchmark problem in process control engineering[14] (see Fig. 1).

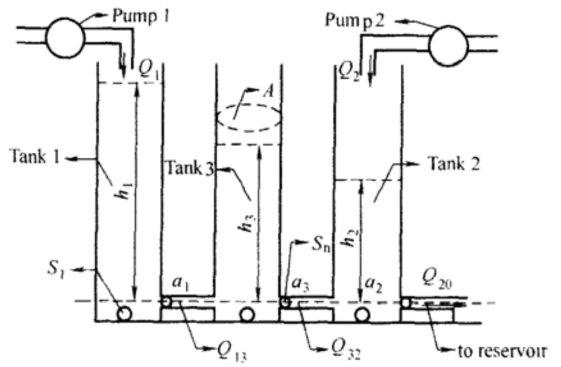

Figure 1 The layout of DTS200 three-tank system

The mathematical model of DTS200 are described as follows:

$$\begin{aligned} A\cdot \mathrm{d}h_1/\mathrm{d}t &= -Q_{13}+Q_1 \\ A\cdot \mathrm{d}h_3/\mathrm{d}t &= Q_{13}-Q_{32} \\ A\cdot \mathrm{d}h_2/\mathrm{d}t &= Q_{32}-Q_{20}+Q_2 \end{aligned} \quad (37a)$$

where

$$\begin{aligned} Q_{13} &= a_1 S_n \mathrm{sign}(h_1-h_3)\sqrt{2g|h_1-h_3|} \\ Q_{32} &= a_3 S_n \mathrm{sign}(h_3-h_2)\sqrt{2g|h_3-h_2|} \\ Q_{20} &= a_2 S_n\sqrt{2gh_2} \end{aligned} \quad (37b)$$

The state is the level of tanks: $\boldsymbol{x}=[h_1,h_2,h_3]^{\mathrm{T}}$; the input is the controlled pump flow $\boldsymbol{u}=[Q_1,Q_2]^{\mathrm{T}}$; and the output is $\boldsymbol{y}=[h_1,h_2]^{\mathrm{T}}$. The actual parameters are $g=9.81\,\mathrm{m\cdot s^{-2}}$, $a_1^0=0.5$, $a_3^0=0.45$, $a_2^0=0.6$, $A=0.0154\,\mathrm{m}^2$, $S_n=5\times10^{-5}\,\mathrm{m}^2$, $h_{\max}=(62\pm1)\,\mathrm{cm}$, $Q_{1\max}=Q_{2\max}=100\,\mathrm{ml\cdot s^{-1}}$. The levels of T1 and T2 are both controlled by PI (proportional integral) controllers, respectively, with parameters $K_{\mathrm{p}}=0.001$ (gain constant) and $T_1=5\,\mathrm{s}$ (integral time constant). It is discretized by the Euler method with sampling time $T=1\,\mathrm{s}$.

Robust FDI strategy of nonlinear systems using UIEKF here is the classical dedicated strategy just like what is generally used in the unknown input observer as before[2]. In fact, both unknown disturbances and faults can be described by $\boldsymbol{E}(\boldsymbol{x}_k)\boldsymbol{d}_k$, where $\boldsymbol{d}_k$ is the magnitude of unknown disturbances or the parameters of faults, and $\boldsymbol{E}(\cdot)$ represents the distribution matrix of unknown disturbances. Assume that all unknown disturbances in the system are $\boldsymbol{E}(\boldsymbol{x}_k)\boldsymbol{d}_k$, and the fault is described by $\boldsymbol{F}(\boldsymbol{x}_k)\boldsymbol{\theta}_k$, where $\boldsymbol{E}(\boldsymbol{x}_k)\in\boldsymbol{R}^{n\times p}$, $\boldsymbol{F}(\boldsymbol{x}_k)=[\boldsymbol{F}^1(\boldsymbol{x}_k)\cdots\boldsymbol{F}^s(\boldsymbol{x}_k)]\in\boldsymbol{R}^{n\times s}$ are assumed to be known, and $\boldsymbol{d}_k\in\boldsymbol{R}^p, \boldsymbol{\theta}_k:=[\theta_k^1\cdots\theta_k^s]^{\mathrm{T}}\in\boldsymbol{R}^s$.

To detect fault we need to design an UIEKF (called UIEKF_0, with state estimation $\hat{\boldsymbol{x}}_{k|k}^0$) which decouples all disturbance $\boldsymbol{E}(\boldsymbol{x}_k)\boldsymbol{d}_k$. The residual signal used for detection are $\boldsymbol{e}_{k+1}^0=\boldsymbol{y}_{k+1}-\boldsymbol{h}(\hat{\boldsymbol{x}}_{k+1|k+1}^0,\boldsymbol{u}_{k+1})$. Faults will be detected by comparing $\|\boldsymbol{e}_{k+1}^0\|$ with a threshold ε^0. For fault isolation, a group of UIEKF is needed to design "structured residuals". The most commonly used scheme in designing the residual set is to make each residual sensitive to all but one fault. Design s UIEKFs (called UIEKF_i, with state estimation $\hat{\boldsymbol{x}}_{k|k}^i, i=1,\cdots,s$) each of which is robust to the i^{th} fault, respectively, besides disturbances. Then UIEKF_i is designed as

$$\begin{cases} \boldsymbol{x}_{k+1}=\boldsymbol{f}(\boldsymbol{x}_k,\boldsymbol{u}_k)+[\boldsymbol{E}(\boldsymbol{x}_k)\boldsymbol{F}^i(\boldsymbol{x}_k)]\cdot\begin{bmatrix}\boldsymbol{d}_k\\ \boldsymbol{\theta}_k^i\end{bmatrix} \\ \boldsymbol{y}_{k+1}=\boldsymbol{h}(\boldsymbol{x}_{k+1},\boldsymbol{u}_{k+1}) \end{cases} \quad (38)$$

The isolation task can be performed using simple threshold testing according to the following logic:

$$\left.\begin{aligned} \|\boldsymbol{e}_{k+1}^i\| &\leqslant \varepsilon^i \\ \|\boldsymbol{e}_{k+1}^j\| &> \varepsilon^j\ \forall j\neq i \end{aligned}\right\} \Rightarrow i^{\mathrm{th}}\ \text{fault occurs} \quad (39)$$

where $e_{k+1}^i = y_{k+1} - h(\hat{x}_{k+1|k+1}^i, u_{k+1})$ and ε^i are corresponding thresholds $(i = 1, \cdots, s)$.

We consider four types of faults in this simulation.

(1) Leakage in T1: $Q_{\text{leak}}^1 = a_1 \pi r_1^2 \sqrt{2gh_1}$.

(2) Leakage in T2: $Q_{\text{leak}}^2 = a_2 \pi r_2^2 \sqrt{2gh_2}$.

(3) Clogging between T1 and T3: $a_1 = (1-\delta_1)a_1^0$.

(4) Clogging between T3 and T2: $a_3 = (1-\delta_3)a_3^0$.

i.e. fault functions and the corresponding fault parameters are

$$\boldsymbol{F}^1(x) = [-a_1\pi\sqrt{2gh_1}/A \quad 0 \quad 0]^{\mathrm{T}}, \theta^1 = r_1^2 \tag{40}$$

$$\boldsymbol{F}^2(x) = [0 \quad -a_2\pi\sqrt{2gh_2}/A \quad 0]^{\mathrm{T}}, \theta^2 = r_2^2 \tag{41}$$

$$\boldsymbol{F}^3(x) = [Q_{13}^0 \quad 0 \quad -Q_{13}^0]^{\mathrm{T}}, \theta^3 = \delta_1 \tag{42}$$

$$\boldsymbol{F}^4(x) = [0 \quad -Q_{32}^0 \quad Q_{32}^0]^{\mathrm{T}}, \theta^4 = \delta_3 \tag{43}$$

where $r_1, r_2 > 0$; $0 < \delta_1, \delta_3 \leqslant 1$; Q_{13}^0 and Q_{32}^0 are the values calculated in (37b) by substituting a_1^0 and a_3^0 into it.

For simplicity we don't consider any other unknown disturbance here. Then we can design a general EKF for fault detection, four UIEKFs for fault isolation are designed, each of which is robust to one specific type of faults, respectively. Fault isolation logic is according to Eq. (39).

In the following simulation, actual initial liquid levels are $\boldsymbol{h}^0 = [0.51 \quad 0.21 \quad 0.36]^{\mathrm{T}}$ (unit: m), and the initial states of all five observers are taken to be $\hat{\boldsymbol{x}}_{0|0} = [0.5 \quad 0.2 \quad 0.35]^{\mathrm{T}}$ (unit: m). And we take constant $\boldsymbol{Q}_k$ and $\boldsymbol{R}_{k+1}$ for simplicity with $\boldsymbol{Q}_k = 10^{-6}$ and $\boldsymbol{R}_{k+1} = 10^{-7}$, respectively. Assume that fault (1) occurs at 100 s with $r_1 = 5$ mm. Residuals of these five observers are illustrated in Fig. 2, and the FDI results are shown in Fig. 3.

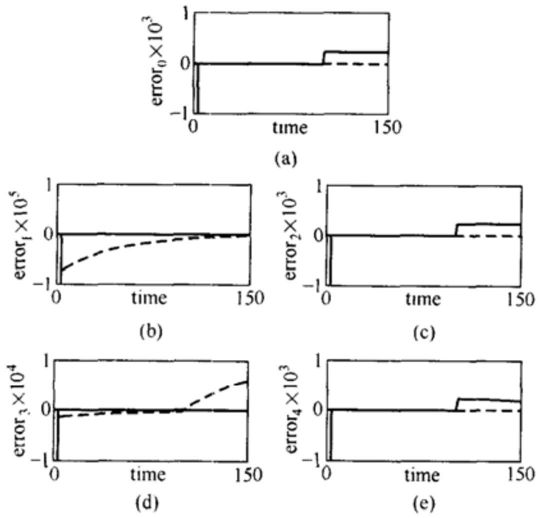

Figure 2 Residuals
(a) Residual of UIEKF_0 (EKF); (b)—(e) Residual of UIEKF_i respectively ($i = 1, 2, 3, 4$)
The solid and dash lines denote the first and second elements of the residual, respectively

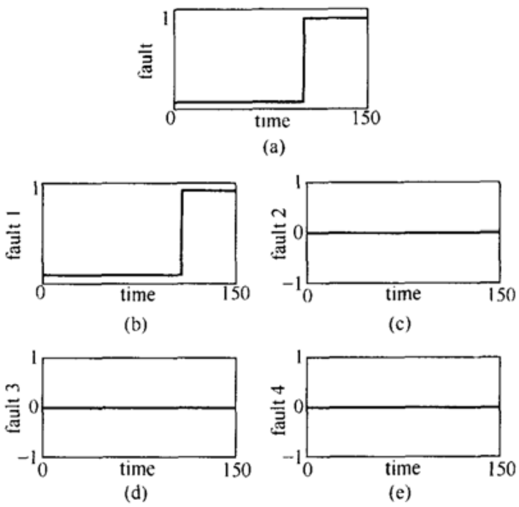

Figure 3 Fault detection and isolation results
(a) Fault detection result; (b)—(e) Fault isolation results of each type of faults, respectively

From Fig. 2(a) it is obvious that after the fault occurs, the residual of EKF increases immediately, then fault is detected as is shown in Fig. 3(a). Fig. 2(b) demonstrates clearly the effectiveness of the disturbance (fault) decoupling by the UIEKF, whose $\boldsymbol{E}(\boldsymbol{x}_k)$ matches the real one. On the other hand, the residuals of other UIEKFs will depart from zero as well as the EKF does, which are illustrated in Figs. 2(c)—(e). From Figs. 3(b)—(e), it can be seen that the fault isolation result is accurate, though there is some time delay to fault detection. Notice that the residuals in Fig. 2 are large in the beginning due to the initial error, which is easy to be excluded from faults. For briefness, simulations of other faults are omitted.

Remark 3. Actually in this example, the distribution matrices can be treated as constant matrices just like below,

$$\overline{\boldsymbol{F}}^1(x) = [1 \quad 0 \quad 0]^{\mathrm{T}}, \quad \overline{\theta}^1 = -a_1\pi\sqrt{2gh_1}/A \cdot r_1^2 \tag{44}$$

$$\overline{\boldsymbol{F}}^2(x) = [0 \quad 1 \quad 0]^{\mathrm{T}}, \quad \overline{\theta}^2 = -a_2\pi\sqrt{2gh_2}/A \cdot r_2^2 \tag{45}$$

$$\overline{\boldsymbol{F}}^3(x) = [1 \quad 0 \quad -1]^{\mathrm{T}}, \quad \overline{\theta}^3 = \delta_1 \cdot Q_{13}^0 \tag{46}$$

$$\overline{\boldsymbol{F}}^4(x) = [0 \quad -1 \quad 1]^{\mathrm{T}}, \quad \overline{\theta}^4 = \delta_3 \cdot Q_{32}^0 \tag{47}$$

And this situation is usual for practical problems. However, it has no effect on the results of the UIEKFs. From Lemma 3 it can be drawn that the numerical values of UIEKF are only dependent on $\boldsymbol{S}_k$, not $\boldsymbol{E}(\boldsymbol{x}_k)$ itself. And it is easy to check that the left-null space of $\boldsymbol{F}^i(x_k)$ and $\overline{\boldsymbol{F}}^i(x_k)$ $(i = 1, 2, 3, 4)$ are identical, which means the results of the UIEKFs using $\boldsymbol{F}^i(x_k)$ and $\overline{\boldsymbol{F}}^i(x_k)$ are the same. It demonstrates that the structure of $\boldsymbol{E}(\boldsymbol{x}_k)$ is the key issue, not the formulation of itself. It means that in this situation there is no error in the estimation of distribution matrix $\boldsymbol{E}(\boldsymbol{x}_k)$ in fact. Moreover, with linear measurements just as in DTS200 model which is also a general case in practice, the unknown disturbances are accurately decoupled from beginning, like that in linear cases.

5 CONCLUSIONS

Robust fault diagnosis of nonlinear systems is an important research area of FDI. However, there are few results in the past. One of the difficulties is to design a proper nonlinear observer. Inspired by unknown input Kalman filter and based on the convergence result of EKF as a nonlinear observer, we propose an algorithm of the UIEKF and analyze its convergence as a nonlinear robust observer theoretically in this paper. Simulation results show the good estimation performance of the UIEKF for nonlinear systems with unknown disturbance, and the nonlinear robust FDI of "DTS200" three-tank system based on the UIEKF is effective.

NOMENCLATURE

A	section of cylinder
a_i	outflow coefficients
$\boldsymbol{B}_k$	the input matrix
$\boldsymbol{d}_k$	the unknown input vector
$\boldsymbol{E}_k$	the distribution matrix of the unknown input
$\boldsymbol{F}_k$	the system matrix
g	earth gravity acceleration
$\boldsymbol{H}_{k+1}$	the measurement matrix
h_i	liquid levels
$\boldsymbol{L}_{k+1}$	the gain matrix
Q_i	supplying flow rates
Q_{ij}	flow rates
$\boldsymbol{Q}_k$	the variance matrix of $\boldsymbol{w}_k$
R^i	real linear vector space of dimension i
$R^{i\times j}$	real matrix space of dimension $i \times j$
$\boldsymbol{R}_{k+1}$	the variance matrix of $\boldsymbol{v}_{k+1}$
$\mathrm{rank}(\boldsymbol{Z})$	rank of matrix $\boldsymbol{Z}$
S_n	section of connection pipe
$\mathrm{sign}(z)$	sign of argument z
$\boldsymbol{u}_k$	the known input vector
$\boldsymbol{v}_{k+1}$	the measurement noises
$\boldsymbol{w}_k$	the system noises
$\boldsymbol{x}_k$	the state vector
$\hat{\boldsymbol{x}}_{k\|k}$	a posteriori estimate of $\boldsymbol{x}_k$
$\hat{\boldsymbol{x}}_{k+1\|k}$	a priori estimate of $\boldsymbol{x}_{k+1}$
$\boldsymbol{y}_k$	the observation vector
$\lambda_{\min}(\boldsymbol{Z})$	the minimum eigenvalue of matrix $\boldsymbol{Z}$

Superscripts

T	transpose operation of a matrix
-1	inverse operation of a matrix

Subscripts

k	current time

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APPENDIX

Proof of Lemma 2. $\boldsymbol{E}$ has a singular decomposition as follows:

$$\boldsymbol{E} = \boldsymbol{U}^{\mathrm{T}} \begin{bmatrix} \Sigma \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{V} \tag{A1}$$

where $\Sigma \in \boldsymbol{R}^{q\times q}$ is a diagonal matrix, and $\boldsymbol{U}$, $\boldsymbol{V}$ are orthogonal matrices with proper dimensions.

Let $\overline{P} = UPU^{\mathrm{T}} = \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{21} & \overline{P}_{22} \end{bmatrix}$ ($\overline{P}_{11} \in R^{q\times q}$), then

$$\begin{aligned} M =& P - PU^{\mathrm{T}} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V(V^{\mathrm{T}}[\Sigma \quad 0]\overline{P} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V)^{-1} V^{\mathrm{T}}[\Sigma \quad 0]UP \\ =& P - PU^{\mathrm{T}} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \Sigma^{-1}\overline{P}_{11}^{-1}\Sigma^{-1}[\Sigma \quad 0]UP \\ =& U^{\mathrm{T}}(\overline{P} - \overline{P} \begin{bmatrix} \overline{P}_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \overline{P})U \\ =& U^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & \overline{P}_{22} - \overline{P}_{21}\overline{P}_{11}^{-1}\overline{P}_{12} \end{bmatrix} U \end{aligned} \tag{A2}$$

On the other hand, any full row rank matrix $S \in R^{(n-q)\times n}$ satisfying $SE = 0$ can be expressed by

$$S = X[0 \quad I_{n-q}]U \tag{A3}$$

where $X \in R^{(n-q)\times(n-q)}$ is any full rank square matrix. Then

$$\begin{aligned} S^{\mathrm{T}}(SP^{-1}S^{\mathrm{T}})^{-1}S =& U^{\mathrm{T}} \begin{bmatrix} 0 \\ I \end{bmatrix} X^{\mathrm{T}} \left(X[0 \quad I]UP^{-1}U^{\mathrm{T}} \begin{bmatrix} 0 \\ I \end{bmatrix} X^{\mathrm{T}} \right)^{-1} X[0 \quad I]U \\ =& U^{\mathrm{T}} \begin{bmatrix} 0 \\ I \end{bmatrix} \left([0 \quad I]U(U^{\mathrm{T}}\overline{P}U)^{-1}P^{\mathrm{T}} \begin{bmatrix} 0 \\ I \end{bmatrix} \right)^{-1} [0 \quad I]U \\ =& U^{\mathrm{T}} \begin{bmatrix} 0 \\ I \end{bmatrix} \left([0 \quad I]\overline{P}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \right)^{-1} [0 \quad I]U \end{aligned} \tag{A4}$$

Since

$$\overline{P}^{-1} = \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{21} & \overline{P}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (\overline{P}_{11} - \overline{P}_{12}\overline{P}_{22}^{-1}\overline{P}_{21})^{-1} & -(\overline{P}_{11} - \overline{P}_{12}\overline{P}_{22}^{-1}\overline{P}_{21})^{-1}\overline{P}_{12}\overline{P}_{22}^{-1} \\ -(\overline{P}_{22} - \overline{P}_{21}\overline{P}_{11}^{-1}\overline{P}_{12})^{-1}\overline{P}_{21}\overline{P}_{11}^{-1} & (\overline{P}_{22} - \overline{P}_{21}\overline{P}_{11}^{-1}\overline{P}_{12})^{-1} \end{bmatrix} \tag{A5}$$

$$S^{\mathrm{T}}(SP^{-1}S^{\mathrm{T}})^{-1}S = U^{\mathrm{T}} \begin{bmatrix} 0 \\ I \end{bmatrix} \left((\overline{P}_{22} - \overline{P}_{21}\overline{P}_{11}^{-1}\overline{P}_{12})^{-1} \right)^{-1} [0 \quad I]U = U^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & \overline{P}_{22} - \overline{P}_{21}\overline{P}_{11}^{-1}\overline{P}_{12} \end{bmatrix} U \tag{A6}$$

It is obvious that $M = P - PE(E^{\mathrm{T}}PE)^{-1}E^{\mathrm{T}}P = S^{\mathrm{T}}(SP^{-1}S^{\mathrm{T}})^{-1}S$.

Proof of Lemma 3. Eq. (30) can be simply obtained by combining the results of Lemma 1 and Lemma 2.

According to Eqs. (11) and (14), it is given by

$$\begin{aligned} R_{k+1}^{-1}(I - H_{k+1}K_{k+1}) =& R_{k+1}^{-1}(I - H_{k+1}P_{k+1|k}H_{k+1}^{\mathrm{T}}V_{k+1}^{-1}) = R_{k+1}^{-1}(I - (V_{k+1} - R_{k+1})V_{k+1}^{-1}) \\ =& R_{k+1}^{-1}(I - V_{k+1}V_{k+1}^{-1} - R_{k+1}V_{k+1}^{-1}) = V_{k+1}^{-1} \end{aligned} \tag{A7}$$

Then Eq. (31) holds. Based on Eq. (31) we can get

$$\begin{aligned} C_{k+1} =& R_{k+1}^{-1}(H_{k+1}P_{k+1|k+1}H_{k+1}^{\mathrm{T}}R_{k+1}^{-1} - I) = R_{k+1}^{-1}(H_{k+1}L_{k+1} - I) \\ =& R_{k+1}^{-1}H_{k+1}\eta_{k+1}\Pi_{k+1} + R_{k+1}^{-1}(H_{k+1}K_{k+1} - I) \\ =& R_{k+1}^{-1}H_{k+1}(I - K_{k+1}H_{k+1})\hat{E}_k[(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1}(H_{k+1}\hat{E}_k)]^{-1}(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1} - V_{k+1}^{-1} \\ =& (R_{k+1}^{-1}H_{k+1}\hat{E}_k - R_{k+1}^{-1}H_{k+1}K_{k+1}H_{k+1}\hat{E}_k)[(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1}(H_{k+1}\hat{E}_k)]^{-1}(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1} - V_{k+1}^{-1} \\ =& R_{k+1}^{-1}(I - H_{k+1}K_{k+1})(H_{k+1}\hat{E}_k)[(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1}(H_{k+1}\hat{E}_k)]^{-1}(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1} - V_{k+1}^{-1} \\ =& V_{k+1}^{-1}(H_{k+1}\hat{E}_k)[(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1}(H_{k+1}\hat{E}_k)]^{-1}(H_{k+1}\hat{E}_k)^{\mathrm{T}}V_{k+1}^{-1} - V_{k+1}^{-1} \end{aligned} \tag{A8}$$

Since condition (4) is satisfied, $H_{k+1}\hat{E}_k$ is full column rank as well as $\hat{E}_k$. Then according to Lemma 2, we have Eq. (32).

Proof of Theorem 1. The candidate Lyapunov function is defined in Eq. (21). Then,

$$\begin{aligned}
V_{k+1} =&(\widetilde{\boldsymbol{x}}_{k+1|k}-\boldsymbol{L}_{k+1}\boldsymbol{\gamma}_{k+1})^{\mathrm{T}}\boldsymbol{P}_{k+1|k+1}^{-1}(\widetilde{\boldsymbol{x}}_{k+1|k}-\boldsymbol{L}_{k+1}\boldsymbol{\gamma}_{k+1})\\
=&(\widetilde{\boldsymbol{x}}'_{k+1|k}-\boldsymbol{L}_{k+1}\boldsymbol{\gamma}'_{k+1})^{\mathrm{T}}\boldsymbol{P}_{k+1|k+1}^{-1}\\
=&(\widetilde{\boldsymbol{x}}'_{k+1|k}-\boldsymbol{L}_{k+1}\boldsymbol{\gamma}'_{k+1})\\
=&(\widetilde{\boldsymbol{x}}'^{\mathrm{T}}_{k+1|k}\boldsymbol{P}_{k+1|k+1}^{-1}\widetilde{\boldsymbol{x}}'_{k+1|k}-\widetilde{\boldsymbol{x}}'^{\mathrm{T}}_{k+1|k}\boldsymbol{P}_{k+1|k+1}^{-1}\boldsymbol{L}_{k+1}\boldsymbol{\gamma}'_{k+1}-\\
&\boldsymbol{\gamma}'^{\mathrm{T}}_{k+1}\boldsymbol{L}^{\mathrm{T}}_{k+1}\boldsymbol{P}_{k+1|k+1}^{-1}\widetilde{\boldsymbol{x}}'_{k+1|k}+\boldsymbol{\gamma}'^{\mathrm{T}}_{k+1}\boldsymbol{L}^{\mathrm{T}}_{k+1}\boldsymbol{P}_{k+1|k+1}^{-1}\boldsymbol{L}_{k+1}\boldsymbol{\gamma}'_{k+1}\\
=&\widetilde{\boldsymbol{x}}'^{\mathrm{T}}_{k+1|k}\boldsymbol{S}_k^{\mathrm{T}}(\boldsymbol{S}_k\boldsymbol{P}_{k+1|k}\boldsymbol{S}_k^{\mathrm{T}})^{-1}\boldsymbol{S}_k\widetilde{\boldsymbol{x}}'_{k+1|k}+\widetilde{\boldsymbol{x}}'^{\mathrm{T}}_{k+1|k}\boldsymbol{H}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{H}_{k+1}\widetilde{\boldsymbol{x}}_{k+1|k}-\widetilde{\boldsymbol{x}}'^{\mathrm{T}}_{k+1|k}\boldsymbol{H}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{\gamma}'_{k+1}-\\
&\boldsymbol{\gamma}'^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{H}_{k+1}\widetilde{\boldsymbol{x}}'_{k+1|k}+\boldsymbol{\gamma}'^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k+1}\boldsymbol{H}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{\gamma}'_{k+1}\\
=&\widetilde{\boldsymbol{x}}'^{\mathrm{T}}_{k+1|k}\boldsymbol{S}_k^{\mathrm{T}}(\boldsymbol{S}_k\boldsymbol{P}_{k+1|k}\boldsymbol{S}_k^{\mathrm{T}})^{-1}\boldsymbol{S}_k\widetilde{\boldsymbol{x}}'_{k+1|k}+(\boldsymbol{\gamma}'_{k+1}-\boldsymbol{H}_{k+1}\widetilde{\boldsymbol{x}}'_{k+1|k})^{\mathrm{T}}\cdot\boldsymbol{R}^{-1}_{k+1}(\boldsymbol{\gamma}'_{k+1}-\boldsymbol{H}_{k+1}\widetilde{\boldsymbol{x}}'_{k+1|k})+\\
&\boldsymbol{\gamma}'^{\mathrm{T}}_{k+1}(\boldsymbol{R}^{-1}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{P}_{k+1|k+1}\boldsymbol{H}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}-\boldsymbol{R}^{-1}_{k+1})\boldsymbol{\gamma}'_{k+1}\\
=&\widetilde{\boldsymbol{x}}'^{\mathrm{T}}_{k+1|k}\boldsymbol{S}_k^{\mathrm{T}}(\boldsymbol{S}_k\boldsymbol{P}_{k+1|k}\boldsymbol{S}_k^{\mathrm{T}})^{-1}\boldsymbol{S}_k\widetilde{\boldsymbol{x}}'_{k+1|k}+(\boldsymbol{\gamma}'_{k+1}-\boldsymbol{H}_{k+1}\widetilde{\boldsymbol{x}}'_{k+1|k})^{\mathrm{T}}\boldsymbol{R}^{-1}_{k+1}(\boldsymbol{\gamma}'_{k+1}-\boldsymbol{H}_{k+1}\widetilde{\boldsymbol{x}}'_{k+1|k})+\\
&\boldsymbol{\gamma}'^{\mathrm{T}}_{k+1}\boldsymbol{C}_{k+1}\boldsymbol{\gamma}'_{k+1}
\end{aligned}\tag{A9}$$

From Lemma 3 it is known that $\boldsymbol{C}_{k+1}$ is semi-negative definite, then,

$$\begin{aligned}
V_{k+1}\leqslant&\widetilde{\boldsymbol{x}}^{\mathrm{T}}_{k|k}[\boldsymbol{F}^{\mathrm{T}}_k\boldsymbol{\beta}_k\boldsymbol{S}^{\mathrm{T}}_k(\boldsymbol{S}_k(\boldsymbol{F}_k\boldsymbol{P}_{k|k}\boldsymbol{F}^{\mathrm{T}}_k+\boldsymbol{Q}_k)\boldsymbol{S}^{\mathrm{T}}_k)^{-1}\boldsymbol{S}_k\boldsymbol{\beta}_k\boldsymbol{F}_k+\\
&(\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k-\boldsymbol{H}_{k+1}\boldsymbol{\beta}_k\boldsymbol{F}_k)^{\mathrm{T}}\boldsymbol{R}^{-1}_{k+1}(\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k-\boldsymbol{H}_{k+1}\boldsymbol{\beta}_k\boldsymbol{F}_k)]\widetilde{\boldsymbol{x}}'_{k|k}
\end{aligned}\tag{A10}$$

If the following inequality holds:

$$\begin{aligned}
V_{k+1}-(1-\zeta)V_k\leqslant&\widetilde{\boldsymbol{x}}^{\mathrm{T}}_{k|k}[-(1-\zeta)\boldsymbol{P}^{-1}_{k|k}+\boldsymbol{F}^{\mathrm{T}}_k\boldsymbol{\beta}_k\boldsymbol{S}^{\mathrm{T}}_k(\boldsymbol{S}_k(\boldsymbol{F}_k\boldsymbol{P}_{k|k}\boldsymbol{F}^{\mathrm{T}}_k+\boldsymbol{Q}_k)\boldsymbol{S}^{\mathrm{T}}_k)^{-1}\boldsymbol{S}_k\boldsymbol{\beta}_k\boldsymbol{F}_k+\\
&(\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k-\boldsymbol{H}_{k+1}\boldsymbol{\beta}_k\boldsymbol{F}_k)^{\mathrm{T}}\boldsymbol{R}^{-1}_{k+1}(\boldsymbol{\alpha}_{k+1}\boldsymbol{H}_{k+1}\boldsymbol{F}_k-\boldsymbol{H}_{k+1}\boldsymbol{\beta}_k\boldsymbol{F}_k)]\widetilde{\boldsymbol{x}}'_{k|k}\leqslant 0
\end{aligned}\tag{A11}$$

where the constant $0<\zeta<1$, then $\{V_k\}_{k=1,\cdots}$ is an exponential decreasing sequence, *i.e.*

$$V_{k+1}-V_k\leqslant-\zeta V_k\tag{A12}$$

According to Schur complement Lemma[15], (A11) is equivalent to LMI (34). Then by Assumption 1,

$$0\leqslant\frac{1}{\mu_1}\widetilde{\boldsymbol{x}}^{\mathrm{T}}_{k|k}\widetilde{\boldsymbol{x}}_{k|k}\leqslant V_k\leqslant(1-\zeta)^kV_0\Rightarrow 0\leqslant\lim_{k\to\infty}(\widetilde{\boldsymbol{x}}^{\mathrm{T}}_{k|k}\widetilde{\boldsymbol{x}}_{k|k})\leqslant\mu_1\lim_{k\to\infty}V_k\leqslant\mu_1V_0\lim_{k\to\infty}(1-\zeta)^k=0\tag{A13}$$

i.e. $\lim\limits_{k\to\infty}\|\widetilde{\boldsymbol{x}}_k\|=0$.