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Gorenstein projective modules

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Abstract: The aim of this lecture note is to outline some basic concept, results, typical proofs and some recent progress on Gorenstein homological algebras.

Key words: Gorenstein projective module; Gorenstein algebra; approximation

Gorenstein 投射模

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摘要：本文简要概述 Gorenstein 同调代数中最基本的概念、结果、典型的证明以及最近的若干进展。

关键词:Gorenstein 投射模;Gorenstein 代数;逼近

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Since the influential work [EM], [ABr], [AR1], and [EJ2], the Gorenstein homological algebra has been developed to an advanced level. The main idea is to replace projectives and injectives respectively by the Gorenstein projectives and the Gorenstein injectives, introduced by Enochs and Jenda [EJ1]. This concept also goes back to the work of Auslander and Bridger [ABr], where they introduced the G-dimension of finitely generated module M over a two-sided noetherian ring: M is Goreinstein projective if and only if the G-dimension of M is zero (Theorem (4.2.6) in [Ch]; also [AM]). Now it was widely used in the singularity, the Tate cohomology, representation theory, triangulated categories, and so on.

The aim of this lecture note is to outline some basic concept, results, typical proofs and some recent progress on this subject. We omit the dual version, i.e., the ones for Gorenstein injectives. Main references are [ABu], [AR1], [EJ2], [Hol1], and [J1].

Throughout, R is an associative ring with 1; modules are left if not specified; R-Mod and R-mod are the categories of R-modules and of finitely generated R-modules, respectively. Let R-Proj, or simply, Proj, be the full subcategory of projective R-modules; and R-proj, or simply, proj, the full subcategory of finitely generated projective R-modules. Note that R-Mod is an abelian category; and if R is a left noetherian ring then R-mod is abelian.

Throughout, $\mathscr{A}$ is an abelian category with enough projective objects. A subcategory of $\mathscr{A}$ always means closed under isomorphisms. We often take $\mathscr{A}$ to be R-Mod, or R-mod with R left noetherian.

The following concept is fundamental. Let $\mathscr{X}$ be a full subcategory of $\mathscr{A}$, and $M\in\mathscr{A}$. Recall from [AR1] (also [EJ1]) that *a right $\mathscr{X}$-approximation* (or, *$\mathscr{X}$-precover*) of M is a morphism $f: X\longrightarrow M$ with $X\in\mathscr{X}$, such that the induced map $\mathrm{Hom}_{\mathscr{A}}(X', X)\longrightarrow \mathrm{Hom}_{\mathscr{A}}(X', M)$ is surjective for any $X'\in\mathscr{X}$. If every module M admits a right $\mathscr{X}$-approximation, then $\mathscr{X}$ is called *a contravariantly finite subcategory in $\mathscr{A}$*, or *$\mathscr{X}$ is a precovering class in $\mathscr{A}$*.

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By a $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact sequence $E^{\bullet}$, we mean that $E^{\bullet}$ itself is exact, and that $\mathrm{Hom}_{\mathscr{A}}(X,E^{\bullet})$ remains to be exact for any $X\in\mathscr{X}$. Dually, we use the terminology $\mathrm{Hom}_{\mathscr{A}}(-,\mathscr{X})$-exact sequence.

If every object M admits a surjective, right $\mathscr{X}$-approximation, then every object M has a (left) $\mathscr{X}$-resolution which is $\mathrm{Hom}_{\mathscr{A}}(\mathscr{X},-)$-exact. Such a resolution is called a *proper (left) $\mathscr{X}$-resolution of M*. Conversely, if every object admits a proper (left) $\mathscr{X}$-resolution, then every object admits a surjective, right $\mathscr{X}$-approximation.

Dually, one has the concept of *a left $\mathscr{X}$-approximation* (or, *a $\mathscr{X}$-preenvelope*) of M, *a covariantly finite subcategory* (or, *a preenveloping class*), and *a coproper (right) $\mathscr{X}$-resolution of M*.

For a full subcategory $\mathscr{X}$ of $\mathscr{A}$, put

$$^{\perp}\mathscr{X}=\{M\in\mathscr{A}\mid \mathrm{Ext}^i_{\mathscr{A}}(M,X)=0,\ \forall\, X\in\mathscr{X},\ \forall\, i\geqslant 1\}.$$

1 Gorenstein projective modules

We recall some basic properties of Gorenstein projective modules.

1.1 Following Enochs-Jenda ([EJ1], [EJ2]), *a complete projective resolution* in $\mathscr{A}$ is a $\mathrm{Hom}_{\mathscr{A}}(-,\mathrm{Proj})$-exact sequence $(\mathscr{P}^{\bullet},d)=\cdots\longrightarrow P^{-1}\xrightarrow{d^{-1}}P^0\xrightarrow{d^0}P^1\xrightarrow{d^1}P^2\longrightarrow\cdots$ of projective objects. An object M is *Gorenstein projective* in $\mathscr{A}$ if there is a complete projective resolution $(\mathscr{P}^{\bullet},d)$ such that $M\cong\mathrm{Im}\,d^{-1}$.

Denote by R-GProj, or simply, GProj, the full subcategory of Gorenstein projective objects in R-Mod. Denote by R-Gproj, or simply, Gproj, the full subcategory of Gorenstein projective objects in R-mod.

Remark 1.1 (ⅰ) *For a full subcategory ω, Auslander–Reiten ([AR1]) considered the following full subcategory*

$$\mathscr{X}_{\omega}=\{M\in R\text{-}\mathrm{Mod}\mid\exists\ \text{an exact sequence}\ 0\longrightarrow M\longrightarrow T^0\xrightarrow{d^0}T^1\xrightarrow{d^1}\cdots,\ \text{with}\ T^i\in\omega,\ \mathrm{Ker}\,d^i\in{}^{\perp}\omega,\ \forall\, i\geqslant 0\}.$$

Note that if $\omega=R$-Proj, then $\mathscr{X}_{\omega}=R$-GProj.

(ⅱ) *If R is an artin algebra, then the Gorenstein projective objects in R-*mod *are also referred as the maximal Cohen-Macaulay modules* (see e.g. [B]).

Facts 1.2 (ⅰ) *If A is a quasi-Frobenius ring, then* GProj = A-Mod.

(ⅱ) *A projective module is Gorenstein projective.*

(ⅲ) *If $(\mathscr{P}^{\bullet},d)$ is a complete projective resolution, then all $\mathrm{Im}\,d^i$ are Gorenstein projective; and $\cdots\longrightarrow P^i\longrightarrow\mathrm{Im}\,d^i\longrightarrow 0$, $0\longrightarrow\mathrm{Im}\,d^i\longrightarrow P^{i+1}\longrightarrow\cdots$ and $0\longrightarrow\mathrm{Im}\,d^i\longrightarrow P^{i+1}\longrightarrow\cdots\longrightarrow P^j\longrightarrow\mathrm{Im}\,d^j\longrightarrow 0$, $i<j$ are $\mathrm{Hom}(-,\mathrm{Proj})$-exact.*

(ⅳ) *If M is Gorenstein projective, then $\mathrm{Ext}^i_R(M,L)=0$, $\forall\, i>0$, for all modules L of finite projective dimension.*

(ⅴ) *A module M is Gorenstein projective if and only if $M\in{}^{\perp}(\mathrm{Proj})$ and M has a right* Proj-*resolution which is* $\mathrm{Hom}(-,\mathrm{Proj})$-*exact; if and only if there exists an exact sequence*

$$0\longrightarrow M\longrightarrow T^0\xrightarrow{d^0}T^1\xrightarrow{d^1}\cdots,\ \text{with}\ T^i\in\mathrm{Proj},\ \mathrm{Ker}\,d^i\in{}^{\perp}(\mathrm{Proj}),\ \forall\, i\geqslant 0.$$

(ⅵ) *For a Gorenstein projective module M, there is a complete projective resolution $\cdots\longrightarrow F^{-1}\xrightarrow{d^{-1}}F^0\xrightarrow{d^0}F^1\longrightarrow\cdots$ consisting of free modules, such that $M\cong\mathrm{Im}\,d^{-1}$.*

(ⅶ) *The projective dimension of a Gorenstein projective module is either zero or infinite. So, if* gl.dim $R<\infty$, *then* R-GProj = R-Proj.

Thus, Gorenstein projective modules make sense only to rings of infinite global dimension.

(ⅷ) *Let A be an artin algebra. Then the functor* $\mathrm{Hom}_A(-,A)$ *induces an equivalence* A-Gproj $\cong A^{op}$-Gproj *with a quasi-inverse* $\mathrm{Hom}_A(-,A)$.

Proof We include the proof of (ⅵ) and (ⅶ).

(ⅵ) There is a $\mathrm{Hom}(-,\mathrm{Proj})$-exact sequence $0\longrightarrow M\longrightarrow P^0\longrightarrow P^1\longrightarrow\cdots$ with each P^i projective. Choose projective modules Q^0, Q^1, $\cdots$, such that $F^0 = P^0 \oplus Q^0$, $F^n = P^n \oplus Q^{n-1} \oplus Q^n$, $n > 0$, are free. By adding $0 \longrightarrow Q^i \xrightarrow{=} Q^i \longrightarrow 0$ to the exact sequence in degrees i and $i+1$, we obtain a $\mathrm{Hom}(-,\mathrm{Proj})$-exact sequence of free modules. By connecting a deleted free resolution of M together with the deleted version of this sequence, we get the desired sequence.

(ⅶ) Let $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow G \longrightarrow 0$ be a projective resolution of a Gorenstein projective R-module M, with n minimal. If $n \geqslant 1$, then by $\mathrm{Ext}_R^n(G, P_n) = 0$ we know $\mathrm{Hom}(P_{n-1}, P_n) \longrightarrow \mathrm{Hom}(P_n, P_n)$ is surjective, which implies $0 \longrightarrow P_n \longrightarrow P_{n-1}$ splits. This contradicts the minimality of n.

1.2 A full subcategory $\mathscr{X}$ of R-Mod is resolving, if $\mathrm{Proj} \subseteq \mathscr{X}$; $\mathscr{X}$ is closed under extensions, the kernels of epimorphisms, and the direct summands.

Theorem 1.3 *For any ring R,* GProj *is resolving, and closed under arbitrary direct sums.*

Proof In fact, this is a special case of [AR1], Proposition 5.1. We include a direct proof given by H. Holm in [Hol1].

Easy to see GProj is closed under arbitrary direct sums; and it is closed under extension by using the corresponding Horseshoe Lemma. Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be a short exact sequence with M, M_2 Gorenstein projective. Then $M_1 \in {}^{\perp}(\mathrm{Proj})$. Construct a $\mathrm{Hom}(-,\mathrm{Proj})$-exact, right Proj-resolution of M_1 as follows. Let $\mathbf{M} = 0 \longrightarrow M \longrightarrow P^0 \longrightarrow P_1 \longrightarrow \cdots$ and $\mathbf{M}_2 = 0 \longrightarrow M_2 \longrightarrow Q^0 \longrightarrow Q_1 \longrightarrow \cdots$ be such resolutions of M and M_2, respectively. By $\mathrm{Hom}(-,\mathrm{Proj})$-exactness of $\mathbf{M}$, $M \longrightarrow M_2$ induces a chain map $\mathbf{M} \longrightarrow \mathbf{M}_2$, with mapping cone denoted by $\mathbf{C}$. Then $\mathbf{C}$ is exact, and $\mathrm{Hom}(-,\mathrm{Proj})$-exact by using the distinguished triangle and the induced long exact sequence. Consider a short exact sequence of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{M}_1 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P^0 & \longrightarrow & M_2 \oplus P^0 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q^0 \oplus P^1 & = & Q^0 \oplus P^1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Since $\mathbf{C}$, $\mathbf{D}$ are exact, so is $\mathbf{M}_1$. By a direct analysis on each row, we have an exact sequence of complexes $0 \longrightarrow \mathrm{Hom}(\mathbf{D}, P) \longrightarrow \mathrm{Hom}(\mathbf{C}, P) \longrightarrow \mathrm{Hom}(\mathbf{M}_1, P) \longrightarrow 0$ for every projective P. Since $\mathrm{Hom}(\mathbf{D}, P)$ and $\mathrm{Hom}(\mathbf{C}, P)$ are exact, so is $\mathrm{Hom}(\mathbf{M}_1, P)$. This proves that M_1 is Gorenstein projective.

It remains to prove GProj is closed under arbitrary direct summands. Using Eilenberg's swindle. Let $X = Y \oplus Z \in \mathrm{GProj}$. Put $W = Y \oplus Z \oplus Y \oplus Z \oplus Y \oplus Z \oplus \cdots$ Then $W \in \mathrm{GProj}$, and $Y \oplus W \cong W \in \mathrm{GProj}$. Consider the split exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus W \longrightarrow W \longrightarrow 0$. Then $Y \in \mathrm{GProj}$ since we have proved that GProj is closed under the kernel of epimorphisms.

1.3 Finitely generated Gorenstein projective modules

The full subcategory of finitely generated, Gorenstein projective modules is $\mathrm{GProj} \cap R\text{-mod}$. Denote by proj the full subcategory of finitely generated projective R-modules; and by Gproj the full category of modules M isomorphic to $\mathrm{Im}\, d^{-1}$, where $\cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots$ is a $\mathrm{Hom}(-,\mathrm{proj})$-exact sequence with each $P^i \in \mathrm{proj}$.

Proposition 1.4 *Let R be any ring. Then* $\mathrm{Gproj} \subseteq \mathrm{GProj} \cap R\text{-}\mathrm{mod}$.

If R is left noetherian, then $\mathrm{GProj} \cap R\text{-}\mathrm{mod} = \mathrm{Gproj}$.

Proof Let $M \in \mathrm{Gproj}$. Then there is a $\mathrm{Hom}(-,\mathrm{proj})$-exact sequence

$$\mathscr{P}^{\bullet} = \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} P^{1} \xrightarrow{d^{1}} P^{2} \longrightarrow \cdots$$

with each $P^i \in \mathrm{proj}$, such that $M \cong \mathrm{Im}\, d^{-1}$. So M is finitely generated. Since each P^i is finitely generated, it is clear that $\mathscr{P}^{\bullet}$ is also $\mathrm{Hom}(-,\mathrm{Proj})$-exact, i.e., $M \in \mathrm{GProj} \cap R\text{-mod}$.

Let R be a left noetherian ring, and $M \in \mathrm{GProj} \cap R\text{-mod}$. By Facts 1.2(ⅵ) one can take an exact sequence $0 \longrightarrow M \xrightarrow{f} F \longrightarrow X \longrightarrow 0$ with F free and X Gorenstein projective. Since M is finitely generated, one can write $F = P^0 \oplus Q^0$ with $\mathrm{Im} f \subseteq P^0$, and P^0 finitely generated. Then we have an exact sequence $0 \longrightarrow M \xrightarrow{f} P^0 \longrightarrow M' \longrightarrow 0$ with $X \cong M' \oplus Q^0$, and hence $M' \in \mathrm{GProj} \cap R\text{-}\mathrm{mod}$ by Theorem 1.3. Repeating this procedure with M' replacing M, we get an exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ with all images in $\mathrm{GProj} \cap R\text{-mod}$. Hence it is a $\mathrm{Hom}(-,\mathrm{proj})$-exact sequence. Om the other hand, since R is left noetherian, M has a finitely generated projective resolution, which is $\mathrm{Hom}(-,\mathrm{proj})$-exact since $M \in {}^{\perp}(\mathrm{proj})$. So $M \in \mathrm{Gproj}$.

1.4 Strongly Gorenstein projective modules

A complete projective resolution of the form $\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$ is said to be *strong*, and $M \cong \mathrm{Ker}\, f$ is called *a strongly Gorenstein projective module* ([BM]). Denote by SGProj the full subcategory of strongly Gorenstein projective modules. Then it is known in [BM] that $\mathrm{Proj} \subsetneqq \mathrm{SGProj} \subsetneqq \mathrm{GProj}$; and that *a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module*. So, a strongly Gorenstein projective module is an analogue of a free module.

Denote by SGproj the full subcategory of all the modules M isomorphic to $\mathrm{Ker}\, f$, where $\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$ is a complete projective resolution with P finitely generated. Then for any ring R we have $(\mathrm{SGProj}) \cap R\text{-mod} = \mathrm{SGproj}$.

2 Proper Gorenstein projective resolutions

A basic problem in Gorenstein homological algebra is, given a ring R, when R-GProj is contravariantly finite in R-Mod; or equivalently, when every module admits *a proper Gorenstein projective resolution*. Similarly, it is fundamental to know when R-Gproj is contravariantly finite in R-mod; or equivalently, when every finitely generated module admits *a proper Gorenstein projective resolution*, with each term in R-Gproj.

2.1 First, we recall a general result due to Auslander and Buchweitz [ABu].

Let $\mathscr{A}$ be an abelian category, $\mathscr{X}$ be a full subcategory of $\mathscr{A}$ closed under extensions, direct summands, and isomorphisms. Let ω be a cogenerator of $\mathscr{X}$, which means ω is a full subcategory of $\mathscr{X}$ closed under finite direct sums and isomorphisms, and for any $X \in \mathscr{X}$, there is an exact sequence $0 \longrightarrow X \longrightarrow B \longrightarrow X' \longrightarrow 0$ in $\mathscr{X}$ with $B \in \omega$. Denote by $\hat{\mathscr{X}}$ the full subcategory of $\mathscr{A}$ consisting of all objects X of finite $\mathscr{X}$-dimension n, i.e., there is an exact sequence $0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow X \longrightarrow 0$ with $X_i \in \mathscr{X}$.

Theorem 2.1 ([ABu], Theorems 1.1, 2.3, 2.5) (ⅰ) *Every object $C \in \hat{\mathscr{X}}$ has a surjective right $\mathscr{X}$-approximation. More precisely, for any $C \in \hat{\mathscr{X}}$ there is a* $\mathrm{Hom}(\mathscr{X}, -)$*-exact sequence $0 \longrightarrow Y_C \longrightarrow X_C \longrightarrow C \longrightarrow 0$ with $X_C \in \mathscr{X}$, $Y_C \in \hat{\omega}$; and $Y_C \in \mathscr{X}^{\perp}$.*

(ⅱ) *Every object $C \in \hat{\mathscr{X}}$ has a injective left $\hat{\omega}$-approximation. More precisely, for any $C \in \hat{\mathscr{X}}$ there is a* $\mathrm{Hom}(-, \hat{\omega})$*-exact sequence $0 \longrightarrow C \longrightarrow Y^C \longrightarrow X^C \longrightarrow 0$ with $X^C \in X$, $Y^C \in \hat{\omega}$; and $X^C \in {}^{\perp}\hat{\omega}$.*

2.2 *A proper Gorenstein projective resolution* of an R-module M is a $\mathrm{Hom}(\mathrm{GProj}, -)$-exact sequence $\mathscr{G}^{\bullet}: \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ with each G_i Gorenstein projective. Note that $\mathrm{Hom}(\mathrm{GProj}, -)$-exactness guarantee the uniqueness of such a resolution in the homotopy category.

2.3 *The Gorenstein projective dimension*, Gpd M, of R-module M is defined as the smallest integer $n \geqslant 0$ such that M has a GProj-resolution of length n.

Theorem 2.2 *Let M be an R-module of finite Gorenstein projective dimension n. Then M admits a surjective right* GProj-*approximation* $\phi: G \longrightarrow M$, *with* pd $\operatorname{Ker}\phi = n-1$ (*if* $n=0$, *then* $K=0$).

In particular, a module of finite Gorenstein projective dimension n admits a proper Gorenstein projective resolution of length at most n.

Proof This follows from Theorem 2.1(i) by letting $\mathscr{X}$ = GProj, ω = Proj. However, we include a direct proof given by H. Holm in [Hol1].

Recall the Auslander–Bridger Lemma ([ABR], Lemma 3.12), which in particular showing that any two "minimal" resolutions are of the same length.

Auslander–Bridger Lemma Let $\mathscr{X}$ be a resolving subcategory of an abelian category $\mathscr{A}$ having enough projective objects. If $0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow A \longrightarrow 0$ are exact sequences with X_i, $Y_i \in \mathscr{X}$, $0 \leqslant i \leqslant n-1$, then $X_n \in \mathscr{X}$ if and only if $Y_n \in \mathscr{X}$.

Coming back to the proof of Theorem 2.2. Take an exact sequence $0 \longrightarrow K' \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with P_i projective. By the Auslander–Bridger Lemma, K' is Gorenstein projective. Hence there is a Hom($-$, Proj)-exact sequence $0 \longrightarrow K' \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \cdots Q^{n-1} \longrightarrow G \longrightarrow 0$, where Q^i are projective, G is Gorenstein projective. Thus there exist homomorphisms $Q^i \longrightarrow P_{n-1-i}$ for $i = 0, \cdots, n-1$, and $G \longrightarrow M$, such that the following diagram is commutative

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & K' & \longrightarrow & Q^0 & \longrightarrow & Q^1 & \longrightarrow & \cdots & \longrightarrow & Q^{n-1} & \longrightarrow & G' & \longrightarrow & 0 \\ & & & & \vdots\downarrow & & \vdots\downarrow & & & & \vdots\downarrow & & \vdots\downarrow & & \\ 0 & \longrightarrow & K' & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Let $C_1^{\bullet}$ and $C_2^{\bullet}$ denote the upper and the lower row, respectively. Then we have a distinguished triangle in the homotopy category $C_1^{\bullet} \xrightarrow{f^{\bullet}} C_2^{\bullet} \longrightarrow \operatorname{Con}(f^{\bullet}) \longrightarrow C_1^{\bullet}[1]$. Since H^0 is a cohomology functor, it follows that $\operatorname{Con}(f^{\bullet})$ is also exact, i.e., we have exact sequence

$$0 \longrightarrow K' \xrightarrow{\alpha} Q^0 \oplus K' \longrightarrow Q^1 \oplus P_{n-1} \longrightarrow \cdots \longrightarrow Q^{n-1} \oplus P_1 \longrightarrow G' \oplus P_0 \longrightarrow M \longrightarrow 0$$

with α splitting mono. It follows that we have exact sequence

$$0 \longrightarrow Q^0 \longrightarrow Q^1 \oplus Q_{n-1} \longrightarrow \cdots \longrightarrow Q^{n-1} \oplus Q_1 \longrightarrow G' \oplus P_0 \xrightarrow{\phi} M \longrightarrow 0$$

with $G' \oplus P_0$ Gorenstein projective, pd $\operatorname{Ker}\phi \leqslant n-1$ (then necessarily pd $\operatorname{Ker}\phi = n-1$). Since $\operatorname{Ext}^i(H, \text{Proj}) = 0$ for $i \geqslant 1$ and Gorenstein projective module H, so in particular $\operatorname{Ext}^1(H, \operatorname{Ker}\phi) = 0$, and hence ϕ is a left GProj-approximation.

Corollary 2.3 *If* $0 \longrightarrow G' \longrightarrow G \longrightarrow M \longrightarrow 0$ *is a short exact sequence with* G', G *Gorenstein projective, and* $\operatorname{Ext}^1(M, \text{Proj}) = 0$, *then M is Gorenstein projective.*

Proof Since Gpd $M \leqslant 1$, by the theorem above there is an exact sequence $0 \longrightarrow Q \longrightarrow E \longrightarrow M \longrightarrow 0$ with E Gorenstein projective and Q projective. By assumption $\operatorname{Ext}^1(M, Q) = 0$, hence M is Gorenstein projective by Theorem 1.3.

Theorem 2.4 *Let R be a left noetherian ring, and M a finitely generated module with* Gpd $M = n < \infty$. *Then M has a surjective left* Gproj-*approximation* $G \longrightarrow M$ *with kernel of projective dimension $n-1$. Hence M has a proper finitely generated Gorenstein projective resolution of length at most n.*

Proof The proof is same as the one of Theorem 2.2. Note that if R is left noetherian, then R-mod is again an abelian category.

2.4 We list some facts on the Gorenstein projective dimension of modules.

Proposition 2.5 (1) ([Hol1]) *We have* $\operatorname{Gpd}(\oplus M_i) = \sup\{\operatorname{Gpd} M_i \mid i \in I\}$.

(2) ([Hol1]) *Let M be R-module of finite Gorenstein projective dimension, and $n \geqslant 0$ be an integer. Then the following are equivalent.*

(ⅰ) Gpd $M \leqslant n$.

(ⅱ) $\mathrm{Ext}^i(M, L) = 0$ *for all* $i > n$ *and modules* L *with finite* pd L.

(ⅲ) $\mathrm{Ext}^i(M, Q) = 0$ *for all* $i > n$ *and projective modules* Q.

(ⅳ) *For every exact sequence* $0 \to K^{-n} \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to M \to 0$ *with all* G^i *Gorenstein projective, then also* K^{-n} *is Gorenstein projective.*

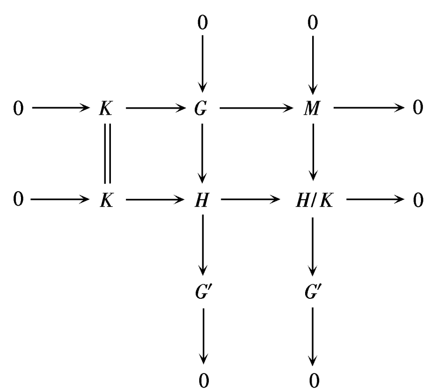
(3) ([Hol1]) *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence. If any two of the modules have finite Gorenstein projective dimension, then so has the third.*

(4) ([Hol2], Theorem 2.2) *If id* $M < \infty$, *then* Gpd M = pd M.

Proof We include the proof of (4) given by H. Holm in [Hol2]. It is clear Gpd $M \leqslant$ pd M. It suffices to prove Gpd $M \geqslant$ pd M. We may assume that Gpd $M = n < \infty$.

First, assume that $n = 0$, i.e., M is Gorenstein projective. Then we have an exact sequence $0 \to M \to P \to M' \to 0$ with P projective and M' Gorenstein projective. So we have exact sequence $0 \to M' \xrightarrow{d_{-1}} P_0 \xrightarrow{d_0} \cdots \to P_m \xrightarrow{d_m} P_{m+1} \to \cdots$ Since id$M < \infty$, it follows that $\mathrm{Ext}^1(M', M) = \mathrm{Ext}^2(\mathrm{Im} d_0, M) = \cdots = 0$. So M is also projective, and hence the equality holds in this case.

Now assume that $n > 0$. Then by Theorem 2.2 we have an exact sequence $0 \to K \to G \to M \to 0$ with pd $K = n - 1$ and G Gorenstein projective. So we have an exact sequence $0 \to G \to H \to G' \to 0$ with H projective and G' Gorenstein projective. Consider the following commutative diagram with exact rows and columns



Since H is projective and pd $K = n - 1$, we get pd$(H/K) \leqslant n$ by the second row. Since G' is Gorenstein projective and id $M < \infty$, with the same argument as before we have $\mathrm{Ext}^1(G', M) = 0$. So the column on right splits, and hence pd $M \leqslant n$. This completes the proof.

2.5 If M has a proper Gorenstein projective resolution $G^{\bullet} \to M \to 0$, then for any module N, *the Gorenstein right derived functor* $\mathrm{Ext}^n_{\mathrm{GProj}}(-, N)$ of $\mathrm{Hom}_R(-, N)$ is defined as $\mathrm{Ext}^n_{\mathrm{GProj}}(M, N) := H^n \mathrm{Hom}_R(G^{\bullet}, N)$. Note that it is only well-defined on the modules having proper Gorenstein projective resolutions. Dually, fix a module M, one has *the Gorenstein right derived functor* $\mathrm{Ext}^n_{\mathrm{GInj}}(M, -)$ of $\mathrm{Hom}_R(M, -)$, which is defined on the modules having coproper Gorenstein injective resolutions.

Theorem 2.6 ([AM], [H3]) *For all modules M and N with* Gpd $_RM < \infty$ *and* Gid $_RN < \infty$, *one has isomorphisms* $\mathrm{Ext}^n_{\mathrm{GProj}}(M, N) \cong \mathrm{Ext}^n_{\mathrm{GInj}}(M, N)$, *which are functorial in M and N; and if either* pd $M < \infty$ *or* id $N < \infty$, *then the group above coincides with* $\mathrm{Ext}^n(M, N)$.

Remark 2.7 (ⅰ) *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a short* Hom(GProj, −)*-exact sequence, where all* M_i *have proper Gorenstein projective resolutions, then for any* N, $\mathrm{Ext}^n_{\mathrm{GProj}}(-, N)$ *induce a desired long exact sequence* ([AM], [V]).

(ⅱ) *If* $0 \to N_1 \to N_2 \to N_3 \to 0$ *is a short* Hom(GProj, −)*-exact sequence, and M has a proper Gorenstein projective resolution, then* $\mathrm{Ext}^n_{\mathrm{GProj}}(M, -)$ *induce a desired long exact sequence* ([EJ2], [AM], [V]).

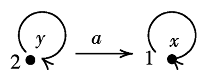
(ⅲ) *Over some special rings (e.g. the Gorenstein rings), the Gorenstein* Ext *groups, the usual Ext groups, and the*

Tate cohomology groups can be related by a long exact sequence ([AM], [J2]).

3 Gorenstein rings

A Iwanaga-Gorenstein ring, or simply, *a Gorenstein ring* R, is a left and right noetherian ring, and id ${}_RR$, id $R_R < \infty$ (see e.g. [EJ2], p. 211). A Gorenstein ring R is n-Gorenstein if id ${}_RR \leqslant n$. In this case id $R_R < \infty$ ([EJ2], 9.1.9). A k-algebra A is a Gorenstein algebra if A is a Gorenstein ring. Then a finite-dimensional k-algebra A is Gorenstein if and only if id ${}_AA < \infty$ and pd $\mathrm{Hom}_k(A_A, k) < \infty$.

3.1 Quasi-Frobenius rings are 0-Gorenstein rings; rings of finite global dimension are Gorenstein. The k-algebra given by the quiver



with relations x^2, y^2, $ay - xa$ (the conjunction of paths is from right to left) is Gorenstein of infinite global dimension, but not self-injective.

Let A be an artin algebra ([ARS]), $T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ with multiplication given by the one of matrices. Then A is n-Gorenstein if and only if $T_2(A)$ is $(n+1)$-Gorenstein (see [FGR]; also [Hap]). Note that $A \otimes_k B$ is Gorenstein if and only if A and B are Gorenstein ([AR2]); and that $\begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$ is Gorenstein if and only if A and B are Gorenstein, and pd ${}_AM < \infty$ and pd $M_B < \infty$ ([Chen2]). Also, cluster tilted algebras are Gorenstein ([KR]).

The following result gives an inductive construction of finitely generated Gorenstein projective modules over Gorenstein artin algebras. For the description of $T_2(A)$-mod we refer to [ARS] and [R].

Theorem 3.1 ([GZ]) *Let A be a Gorenstein artin algebra. Then a finitely generated $T_2(A)$-module $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$ is Gorenstein projective if and only if X, Y and* Coker ϕ *are finitely generated Gorenstein projective A-modules and $\phi: Y \longrightarrow X$ is injective.*

If A is self-injective, then every A-module is Gorenstein-projective. So we have

Corollary 3.2 ([GZ]) *Let A be a self-injective algebra. Then $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$ is a finitely generated Gorenstein projective $T_2(A)$-module if and only if ϕ is injective.*

3.2 One has the following basic property of an n-Gorenstein ring.

Theorem 3 (Iwanaga, 1980) *Let R be an n-Gorenstein ring, and M a left R-module. Then the following are equivalent.*

(i) id $M < \infty$.

(ii) id $M \leqslant n$.

(iii) pd $M < \infty$.

(iv) pd $M \leqslant n$.

(v) fd $M < \infty$.

(vi) fd $M \leqslant n$.

Proof Before proving we recall some useful facts.

(A) If R is a left noetherian ring, id ${}_RR \leqslant n$, then id ${}_RP \leqslant n$ for any left projective module P.

In fact, P may be a direct summand of an infinite direct sum $R^{(I)}$. It suffices to see id $R^{(I)} \leqslant n$. This follows from the following result: a ring R is a left noetherian ring if and only if every direct sum of injective R-modules is injective (see e.g. Theorem 3.1.17 in [EJ2]).

(B) A module is flat if and only if it is a direct limit of finitely generated projective modules.

The sufficiency follows from $\mathrm{Tor}_i(X, \varinjlim N_j) = \varinjlim \mathrm{Tor}_i(X, N_j)$, $\forall i \geqslant 1$, $\forall X_R$. The first proof of the necessity was given by V.E. Govorov in [G]; and then also by D. Lazard in [L]. See also [O], Theorem 8.16.

(C) If R is left noetherian, and id ${}_RR \leqslant n$, then id ${}_RF \leqslant n$, for any left flat module F.

In fact, by (B) we have id F = id $\varinjlim P_i$. Since R is left noetherian, we have

$$\mathrm{Ext}_R^i(M, \varinjlim N_i) = \varinjlim \mathrm{Ext}_R^i(M, N_i)$$

for any finitely generated module M. It follows that id $\varinjlim N_i \leqslant \sup\{\mathrm{id}\ N_i\}$. Now the assertion follows from (A).

(iii)⇒(ii): Since pd $M < \infty$, and id $P \leqslant n$ for any projective module P by (A), it follows that id $M \leqslant n$.

(v)⇒(iv): Let m = fd $M < \infty$. Take a projective resolution of M

$$\cdots \longrightarrow P_t \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

If $m > n$, then by dimension shift we see $F = \mathrm{Im}(P_m \longrightarrow P_{m-1})$ is flat, and hence id $F \leqslant n$ by (C). Then $\mathrm{Ext}^m(M, F) = 0$, which implies $\mathrm{Hom}(P_{m-1}, F) \longrightarrow \mathrm{Hom}(F, F)$ is surjective, and hence $F \hookrightarrow P_{n-1}$ splits, say $P_{n-1} = F \oplus G$. Hence we have exact sequence

$$0 \longrightarrow G \longrightarrow P_{m-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with G projective. If $m - 1 = n$ then we are done. If $m - 1 > n$ then we repeat the procedure with G replacing F. So we see pd $M \leqslant n$.

If $m \leqslant n$, then we also have pd $M \leqslant n$. Otherwise d = pd $M > n$, then one can choose $d' < \infty$, $n < d' \leqslant d$. Note that $F' = \mathrm{Im}(P_{d'} \longrightarrow P_{d'-1})$ is again flat by dimension shift, and hence id $F' \leqslant n$ by (C). Then $\mathrm{Ext}^{d'}(M, F') = 0$, which implies $\mathrm{Hom}(P_{d'-1}, F') \longrightarrow \mathrm{Hom}(F', F')$ is surjective, and hence $F' \hookrightarrow P_{d'-1}$ splits, say $P_{d'-1} = F' \oplus G'$. Hence we have exact sequence

$$0 \longrightarrow G' \longrightarrow P_{d'-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with G' projective. This contradicts d = pd M.

(i)⇒(vi): It suffices to prove fd $I \leqslant n$ for any injective left R-module.

Note that $I^+ = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is flat right R-module. Then by the right version of (C) one has id $I^+ \leqslant n$, and hence fd $I^{++} \leqslant n$. However, I is a pure submodule of I^{++}, it follows that (for details see Appendix) fd $I \leqslant$ fd $I^{++} \leqslant n$.

Now we have (i)⇒(vi)⇒(v)⇒(iv)⇒(iii)⇒(ii)⇒(i).

3.3 As we will see, for an n-Gorenstein ring, the full subcategory of Gorenstein projective modules is exactly the left perpendicular of projective modules.

Lemma 3.4 ([EJ2], Lemma 10.2.13) *Let R be a Gorenstein ring. Then every module M has an injective left $\mathscr{L}$-approximation $f: M \longrightarrow L$, where $\mathscr{L}$ is the full subcategory of R-modules of finite injective dimension.*

Theorem 3.5 ([EJ2], Corollary 11.5.3) *Let R be a Gorenstein ring. Then* GProj = ${}^{\perp}$(Proj); *and* Gproj = $\{X \in R\text{-mod} \mid \mathrm{Ext}^i(X, R) = 0, \forall i \geqslant 1\}$.

Proof Note that GProj ⊆ ${}^{\perp}$(Proj). Assume $M \in {}^{\perp}$(Proj). By Lemma 3.4 M has an injective left $\mathscr{L}$-approximation $f: M \longrightarrow L$. Take an exact sequence $0 \longrightarrow K \longrightarrow P^0 \stackrel{\theta}{\longrightarrow} L \longrightarrow 0$ with P^0 projective. By Theorem 3.3 K has finite projective dimension. It follows from this and the assumption $M \in {}^{\perp}$(Proj) that $\mathrm{Ext}^i(M, K) = 0$ for $i \geqslant 1$. In particular $\mathrm{Ext}^1(M, K) = 0$. Thus, θ induces a surjective map $\mathrm{Hom}_R(M, P^0) \longrightarrow \mathrm{Hom}_R(M, L)$. Hence we get $g: M \longrightarrow P^0$ such that $f = \theta g$. Since f is an injective left $\mathscr{L}$-approximation and $P^0 \in \mathscr{L}$, we deduce that g is also an injective left $\mathscr{L}$-approximation, and hence $\mathrm{Ext}^i(P^0/M, \mathrm{Proj}) = 0$ for $i \geqslant 1$.

Applying the same argument to P^0/M and continuing this process, we obtain a long exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$, which is Hom(−,Proj)-exact. Putting a (deleted) projective resolution of M together with (the deleted version of) this exact sequence, we see M is Gorenstein projective. This proves GProj = ${}^{\perp}$(Proj).

It remains to prove Gproj = $\{X\in R\text{-mod} \mid \mathrm{Ext}^i(X,R)=0,\ \forall\, i\geq 1\}$. Since Gproj = GProj $\cap$ R-mod = ${}^{\perp}(\mathrm{Proj})\cap R$-mod, it suffices to prove that if $X\in R$-mod with $\mathrm{Ext}^i(X,R)=0$, $\forall\, i\geq 1$, then $\mathrm{Ext}^i_R(X,P)=0$ for $i\geq 1$ and any projective module P. Since P is a direct summand of a direct sum $R^{(I)}$ (may be infinite), it suffices to prove $\mathrm{Ext}^i_R(X,R^{(I)})=0$ for $i\geq 1$, while this true since X is finitely generated and $\mathrm{Ext}^i_R(X,R^{(I)})=\mathrm{Ext}^i_R(X,R)^{(I)}=0$ for $i\geq 1$.

Remark If R is in addition artin, then GProj = $\{X\in R\text{-Mod} \mid \mathrm{Ext}^i(X,R)=0,\ \forall\, i\geq 1\}$. See [B], or [HHT].

Corollary 3.6 *Let R be an n-Gorenstein ring, and*

$$0\longrightarrow K\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

be an exact sequence with P_i projective. Then K is Gorenstein projective.

Proof Note that $\mathrm{Ext}^i_R(K,\mathrm{Proj})=\mathrm{Ext}^{n+i}_R(M,\mathrm{Proj})=0$ for all $i\geq 1$, by Theorem 3.3. Then K is Gorenstein projective by Theorem 3.5.

3.4 The following shows, in particular, that for an n-Gorenstein ring, the subcategory of Gorenstein projective modules is covariantly finite; and moreover, that any module has a proper Gorenstein projective resolution of bounded length n.

Theorem 3.7 ([EJ2]) *Let R be an n-Gorenstein ring. Then*

(i) *Every R-module M has a surjective left* GProj-*approximation $\phi: G\longrightarrow M\longrightarrow 0$, with* pd Ker $\phi\leq n-1$ (*if $n=0$ then* Ker $\phi=0$).

(ii) *Every finitely generated R-module M has a surjective left* Gproj-*approximation $\phi: G\longrightarrow M\longrightarrow 0$, with* pd Ker $\phi\leq n-1$.

Thus, every R-module has Gorenstein projective dimension at most n.

Proof By the corollary above there is an exact sequence

$$0\longrightarrow K\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

with every P_i projective and K Gorenstein projective. Since K is Gorenstein projective, there is a Hom(−, Proj)-exact sequence

$$0\longrightarrow K\longrightarrow P^0\longrightarrow P^1\longrightarrow\cdots\longrightarrow P^{n-1}\longrightarrow G'\longrightarrow 0 \qquad (*)$$

with every P^i projective and G' Gorenstein projective. Since (*) is Hom(−, Proj)-exact, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & K & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & \cdots & \longrightarrow & P^{n-1} & \longrightarrow & G' & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

With the same argument as in the proof of Theorem 2.2 we get a surjective left GProj-approximation $\phi: G\longrightarrow M\longrightarrow 0$, with pd Ker $\phi\leq n-1$.

Together with Theorem 2.2 we have

Corollary 3.8 *Let R be an n-Gorenstein ring. Then every module has a proper Gorenstein projective resoluiton of length at most n; and every finitely generated module has a proper Gorenstein projective resoluiton of length at most n, by finitely generated Gorenstein projective modules.*

4 Some recent results

We state more recent results.

4.1 Let A, B be k-algebra. *A dualizing dualizing complex* (see [Har], [YZ] and [WZ]) ${}_B D^{\bullet}_A$ is a bounded complex of B-A-bimodules, such that

(i) All the cohomology modules of $D^{\bullet}$ are finitely generated over B and A^{op};

(ii) $D^{\bullet}\cong I^{\bullet}\in D^b(B\otimes_k A^{op})$, where each component of $I^{\bullet}$ is injective both over B and A^{op}; and

(iii) the canonical maps $A\longrightarrow \mathrm{RHom}_B(D^{\bullet},D^{\bullet})$, $B\longrightarrow \mathrm{RHom}_{A^{op}}(D^{\bullet},D^{\bullet})$ are isomorphisms in the derived categories $D(A^e)$ and in $D(B^e)$, respectively, where $A^e = A \otimes_k A^{op}$.

Theorem 4.1 ([J1, Theorems 1.10 and 2.11]) *If a ring A satisfies one of the following two conditions, then* A-GProj *is contravariantly finite in* A-Mod.

(i) *A is a noetherian commutative ring with a dualizing complex.*

(ii) *A is a left coherent and right noetherian k-algebra for which there exists a left noetherian k-algebra B and a dualizing complex* ${}_B D_A$.

If A is a finite-dimensional k-algebra, then A has a dualizing complex $A^* = \mathrm{Hom}_k(A, k)$. So A-GProj is contravariantly finite in A-Mod, in other words, each A-module (not necessarily finitely generated) admits a left proper Gorenstein projective resolution. It should be stress that it is open whether or not A-Gproj is contravariantly finite in A-mod. Auslander and Reiten pointed out (Proposition 6.12 in [AR1]) that a positive answer is equivalent to the Gorenstein Symmetric Conjecture. For more information see [B] and [BR]. Note that there exists a ring R such that R-Gproj is not contravariantly finite in R-mod([T]).

4.2 Recall from [B] that a ring R is called *Cohen-Macaulay finite* (or, *CM-finite* for short) if there are only finitely many isomorphism classes of finitely generated indecomposable Gorenstein projective R-modules.

Theorem 4.2 ([Chen1]) *Let A be a Gorenstein artin algebra. Then A is CM-finite if and only if every Gorenstein projective module is a direct sum of finitely generated Gorenstein projective modules.*

4.3 *A class* $\mathscr{T}$ of modules is called a cotilting class if there is a (generalized) cotilting module $\mathscr{T}$ such that $\mathscr{T} = {}^{\perp}T$.

Theorem 4.3 ([HHT]) *Let R be a two sided noetherian ring. Denote by* GProj-R *the category of right Gorenstein projective R-modules. The following statements are equivalent:*

(i) *R is a Gorenstein ring;*

(ii) *Both* R-GProj *and* GProj-R *are cotilting classes.*

4.4 A left artin ring R with ${}_R R$ being an injective module is called *a left quasi-Frobenius ring*. Similarly, one has the concept of *a right quasi-Frobenius ring*. A ring is left quasi-Frobenius if and only if it is right quasi-Frobenius. A left artin ring is quasi-Frobenius if and only if the map ${}_R M \mapsto \mathrm{Hom}_R(M, R)$ defines a duality between the categories of left and right finitely-generated R-modules.

Theorem 4.4 ([BMO]) *Let R be a commutative ring. Then the following are equivalent.*

(i) *Any Gorenstein projective module is Gorenstein injective.*

(ii) *Any Gorenstein injective module is Gorenstein projective.*

(iii) *Ris quasi-Frobenius.*

4.5 The following shows that, over a commutative ring, replacing projectives in the definition of Gorenstein projective modules by Gorenstein projectives does not produce new kind of modules.

Theorem 4.5 ([W]) *Let R be a commutative ring. Given an exact sequence of Gorenstein projective R-modules* $G^\bullet = \cdots \longrightarrow G^{-2} \longrightarrow G^{-1} \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$ *such that the complexes* $\mathrm{Hom}_R(G^\bullet, H)$ *and* $\mathrm{Hom}_R(H, G^\bullet)$ *are exact for each Gorenstein projective R-module H. Then all the images are Gorenstein projective.*

4.6 If R is a Gorenstein ring, then R-Mod $=$ R-Proj if and only if $\mathrm{gl.dim} R < \infty$.

Theorem 4.6 ([LH]) *Let R be a commutative artin ring, and G a finitely generated Gorenstein projective R-module. Then G is projective if and only if* $\mathrm{Ext}_R^i(G, G) = 0$, $\forall\, i \geq 1$.

5 Appendix I: Character modules

5.1 An injective module E is *an injective cogenerator*, if $\mathrm{Hom}(M, E) \neq 0$, $\forall\, M \neq 0$, or equivalently, for any $M \neq 0$ and $0 \neq x \in M$, there is $f \in \mathrm{Hom}(M, E)$ such that $f(x) \neq 0$. For example, $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathbb{Z}$-modules.

For any right R-module $M \neq 0$, the nonzero left R-module $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called *the character module* of M.

By the adjoint pair one can see that R^+ is an injective left R-modules. So R^+ is an injective cogenerator of left R-modules since $\mathrm{Hom}(M, R^+) \cong M^+$.

5.2 Any module M is a submodule of M^{++}: in fact, the R-map $M \longrightarrow M^{++} = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ given by $m \mapsto f$: "$g \mapsto g(m)$" is injective.

A submodule N of M is pure if $0 \longrightarrow X \otimes N \longrightarrow X \otimes M$ is exact for any right R-module X. We have

Lemma 5.1 *For any module M, M is a pure submodule of M^{++}.*

Proof Applying $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ to the canonical injection $M \hookrightarrow M^{++}$, we get surjection $\pi: M^{+++} \twoheadrightarrow M^+$. Put $\sigma: M^+ \hookrightarrow M^{+++}$. Then by direct verification one see $\pi\sigma = \mathrm{Id}_{M^+}$, which implies M^+ is a direct summand of M^{+++}. Thus for any right module X we have exact sequence $\mathrm{Hom}(X, M^{+++}) \longrightarrow \mathrm{Hom}(X, M^+) \longrightarrow 0$, i.e., we have exact sequence (by the adjoint pair) $(X \otimes M^{++})^+ \longrightarrow (X \otimes M)^+ \longrightarrow 0$; and then it is easily verified that $0 \longrightarrow X \otimes M \longrightarrow X \otimes M^{++}$ is exact.

Lemma 5.2 *Let N be a pure submodule of M. Then* fd $N \leqslant$ fd M.

In particular, fd $M \leqslant$ fd M^{++} *for any module M.*

Proof We may assume fd $M = n < \infty$. For any right module X, take a partial projective resolution $0 \longrightarrow K \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$. Consider the commutative diagram

$$\begin{array}{ccccc} & & K \otimes N & \longrightarrow & P_n \otimes N \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K \otimes M & \longrightarrow & P_n \otimes M. \end{array}$$

Since $\mathrm{Tor}_{n+1}(X, M) = 0$, the bottom row is exact. Two vertical maps are injective since N is a pure submodule of M. It follows that $K \otimes N \longrightarrow P_n \otimes N$ is injective, and hence $\mathrm{Tor}_{n+1}(X, N) = 0$. So fd $N \leqslant n$.

5.3 Flat modules can be related with injective modules as follows.

Proposition 5.3 *Let M be a R-S-bimodule, E an injective cogenerator of right S-modules. Then*

(ⅰ) fd ${}_RM$ = id $\mathrm{Hom}_S(M, E)_R$.

In particular, ${}_RM$ *is flat* $\Longleftrightarrow$ $(M^+)_R$ *is injective.*

(ⅱ) *If furthermore R is left noetherian, then* id ${}_RM$ = fd $\mathrm{Hom}_S(M, E)_R$.

In particular, ${}_RM$ *is injective* $\Longleftrightarrow$ $(M^+)_R$ *is flat.*

Proof The assertion (ⅰ) follows from the identity

$$\mathrm{Hom}_S(\mathrm{Tor}_i^R(X, M), E) \cong \mathrm{Ext}_R^i(X, Hom_S(M, E)), \ \forall X_R.$$

For (ⅱ), note that if R is left noetherian, then for any finitely presented module ${}_RX$ there holds

$$\mathrm{Tor}_i^R(\mathrm{Hom}_S(M, E), X) \cong \mathrm{Hom}_S(\mathrm{Ext}_R^i(X, M), E);$$

also, by using direct limit one has id $M \leqslant n$ if and only if $\mathrm{Ext}_R^i(X, M) = 0$, $\forall\, i \geqslant n+1$, and for all finitely generated modules X.

6 Appendix Ⅱ: Direct products (sums), direct (inverse) limits

For the convenience, we list some frequently used properties in R-Mod.

(1) We have

$$\mathrm{Ext}^n(M, \prod_{i \in I} N_i) \cong \prod_{i \in I} \mathrm{Ext}^n(M, N_i), \ n \geqslant 0.$$

$$\mathrm{Ext}^n(\bigoplus_{i \in I} M_i, N) \cong \prod_{i \in I} \mathrm{Ext}^n(M_i, N), \ n \geqslant 0.$$

$$\mathrm{Tor}_n(\bigoplus_{i\in I} M_i, N) \cong \bigoplus_{i\in I} \mathrm{Tor}_n(M_i, N), n \geqslant 0.$$

$$\mathrm{Tor}_n(M, \bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I} \mathrm{Tor}_n(M, N_i), n \geqslant 0.$$

(2) If M is a finitely generated R-module, then

$$\mathrm{Ext}^n(M, \bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I} \mathrm{Ext}^n(M, N_i), n \geqslant 0.$$

(3) A ring is *left coherent*, if every finitely generated left idea is finitely presented, or equivalently, every finitely generated submodule of finitely presented left module is finitely presented. A left noetherian ring is a left coherent ring.

If R is left coherent and N is a finitely presented left R-module, then

$$\mathrm{Tor}_n(\prod_{i\in I} M_i, N) \cong \prod_{i\in I} \mathrm{Tor}_n(M_i, N), \ n \geqslant 0.$$

If R is right coherent and M is a finitely presented right R-module, then

$$\mathrm{Tor}_n(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Tor}_n(M, N_i), \ n \geqslant 0.$$

(4) The direct limit of a direct system of modules always exists.

The inverse limit of an inverse system of modules always exists.

(5) Each module M is the direct limit of all the finitely generated submodules of M.

(6) If for each $j \in J$, $((M_{ij}), (f_{ij}))_{i\in I}$ is a direct system of modules, then

$$\varinjlim_{i\in I}(\bigoplus_{j\in J} M_{ij}) \cong \bigoplus_{j\in J}(\varinjlim_{i\in I} M_{ij}); \text{ but } \varinjlim_{i\in I}(\prod_{j\in J} M_{ij}) \ncong \prod_{j\in J}(\varinjlim_{i\in I} M_{ij}) \text{ in general.}$$

(7) If for each $j \in J$, $((M_{ij}), (f_{ij}))_{i\in I}$ is an inverse system of modules, then

$$\varprojlim_{i\in I}(\prod_{j\in J} M_{ij}) \cong \prod_{j\in J}(\varprojlim_{i\in I} M_{ij}); \text{ but } \varprojlim_{i\in I}(\bigoplus_{j\in J} M_{ij}) \ncong \bigoplus_{j\in J}(\varprojlim_{i\in I} M_{ij}) \text{ in general.}$$

(8) We have

$$\mathrm{Ext}^n(M, \varprojlim_{i\in I} N_i) = \varprojlim_{i\in I} \mathrm{Ext}^n(M, N_i), n \geqslant 0.$$

$$\mathrm{Tor}^n(\varinjlim_{i\in I} M_i, N) = \varinjlim_{i\in I} \mathrm{Tor}^n(M_i, N), n \geqslant 0.$$

$$\mathrm{Tor}^n(M, \varinjlim_{i\in I} N_i) = \varinjlim_{i\in I} \mathrm{Tor}^n(M, N_i), n \geqslant 0.$$

(9) We have $\mathrm{Hom}(\varinjlim_{i\in I} M_i, N) = \varprojlim_{i\in I} \mathrm{Hom}(M_i, N)$; but in general $\mathrm{Ext}^n(\varinjlim_{i\in I} M_i, N) \ncong \varprojlim_{i\in I} \mathrm{Ext}^n(M_i, N), n \geqslant 1$.

(10) If R is left noetherian and M is a finitely generated left R-module, then

$$\mathrm{Ext}^n(M, \varinjlim_{i\in I} N_i) = \varinjlim_{i\in I} \mathrm{Ext}^n(M, N_i), n \geqslant 0.$$

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