SAMPLING AND RECONSTRUCTION FOR BIOMEDICAL IMAGE REGISTRATION

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ABSTRACT

We investigate the problem of sampling and reconstruction of signals in the context of biomedical image registration. Given a finite energy input signal, a band limited point spread function, and Nyquist sampling scheme, we characterize the basis functions that can be used in reconstructing the signals so that the shift (pure translation) between two such input signals can be recovered exactly. Computational results using reconstruction using B-spline function spaces show that perfect reconstruction nor interpolation are not necessary for exact recovery of shifts between two signals.

***Index Terms*—** Image registration, interpolation, sub-pixel, reconstruction

1. INTRODUCTION

Due to their applications to data fusion, motion and distortion correction, computational anatomy, and other related problems in biomedical imaging, automated image registration (alignment) has become a major topic of image processing research. Precision, in conjunction with robustness to noise and computational efficiency, are common requirements of many applications. One possible way through which precision and accuracy of current registration methods can be assessed, and perhaps improved, is by including as much prior information from the image acquisition system in the consideration of the problem. Here we examine the problem of recovering uniform shifts between two images (signals) by considering the sampling and reconstruction operations normally present in digital imaging systems and registration algorithms. We offer a characterization result for reconstruction basis functions that allow one to determine a shift between two digital images, exactly, so long as the images are not aliased. We show computational results demonstrating that perfect reconstruction is not necessary for recovering the shift between two signals to sub-pixel accuracy.

1.1. Prior work

Intensity based image registration methods are plentiful and have been applied to a variety of imaging problems, in addition to biomedical ones (see [1] for a review). Most algorithms developed to date, however, assume implicitly or explicitly that straightforward image interpolation or approximation is sufficient for recovering spatial transformations (translations, rotations, and higher order nonlinear mappings) between two or more images. Unser et al., for example, work in the function space generated by integer shifts of B-spline basis functions [2], while Foroosh et al [3] use band-limited interpolation to recover shifts between images to sub-pixel accuracy. In the following sections we investigate these assumptions more closely.

2. THEORY

2.1. Preliminaries: band-limited signal acquisition

We consider the Hilbert space of square integrable images (signals) $s(x) \in L_2$, $x \in \mathbb{R}^d$ with the usual norm

$$\|s\|_{L_2}^2 = \langle s, s \rangle_{L_2} = \int_{\mathbb{R}^d} s(x) s^*(x) dx \qquad (1)$$

with s^* denoting the complex conjugate in the case of complex images, although in this work we consider exclusively real valued images and signals.

We briefly describe the image acquisition system we assume in subsequent sections. In typical image and signal acquisition devices, and incoming signals is "filtered" prior to sampling and digitization. In time domain signals flow pass filtering is typically performed so as to avoid aliasing, while in imaging devices inherent limitations (features) of typical devices (diffraction limited lenses, magnetic resonance imaging, etc.) prevent arbitrarily high spacial frequencies from being acquired. We represent such a filtering operation as a continuous convolution between the incoming continuous signal (image) s and the point spread function h: $g(x) = h * s(x) = \int_{\mathbb{R}^d} s(u)h(x-u)du$. We assume h is band limited:

$$\hat{h}(\omega) = \int_{\mathbb{R}^d} h(x) e^{-j2\pi\omega x} dx = 0, \qquad (2)$$

for $|\omega| > 1/2$. Next the signal is sampled, preferably at or above the Nyquist rate, and the signal information is then stored using a finite number of bits yielding (assuming $h \in$

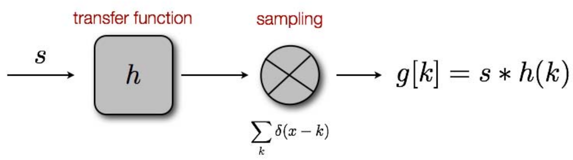

Fig. 1. Linear system representation of image acquisition system. An input signal s is "blurred" according to a point spread function h and then sampled at a regular grid. See text for more details.

L_2) a square summable sequence $g = h * s|_{k\in\mathbb{Z}^d}$. Thus $g = h * s|_{k\in\mathbb{Z}^d}$, with $*$ denoting continuous convolution (in d dimensions), is a square summable (ℓ_2) sequence and represents a filtered and sampled multi-dimensional infinite sequence. Figure 1 depicts the system features described above.

2.2. Image registration

Let s denote a square integrable signal as described above and $T_\tau s = s_\tau = s(\cdot - \tau)$ represent the a version of s shifted by some arbitrary $\tau \in \mathbb{R}^d$. Given s and $T_\tau s$ the goal in the registration problem we consider is to recover τ from sampled data $g = h * s|_{k\in\mathbb{Z}^d}$ and $g^\tau = h * T_\tau s|_{k\in\mathbb{Z}^d}$. We note that since we assume h to be band limited, the Shannon-Whittaker sampling theorem allows perfect reconstruction of the the continuous functions $g = h * s$ and $g^\tau = h * T_\tau s$, not of s and $T_\tau s$. Thus, pending further analysis shown below, at this point it is not clear whether perfect reconstruction using sinc basis functions is enough, or required, for estimating the shift τ.

Let real valued function $\tilde{s}$ represent a reconstruction of s via:

$$\tilde{s}(x) = \sum_{k\in\mathbb{Z}^d} g[k]\varphi(x-k) = \sum_{k\in\mathbb{Z}^d} h * s[k]\varphi(x-k) \quad (3)$$

where φ is a symmetric L_2 basis function. If the signal s suffered a translation prior to sampling its reconstruction from its sampled values is denoted:

$$\tilde{s}^\tau(x) = \sum_{k\in\mathbb{Z}^d} g^\tau[k]\varphi(x-k) = \sum_{k\in\mathbb{Z}^d} h * s_\tau[k]\varphi(x-k) \quad (4)$$

which is to be differentiated from the translated reconstruction

$$T_\lambda\tilde{s}(x) = \sum_{k\in\mathbb{Z}^d} g[k]\varphi(x-\lambda-k) = \sum_{k\in\mathbb{Z}^d} h*s[k]\varphi(x-\lambda-k).$$

Given two reconstructions $\tilde{s}$ and $\tilde{s}^\tau$ we seek to recover τ via

$$\tau^{op} = \arg\min_\lambda \|T_\lambda\tilde{s} - \tilde{s}^\tau\|^2_{L_2}. \quad (5)$$

Clearly, the sampling and reconstruction procedure with φ is appropriate ($\tau^{op} = \tau$) if and only if:

$$\|T_\lambda\tilde{s} - \tilde{s}^\tau\|^2_{L_2} \geq \|T_\tau\tilde{s} - \tilde{s}^\tau\|^2_{L_2} \quad (6)$$

or equivalently

$$\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} \leq \langle T_\tau\tilde{s}, \tilde{s}^\tau\rangle_{L_2}. \quad (7)$$

with equality only when $\tau = \lambda$. Next we offer a sufficient and necessary conditions on φ so that the shift can be recovered exactly for any τ, s, and bandlimited h.

Theorem 2.1 *Let*

$$\Upsilon(\omega,\tau,\lambda) = \sum_{q\in\mathbb{Z}^d} |\hat{\varphi}(\omega+q)|^2 \cos\left(2\pi\omega\cdot(\tau-\lambda) - 2\pi\lambda\cdot q\right). \quad (8)$$

Then $\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} \leq \langle T_\tau\tilde{s}, \tilde{s}^\tau\rangle_{L_2}$ *iff* $\Upsilon(\omega,\tau,\tau) \geq \Upsilon(\omega,\tau,\lambda)$, $\forall\omega \in [-1/2, 1/2]^d$.

Proof:
Substituting the definitions we have:

$$\begin{aligned}\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} &= \sum_{k\in\mathbb{Z}^d}\sum_{p\in\mathbb{Z}^d}(h*s[k])(h*s_\tau[p]) \\ &\quad \int_{\mathbb{R}^d}\varphi(x-\lambda-k)\varphi(x-p)dx \\ &= \sum_{k\in\mathbb{Z}^d}\sum_{p\in\mathbb{Z}^d}(h*s[k])(h*s_\tau[p])\varphi*\varphi^\vee(k-p+\lambda) \\ &= \sum_{k\in\mathbb{Z}^d} h*s[k]\sum_{p\in\mathbb{Z}^d} h*s_\tau[p]\left(\varphi*\varphi^\vee_{-\lambda}(k-p)\right)\end{aligned}$$

which can be viewed as an ℓ_2 inner product between the sequences $h * s[k]$ and $\sum_{p\in\mathbb{Z}^d} h * s_\tau[p]\left(\varphi*\varphi^\vee_{-\lambda}(k-p)\right)$. Note that in the above $\varphi^\vee$ denotes the time reversed function φ. This can be written in Fourier domain with the aid of the discrete time Fourier transform (DTFT):

$$\begin{aligned}\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} &= \int_{[-1/2,1/2]^d}\text{DTFT}\{h*s\}^*(\omega) \\ &\quad \text{DTFT}\left\{h*s_\tau\otimes\varphi*\varphi^\vee_{-\lambda}\right\}(\omega)d\omega \quad (9)\end{aligned}$$

where $\otimes$ denotes the discrete convolution operation between two ℓ_2 sequences. We have

$$\text{DTFT}\{h*s_\tau\otimes\varphi*\varphi_{-\lambda}\}(\omega) = \text{DTFT}\{h*s_\tau\}\,\text{DTFT}\left\{\varphi*\varphi^\vee_{-\lambda}\right\}(\omega)$$

while, via the Poisson summation formula, we have

$$\text{DTFT}\{h*s\}(\omega) = \sum_{k\in\mathbb{Z}^d}\hat{h}(\omega+k)\hat{s}(\omega+k),$$

$$\text{DTFT}\{h*s_\tau\}(\omega) = \sum_{k\in\mathbb{Z}^d}\hat{h}(\omega+k)\hat{s}(\omega+k)e^{-j2\pi\omega\tau\cdot(\omega+k)},$$

and

$$\text{DTFT}\left\{\varphi*\varphi^\vee_{-\lambda}\right\}(\omega) = \sum_{k\in\mathbb{Z}^d}|\hat{\varphi}(\omega+k)|^2 e^{j2\pi\omega\lambda\cdot(\omega+k)}.$$

Inserting these into (9), and keeping in mind that h is bandlimited, we have

$$\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} = \int_{[-1/2,1/2]^d}|\hat{h}(\omega)|^2|\hat{s}(\omega)|^2\Upsilon(\omega,\tau,\lambda)d\omega \quad (10)$$

with $\Upsilon(\omega,\tau,\lambda) = \sum_{q\in\mathbb{Z}^d}|\hat{\varphi}(\omega+q)|^2\cos\left(2\pi\omega\cdot(\tau-\lambda) - 2\pi\lambda\cdot q\right)$.

It is clear that if $\Upsilon(\omega,\tau,\tau) \geq \Upsilon(\omega,\tau,\lambda), \forall\omega \in [-1/2,1/2]^d$ then $\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} \leq \langle T_\tau\tilde{s}, \tilde{s}^\tau\rangle_{L_2}$ holds. The converse is also true. Suppose $\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} \leq \langle T_\tau\tilde{s}, \tilde{s}^\tau\rangle_{L_2}$ then $\Upsilon(\omega,\tau,\tau) \geq \Upsilon(\omega,\tau,\lambda), \forall\omega \in [-1/2,1/2]^d$ must hold. If it is not the case, one can always choose $|\hat{h}(\omega)|^2|\hat{s}(\omega)|^2$ so that $\langle T_\lambda\tilde{s}, \tilde{s}^\tau\rangle_{L_2} \leq \langle T_\tau\tilde{s}, \tilde{s}^\tau\rangle_{L_2}$ is not the case.

As a simple corollary we have that

Corollary 2.2 *If φ is also band-limited (its Fourier transform is zero outside $[-1/2, 1/2]^d$), then $\langle T_\lambda \tilde{s}, \tilde{s}^\tau \rangle_{L_2} \leq \langle T_\tau \tilde{s}, \tilde{s}^\tau \rangle_{L_2}$.*

The result above implies that the orthogonal projection of s onto the space of band limited functions can be used to recover the shift τ without error, so long as the projection is not zero.

3. COMPUTATIONAL EXAMPLE

Using $\Upsilon(\omega, \tau, \lambda)$ defined in Theorem 2.1 we have that

$$\langle T_\lambda \tilde{s}, \tilde{s}^\tau \rangle_{L_2} = \int_{[-1/2,1/2]^d} |\hat{h}(\omega)|^2 |\hat{s}(\omega)|^2 \Upsilon(\omega, \tau, \lambda) d\omega$$

while for registration the basic requirement on the reconstruction procedure is that φ is $\langle T_\lambda \tilde{s}, \tilde{s}^\tau \rangle_{L_2} \leq \langle T_\tau \tilde{s}, \tilde{s}^\tau \rangle_{L_2}$. If $|\hat{h}(\omega)|^2 |\hat{s}(\omega)|^2$ decays fast enough such that the Poisson sum terms in $\Upsilon(\omega, \tau, \lambda)$ (other than the zeroth term) do no overlap the portion of the Fourier domain to which $|\hat{h}(\omega)|^2 |\hat{s}(\omega)|^2$ belongs, φ is not required to be band-limited for exact recovery of the shift τ. We illustrate this with the following computational example.

The top portion of Figure 2 displays one signal (solid line) composed of a linear combination of sinc basis functions, with the coefficients of the linear combination composed of a sample dGaussian function. The signal is by construction band-limited and represents $g = h * s$ as discussed earlier. The signal is to be matched to its shifted version $g^\tau = h * T_\tau s$, where in this case $\tau = 2.61$ samples (pixels). As the system diagram shown in Figure 1 illustrates, the signals are then sampled (at the unit sample regular grid). The corresponding discrete signals are shown in the middle portion of Figure 2. Lastly, the signals $\tilde{s}$ and $\tilde{s}^\tau$ are reconstructed according to equations (3) and (4). The reconstruction basis functions were chosen to be B-splines of degree 2, $\varphi = \beta^2$:

$$\beta^2(x) = \begin{cases} 0 & \text{if } |x| > 1.5; \\ (3/2 - |x|)^2/2 & \text{if } 0.5 < |x| < 1.5; \\ 0.75 - x^2 & \text{if } |x| \leq 0.5 \end{cases}$$

The signal reconstructions are shown at the bottom of Figure 2. Using these, the inner product $\langle T_\lambda \tilde{s}, \tilde{s}^\tau \rangle_{L_2}$ can be computed analytically:

$$\langle T_\lambda \tilde{s}, \tilde{s}^\tau \rangle_{L_2} = \sum_{k \in \mathbb{Z}^d} (g^\tau[k])(g \otimes T_\lambda \beta^3[k]). \tag{11}$$

with

$$\beta^3(x) = \begin{cases} 0 & \text{if } |x| \geq 2; \\ \frac{(2-|x|)^2}{6} & \text{if } 1 \leq |x| < 2; \\ \frac{2}{3} - |x|^2 + \frac{|x|^3}{2} & 0 \leq |x| < 1. \end{cases}$$

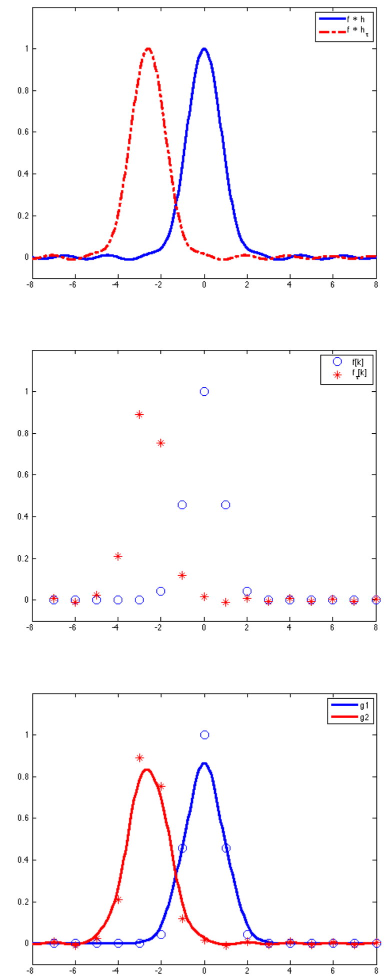

Fig. 2. Top: two band-limited functions g and g^τ. Middle: the functions on the top portion sampled at the unit regular grid. Bottom: the original functions reconstructed with B-splines of degree 2.

Alternatively, the inner product in (11) can be approximated through numerical integration. However, this is not recommended since it can give rise to so called ”interpolation artifacts” [4].

We plot equation (11) in Figure 3 where, for this particular simulation, the correct shift $\tau_{\text{opt}} = 2.61$ can be identified to two decimal places. The accuracy of the simulation can be increased by augmenting the degree of the reconstruction basis function. We note that contrary to previous works, interpolation or least squares projections into function spaces are not a requirement for near perfect recovery of the shift. In fact the B-spline basis functions used in this computational example are non interpolating (as shown in the reconstructions displayed in the bottom portion of Figure 2).

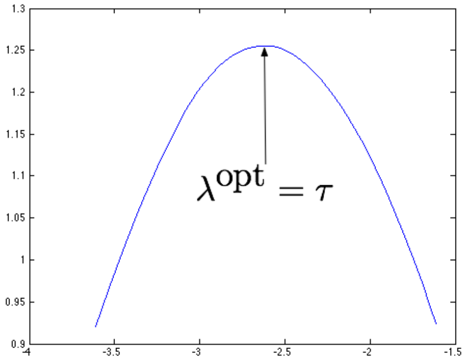

Fig. 3. $\langle T_\lambda \tilde{s}, \tilde{s}^\tau \rangle_{L_2}$ computed using B-splines of degree 2 for reconstructing the signals. See text for more details.

4. SUMMARY AND CONCLUSIONS

In image registration problems, and other image processing problems in general, signal (image) reconstruction is not the end goal. Rather, in the case of image registration, we seek to recover a spatial transformation that aligns two images. Often times sub-pixel shifts are desired, and our goal above was to characterize necessary and sufficient conditions on reconstruction basis functions for recovering the shift between two images.

We presented a result characterizing the reconstruction basis functions φ that enable one to recover the shift between two signals exactly. We also show that the projection of two square integrable signals onto the space of band-limited functions contains enough information to recover sub-pixel shifts exactly. From a practical standpoint, the results above have important applications. Firstly, the characterization result may be used to constructing basis functions φ that are computationally efficient for evaluating the registration functions involved (inner products defined above), and may not necessarily be band-limited themselves. Secondly, as the computational example showed, computationally efficient solutions may be obtained by limiting the frequency content of the signal via the application of a filter $\hat{h}$ such that the frequency content of $\hat{\varphi}(\omega + k), k \neq 0$ does not intersect significantly with $|\hat{h}(\omega)|^2 |\hat{s}(\omega)|^2$. The filtering operation may be performed in hardware (through the effect of lenses, for example) or by post-processing the data.

Finally, we note that although our computational example we provide above was performed in one dimension, the theory and methods described above are valid for multiple dimensions. The computational example can be extended to two or more dimensions by using the cross product of B-splines.

5. REFERENCES

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