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Numerical Methods in Medieval Chinese Mathematical Astronomy†

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Abstract The present paper is a brief history of the numerical method in medieval Chinese mathematical astronomy, The main numerical algorithms invented by Chinese calendar-makers after 600A. D. are listed as follows; polynomial interpolation of many kinds, polynomial functions and equations, compound and inverse functions. Their constructive thought are discussed,

Key words numerical method; China; calendar; interpolation

Chinese Books Classfication [illegible]

Ancient and medieval China's mathematical astronomy was used to compile accurate calendars. Establishing a calendar required great precision. Chinese calendar-makers worked hard to improve the old methods of calculation or to devise a kind of new numerical algorithm rather than constructing a plausible celestial model. The reason for this neglect of cosmic model building might be because none of the available cosmic models at that time could provide a more precise basis for calendar-making than an old or revised numerical method.

People in ancient and medieval China judged a new calendar by its ability to predict of eclipses, especially solar eclipses. In order to improve the accuracy of prediction, calendar-makers devised many new numerical methods to substitute for the method used in old calendar. This inspired people to make ceaseless efforts at compiling a calendar to substitute for the old one,

Before 3rd century A. D., each celestial body was considered at a mean motion in Chinese calendars, its algorithm was simple. The length of a tropical year was considered to be equal to a sidereal year, which divided the big circle on a celestial sphere. Since the mean solar motion was one *du* per day, the circle was divided into about 365, 25 *du* instead of the normal 360°, This system lasted more than two thousand years until modern European astronomical methods entered into traditional Chinese calendar-making system in the 17th century (Qing Dynasty).

When irregular lunar motion derived from the calendar in the 3rd century A. D., calendar-makers used linear interpolation methods to deal with it. After the phenomena of irregular solar and planetary motion was found by Zhang Zixin 张子信 in about 560 A. D., Liu Xiaosun came up with this item in his unused and almost lost *Wuping Calendar* 《武平历》(576A. D.)[1],and then there were a lot of numerical methods made to approximate the positions of the various celestial bodies. The following two sections in this paper is a list of those used in medieval Chinese mathematical astronomy between 600A. D. and 1644A. D.

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1 Polynomial Interpolation

Interpolation may be the oldest method of approximation worldwide. Linear interpolation had been used for calculating the irregular solar or lunar motion in Babylonian astronomy before the first century B. C. In about 600A. D. , Liu Zhuo 刘焯, a Chinese calendar-maker who lived in the Sui Dynasty(581A. D. ~618A. D.), devised what may be termed a piecewise quadratic interpolation in his *Huangji Calendar* 《皇极历》. Liu's algorithm soon became the most important numerical method in later Chinese calendar-making system. Several kinds of polynomial interpolation were derived from his method.

1.1 Liu's Quadratic Interpolation (600A. D.)

As an example, let us use solar motion to explain the construction of Liu Zhuo's interpolation in his *Huangji Calendar*.

In traditional Chinese mathematical astronomy, there is an important concept: *qi* 气. *Qi* is such a special term, when the ecliptic is divided into 24 parts, each parts consists of 15°, Chinese astronomer named each of the 24 nodal points (*qi*) with a special term which mostly corresponds to a natural seasonal phenomenon when the sun passes through it. The 24 solar terms are together called *qi*. A solar year usually adopted the winter solstice as the first *qi*. When the length between two consecutive *qi* equals 1/24 of a tropical year, it is called a length of mean *qi* (平气). For the irregular soalr motion, *qi* may be termed actual *qi* (定气) which gives the real position of the sun on the ecliptic, then the length between two actual *qi* does not equal 1/24 of a tropical year at all. Sometimes the concept of *qi* is confused with a interval. People actually did not distinguish them even in some ancient texts.

In the *Huangji Calendar*, Liu Zhuo adopted the mean *qi* for constructing his interpolation. First of all, making use of 24 *qi* to divide a tropical year into 24 parts, and observed that there were 24 figures for the solar motion to be given, one for each interval. Then, the data was used for constructing the quadratic interpolation function on each interval. Suppose the length of each interval equals n, which is the length between two consecutive mean *qi*, and $f(x)$ is the quadratic interpolation function to be constructed on the first interval of a year. $f(0)$, $f(n)$ and $f(2n)$ are knowns. Let $\Delta_i = f(in) - f((i+1)n)$, $\Delta^2 = \Delta_1 - \Delta_0$.

There is the function as follows: $f(x) = f(0) - [\frac{\Delta_0}{n} - \frac{\Delta^2}{2n}] \cdot x - \frac{\Delta^2}{2n^2} \cdot x^2$. (1)

The construction of (1) consists of two steps(see[2], p. 185~195):

1. Use $f(in)$ to deduce an arithmetic sequence $f(m) - f(m+1)$, $(m = 0, 1, 2, \ldots)$;
2. Sum up the series: $f(x) = f(0) - \sum_{m=0}^{x-1}[f(m) - f(m+1)]$.

This method was used widely to deal with the irregular motion of the sun, the moon and the planets in later calendars.

1.2 Yixing's Anisometric Interpolation (724A. D.)

A century after Liu Zhuo, a similar idea was adopted by a Buddhist monk Yixing for constructing a kind of anisometric interpolation on the basis of a partition made by 24 actual *qi* in his *Dayan Calendar* 《大衍历》(724A. D.):

The lengths of 24 small intervals did not equal each other at all in Yixing's algorithm. Suppose n_i $(i=1, 2, 3, \cdots)$ is the length of an interval divided by 24 actual *qi*. The data $f(n)$, $f(n+n_1)$, $f(n+n_1+n_2)\cdots$ are known numbers. Let $\Delta_0 = f(n) - f(n+n_1)$, $\Delta_1 = f(n+n_1) - f(n+n_1+n_2)$.

Two sub-algorithms in Liu Zhuo's method were used here for constructing Yixing's interpolation function $f(n+x)$ $(n<x<n+n_1)$ (see[2], p. 189~191):

1. Construct an arithmetric sequence $f(n+m)-f(n+m+1)$, $(m=0,1,2,3,\cdots)$;

2. Sum up its arithmetric series $f(n+x)=f(n)+\sum_{m=0}^{x-1}[f(n+m)-f(n+m+1)]$.

The idea used by Yixing for constructing his interpolation on each interval was not different from Liu's, although his function seemed more complex than (1).

1.3 A Kind of Simple Interpolation in the *Futian Calendar* 《符天历》(780A. D.)

There were a lot of kinds of quadratic function constructed in medieval Chinese calendars. The simplest one of them was as follows: $f(x)=x\cdot(a-x)/b$, $0<x<a$. (2)

The first function like (2) in Chinese calendar was devised by an amateur calendar-maker, Cao Shiwei 曹士蔿. His function was reconstructed from a solar motion table of Cao's *Futian Calendar* (780A. D.) by Nakayama[3].

It is known that the quadratic interpolation usually make use of 3 interpolation points for constructing a quadratic function. Since $f(0)=f(a)=0$, function (2) will be deduced if, and only if, there is another interpolation point to be chosen in $(0, a)$. Therefore, the construction of (2) is very simple.

In Bian Gang's 边冈 *Chongxuan Calendar* 《崇玄历》(892A. D.), a function like (2) is found for calculating the irregular solar motion. (see [1], p. 2 358) In the *Yitian Calendar* 《仪天历》(1001A. D.), there is a text which gives an explanation for how a function like (2) to be constructed. (see [1], p. 2 461) A study on this text shows that the idea for constructing a model of (2) is similar to Liu's method for (1). (see [2], p. 207~210)

Compared with Liu's piecewise interpolation, a function like (2) is not only easy to construct but also suitable for application. Therefore, after the *Yitian Calendar*, many calendar-makers devised this kind of functions for calculating irregular movements of the sun and the moon[4].

1.4 Bian Gang's Piecewise Iterated Interpolation (892A. D.)

The problem of Liu's algorithm is that the partition was so small that the function on each interval seemed too complex. But a function like (2) sometimes seemed so simple that it could not be sufficiently precise. In order to solve such a problem, Bian Gang devised a new algorithm in his *Chongxuan Calendar*.

Suppose $f(x)$ is the function to be constructed on the interval $[0, a]$. Let

$$f(0)=f(a)=0, \text{and } f(a_1)=f_1(a_1) \qquad (0<a_1<a),$$

there is a function like (2): $f_1(x)=x.(a-x)/b$.

Whether $f_1(x)$ is a good function approximate to $f(x)$ or not, we can choose a new interpolation point in $(0, a_1)$ or (a_1, a) to reckon the difference between $f(x)$ and $f_1(x)$. For instance, we choose $x=a_2$ in $(0, a_1)$ as a new interpolation point. In case $f(a_2)-f_1(a_2)$ is so much that we cannot neglect it, a function $f_2(x)$ on $[0, a_1]$ could be derived to pile up $f_1(x)$:

$$f_2(x)=x.(a_1-x)/b_1,\ b_1=a_2.(a_1-a_2)/[f(a_2)-f_1(a_2)].$$

It is clear that $f_1(x)+f_2(x)$ should be more approximate to $f(x)$ on $[0, a_1]$ than $f_1(x)$.

The method devised by Bian Gang as above is a kind of succe si e approximation. We call it a piecewise iterated quadratic interpolation[5].

Because making an appropriate partition for constructing a piecewise interpolation function on a big interval is still a problem to be improved in applied mathematics, Bian Gang's discovery of a numerical appoximate algorithm provided a valuable idea even from the perspective of modern science[6].

1.5 Cubic Interpolation in the *Shoushi Calendar* ≪授时历≫(1280A. D.)

Since mathematicians usually made use of geometric figures to construct their algorithm in medieval China, the problem for those who wanted to construct a cubic interpolation was that no such a model could be used for that purpose directly. So it took many generations of mathematicians nearly 700 years to come upon the new algorithm after Liu's quadratic interpolation, although the third-order difference table had already been used for hundreds of years.

In 1280A. D., a great idea was raised by Wang Xun 王恂 and Guo Shoujing 郭守敬 in their *Shoushi Calendar*. They used a purely algebraic method to transform the third-order problem into a second-order one, and devised a method similar to Liu's to deduce the second-order function. Then a cubic function was constructed.

Suppose $f(x)$ is the cubic interpolation function to be constructed by Wang and Guo, it consisted of 3 steps[7]:

1. Reduce the order of the function: It is easy to choose a problem which makes $f(0) = 0$, let $f(x) = g(x) \cdot x$;

2. Construct an arithmetic sequence: Let $x = n, 2n$, and $3n$ be the interpolation points, $g(in) = f(in)/in \quad (i = 1, 2, 3)$ are knowns. By the use of $g(in)$, $g(0)$ can be deduced, and the arithmetic sequence $\{g(m)\text{-}g(m+1)\} \quad (m = 0, 1, 2, \cdots)$ is derived;

3. Sum up the series: $g(x) = g(0) - \sum_{m=0}^{x-1}[g(m) - g(m+1)]$.

Then, the cubic interpolation function $f(x) = g(x) \cdot x$ is constructed.

The success with which Wang and Guo constructed a kind of cubic interpolation was due to a remarkable idea in which they transformed the problem from a cubic into a quadratic interpolation. This was the key thought which was also used by people to construct Newton's interpolation formula.

2 Other Numerical Methods

The polynomial interpolation was the mainstream of numerical method in medieval Chinese mathematical astronomy. Besides that, many other kinds of numerical methods also palyed a very important role in the development of algorithms in old China.

2.1 Cubic Function

The length of solar shadow was difficult to deal with in ancient and medieval mathematical astronomy. Yixing's *Dayan Calendar* has been regarded as a contribution that people have verified some data equivalent to a tangent table for calaulating the solar shadow. In his *Chongxuan Calendar* (892A. D.), Bian Gang constructed two similar cubic functions for calculating the length of solar shadow. This is the earliest cubic function which appeared in the history of Chinese mathematical astronomy(see [4], p. 127).

It was considered that Bian's cubic function was an evidence which showed the cubic interpolation method was already used in 9th century. In fact, although the result of a cubic interpolation is usually a cubic function, it is by no means that a cubic function must be constructed by the interpolation method. As an example, let us see how Bian Gang devised his cubic function.

Suppose $f(x)$ is a length of solar shadow after (or before) the winter solstice x days, such a function is found in Bian Gang's *Chongxuan Calendar*: $f(x) = f(0) - (a - b \cdot x) \cdot x^2$. (3)

$f(0)$ is the length of solar shadow on the day of winter solstice. The construction of (3) consisted of following two steps (see [5]):

1. Let $[f(0) - f(x)]/x^2 = a - b \cdot x$;

2. Make use of measuring data of $f(x)$, the linear function $a - b \cdot x$ is determined.

Then the function (3) is derived.

As with other algorithms, what Bian Gang derived in his calendar, the method used to construct the solar shadow formula also had a great influence upon later calendar-makers (see[4], 129～32).

2.2 Compound Functions

Let $\delta(x)$ be a declination of the sun on the day after the winter (or summer) solstice x days. As a given constant, let ε be the obliquity of ecliptic. Bian Gang constructed another function $g(x)$ for calculating $\delta(x)$ in his *Chongxuan Calendar*(892A.D.): $\delta(x)=|\varepsilon-g(x)|$, $(0<x<91.31)$.

Because the declination $\delta(x)$ is symmetry in two quadrants around the winter or summer solstice, Bian Gang discussed only the case on one of the four quadrants here. Let $n=91.31$ express a quarter tropical year, $g(x)$ was geven by Bian Gang as follows (see [4], 171):

$$g(x)=\{\frac{x^2}{n^2}+[(1-\frac{x^2}{n^2})\cdot\frac{x^2}{n^2}]/3.6\}. \tag{4}$$

In fact, the biquadratic function (4) is a kind of compound function:

$$g(x)=y+[(1-y)\cdot y]/3.6, \qquad y(x)=x^2/n^2.$$

According to a seperate study, we came to a conclusion that Bian Gang might have made use of a geometric figure to construct the biquadratic function $g(x)$ (see [5]).

After Bian Gang, functions like (4) were often constructed for calculating a declination of the sun (see [4], 172～176).

In 1106A.D., a complex new algorithm was devised by Yao Shunfu in his *Jiyuan Calendar* for calculating a declination of the sun and a celestcial pole's latitude of the moon (because in ancient Chinese astronomy people used the pole of equator as the pole of ecliptic, the celestial latitude in ancient and medieval China was different from that used in modern astronomy. Yabuuchi K. suggested it a celestial pole's latitude). A kind of compund function as follows is found in the *Jiyuan Calendar* for calculating $\delta(x)$ $(0<x<n)$ (see [4], 177):

$$\delta(x)=\varepsilon\cdot\frac{(2n-y)\cdot y}{n^2}, \qquad y(x)=n-(\frac{n-x}{c}+x),$$

$n=91.3109$ equals a quarter tropical year; ε is the obliquity of ecliptic; c is a constant to be determined. Yao's compound functions were adopted by later calendar-makers in their new calendars for many years. The construction of the compound function $\delta(x)$ might also have made use of a plane geometric figure (see [2], p. 219～223).

2.3 Inverse Function and Numerical Equation

Ancient and medieval Chinese mathematicians usually did not make use of the root formula of any higher order equation, although a couple of cases in the second order equation were discussed in *Nine Chapters on the Mathematical Art* 《九章算术》(ca. 50B.C.) and in other places. For instance, a positive root formula of an equation as follows is found in Yixing's *Dayan Calendar* (724A.D.) (see [1], p. 2 270): $x^2+bx-c=0$ $(b>0, c>0)$. (5)

In *Jiyuan Calendar* (1106A.D.), there are two quadratic functions used for calculating the celestial longitude. The independent variable of them is equivalent to the right ascension. By just relying on them, the calendar-maker let the longitude be the independent variable and gave the right ascension as a positive root of each of the two equations. He stipulated that the fields of value of original functions were the fields of definition of new functions, which were simultaneously the root formulas. Two inverse functions are derived here[8].

In the mid-11th century, Chinese mathematicians developed a numerical method for calculating the positive root of an arbitrary higher order equation. But normally calendar-makers did not devised equations in their calendars at that time. The problem might be there was not a powerful method for the calendar-makers to transform an astronomical question into an algebraic equation.

About 200 years later, Li Ye 李冶 completed *tianyuan shu* 天元术 which was an algebraic method to be used for constructing polynomial equation. A couple of quadratic equations and a quartic equation were constructed by the use of tianyuan shu in Wang Xun and Guo Shoujing's *Shoushi Calendar*(1280A. D.).

3 Conclusion

Those listed above are standard for the main algorithms in medieval Chinese mathematical astronomy. Most of them were algebraic functions or equations. Basically, although the numerical methods to be constructed in calendars usually made use of a special kind of plane geometric model, none of them had the same geometric meaning with modern spherical astronomy. Therefore it is said that mathematical astronomy in medieval China was an algebraic astronomy instead of a geometric astronomy.

The algorithms to be devised in calendars included quadratic(600A. D.) and cubic (1280A. D.) interpolation, quadratic iterated function (892A. D.), cubic function (892A. D.), higher order compound functions(892A. D., 1106A. D.), inverse function(724A. D., 1106A. D.), and higher order equation(1280A. D.).

The most important algorithm in the history of medieval Chinese mathematical astronomy was the construction of quadratic interpolation. Liu Zhuo in his *Huangji Calendar*(600A. D.), Yixing in his *Dayan Calendar*, Bian Gang in his *Chongxuan Calendar* (892A. D.), Yao Shunfu in his *Jiyuan Calendar* (1106A. D.) and Wang Xun and Guo Shoujing in their *Shoushi Calendar*(1280A. D.) made major contrabutions to the development of algorithm in mathematical astronomy of old China.

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中国古代数理天文学中的数值计算方法†

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摘 要 论述隋唐以降中国古代历法家使用的各种多项式插值算法的构造思想,以及当时的历算家所采用的其他一些数值计算方法,如高次多项式函数,复合函数与反函数等等。

关键词 数值方法;中国;历法;插值法

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