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Partially invariant solutions of the 2+1 dimensional nonlinear Schrödinger equations

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Abstract: The results of the 2+1 dimensional nonlinear schrödinger equations for the wave amplitude deep water waves have been extended. Partially invariant solutions of a class of 2+1 dimensional nonlinear Schrödinger equations are explicitly obtained by using a general and systematic approach based on subgroup classification methods. More solutions can be obtained by this method than by the classical method and direct method.

Key words: symmetry group; partially invariant solution; nonlinear Schrödinger equations

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Zakharov[1] first derived the 2+1 dimensional nonlinear Schrödinger equation for the wave amplitude in deep water waves, which is equivalent to the form

$$iu_t + u_{xy} + |u|^2 u = 0. \quad (1)$$

As a natural extension of (1), we consider more general 2+1 dimensional nonlinear Schrödinger equations(NLSEs)

$$iu_t + u_{xy} = f(|u|)u, \quad (2)$$

where $u(x,t)$ is a complex function of real variables, and f is real function of $|u|$.

The purpose of the present paper is to study partially invariant solutions of (2), the concept of partially invariant solutions of systems of partial differential equations (PDEs) was introduced by Ovsiannikov some time ago[3]. A systematic study of such solutions were begun in recent articles[4~5], which was devoted to partially invariant solutions of complex nonlinear Klein-Gordon ($\varepsilon=-1$) or Laplace ($\varepsilon=+1$) equations of the form

$$u_{tt} + \varepsilon u_{xx} = f(|u|)u, \ \varepsilon = \pm 1, \quad (3)$$

and a class of 1+1 dimensional nonlinear Schrödinger equations

$$iu_t + u_{xx} = (F + ik)u + (G + iL)u_x, \quad (4)$$

with four functions F, K, G and L of $|u|$ and $|u|_x$.

The partially invariant solutions of (3) and (4) are obtained in[4] and [5] provided the functions of equations satisfy some compatibility conditions. For (2), the existence of partially invariant solutions of different forms also depend on the function $f(|u|)$.

We shall now show that partially invariant solutions of Eqs. (2) exist for certain function $f(|u|)$ and we obtain them explicitly.

1 Symmetey groups of equations (2)

The 2+1 dimensional NLSEs (2) can be rewritten as a system of two PDEs for the modulus ρ, and the phase $\varphi(x,t)$ of u, namely

$$-\rho\varphi_t + \rho_{xy} - \rho\varphi_x\varphi_y - f(\rho)\rho = 0,$$
$$\rho_t + \rho_x\varphi_y + \rho_y\varphi_x + \rho\varphi_{xy} = 0, \quad (5)$$

Using the general schedule[7], we see that the symmetry groups of Eqs. (2) for arbitrary function f are time t, space x, y, and phase φ translations,

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$$P_0 = \partial_t,\ P_1 = \partial_x,\ P_3 = \partial_\varphi,$$
$$B_1 = t\partial_x + y\partial_\varphi,\ B_2 = t\partial_y + x\partial_\varphi. \qquad (6)$$

Subalgebras that lead to partially invariant solutions are

$\{P_0,P_3\}$, $\{P_1,P_3\}$, $\{P_2,P_3\}$, $\{P_0,P_1,P_3\}$, $\{P_1,P_2,P_3\}$, $\{P_0,P_2,P_3\}$, $\{P_0+aP_1,P_3\}$, $\{P_0+aP_2,P_3\}$, $\{P_1+aP_2,P_3\}$, $\{P_0+aP_1,P_2,P_3\}$, $\{P_0+aP_2,P_1,P_3\}$, $\{P_1+aP_2,P_0,P_3\}$, $\{P_0,P_1,P_3,B_1\}$, $\{P_0,P_2,P_3,B_2\}$, $\{P_2,P_3,B_1\}$, $\{P_1,P_3,B_2\}$, $\{P_1,P_3,B_1\}$, $\{P_2,P_3,B_2\}$

Let us now run through the individual cases.

2 Partially invariant solutions of (2)

1) The subalgebra $\{P_1,P_3\}$

The invariants are t, y and ρ, and the reduction formulas are simply

$$\rho = \rho(t,y),\quad \varphi = \varphi(t,x,y), \qquad (7)$$

where

$$\rho = \frac{h(\zeta)}{1+t\psi'(\zeta)}, \qquad (8)$$

$$\varphi = -\psi(\zeta)x + d_2(\zeta), \qquad (9)$$

where $h(\zeta)$ and $d_2(\zeta)$ are two arbitrary functions of ζ.

2) The subalgebra $\{P_2,P_3\}$

The result in this case can be obtained directly from 1) by replacing

$$x \leftarrow\rightarrow y.$$

3) The subalgebra $\{P_0,P_1,P_3\}$

The solutions have the form

$$\rho = \rho(y),\ \varphi = \varphi(x,y,t). \qquad (10)$$

Equations (5) can be rewritten as

$$\varphi_t = -\varphi_x\varphi_y - f(\rho), \qquad (11a)$$

$$\rho\varphi_{xy} + \rho_y\varphi_x = 0. \qquad (11b)$$

Integrating second equation with respect to x and y, we obtain

$$\varphi = \frac{d_1(x,t)}{\rho} + d_2(y,t), \qquad (12)$$

with two functions d_1 and d_2 to be determined.

The substitution of (12) into (11,a), which implies

$$d_{1t} - \frac{\rho_y}{\rho_2}d_1 d_{1x} + d_{1x}d_{2y} + \rho(d_{2t} + f(\rho)) = 0. \qquad (13)$$

To obtain the solutions of (13), we consider two subcases:

(i) $\frac{d}{dy}\left(\frac{\rho_y}{\rho^2}\right) = 0.$

We immediately obtain

$$\frac{\rho_y}{\rho^2} = \lambda,\ d_{2y} = \mu t + \eta,\ d_{2t} + f(\rho) = 0. \quad (14)$$

Solving (14), we have

$$\begin{aligned} &\rho = (\gamma - \lambda y)^{-1}, \\ &d_2 = (\mu t + \eta)y + \alpha t + \beta, \\ &f(\rho) = \frac{\mu}{\lambda}\rho^{-1} + \lambda_0. \end{aligned} \qquad (15)$$

So d_1 satisfies

$$d_{1t} - \lambda d_1 d_{1x} + (\mu t + \eta)d_{1x} = 0, \qquad (16)$$

which can be solved by the characteristic method. Thus we have obtained the solutions of (11)

$$\begin{aligned} &\rho = (\gamma - \lambda y)^{-1}, \\ &\varphi = \frac{d_1(x,t)}{\rho} + (\mu t + \eta)y + \alpha t + \beta. \end{aligned} \qquad (17)$$

where $\gamma,\lambda,\alpha,\beta,\mu$ and η are arbitrary constants.

(ii) $\frac{d}{dy}\frac{\rho_y}{\rho^2} \neq 0$. From (13), we have

$$d_{1t} = 0,\ d_1 d_{1x} = C_1,\ d_{2y} = 0,\ d_{2t} = C_2. \quad (18)$$

It is easy to see that

$$d_1 = (2C_1 x + C_3)^{\frac{1}{2}},\ d_2 = C_2 t + C_4, \quad (19)$$

and ρ is given by a quadrature

$$\int \frac{d\rho}{\rho^3(f(\rho) + C_2)} = \frac{y}{C_1}, \qquad (20)$$

where C_1, C_2, C_3 and C_4 are arbitary constants.

4) The subalgebra $\{P_0,P_2,P_3\}$

Considerations for this case are completely analogous to 3).

5) The subalgebra $\{P_1,P_2,P_3\}$

The reduction formulas are

$$\rho = \rho(t),\ \varphi = \varphi(t,x,y), \qquad (21)$$

where

$$\begin{aligned} &\rho = \frac{\rho_0}{t+t_0}, \\ &\varphi = -\frac{1}{t+t_0}xy + \frac{\bar{\lambda}x + \bar{\mu}y}{t+t_0} - \\ &\frac{\bar{\mu}\bar{\lambda}}{t+t_0}\int_0^t f\left(\frac{\rho_0}{t+t_0}\right)dt, \end{aligned} \qquad (22)$$

with arbitrary constants $\rho_0, t_0, \bar{\lambda}$ and $\bar{\mu}$.

6) $\{P_0+aP_1,P_2,P_3\}$

The solutions take the form

$$\rho = \rho(\zeta),\ \varphi = \varphi(y,\zeta t),\ \zeta = x - \alpha t, \quad (23)$$

which can be obtained directly from 3) by replacing

$$y \longrightarrow \zeta, \varphi \longrightarrow \varphi - ay, x \longrightarrow y.$$

7) The subalgebra $\{P_0+aP_2, P_1, P_3\}$

Replacing in 6). $x \longleftrightarrow y$.

8) The subalgebra $\{P_0, P_1, P_3, B_1\}$

The solutions take the form

$$\rho = \rho(y), \varphi = \frac{xy}{t} + \psi(y,t). \qquad (24)$$

The substitution of (24) into (5), we obtain

$$\rho = \frac{\rho_0}{y}, \psi = -\frac{C}{2}ty + \frac{xy}{t}, f(\rho) = C\rho, \qquad (25)$$

$$\text{or} \quad \rho = -\frac{\rho_0}{y}, \psi = \frac{y}{t}, f \equiv 0. \qquad (26)$$

9) The subalgebra $\{P_0, P_2, P_3, B_2\}$

Replacing $x \longleftrightarrow y$ in 8).

10) The subalgebra $\{P_2, P_3, B_1\}$

The solutions take the form

$$\rho = \rho(t,x), \varphi = \frac{xy}{t} + \psi(y,t). \qquad (27)$$

The substitution of (27) into (5), we have

$$t\psi_t + y\psi_y = -f(\rho)t,$$
$$t\rho_t + (\rho x)_x + t\rho_x\psi_y = 0. \qquad (28)$$

The solutions are

$$\psi = \frac{y}{t}, \rho = [t^2 + (x+1)^2]^{-\frac{1}{2}}, f \equiv 0. \qquad (29)$$

11) The subalgebra $\{P_1, P_3, B_1\}$

The solutions take the form

$$\rho = \rho(t,y), \varphi = \frac{xy}{t} + \psi(y,t). \qquad (30)$$

The substitution of (30) into (5) gives

$$\rho = [t^2 + y^2]^{-\frac{1}{2}}, \psi = \frac{y}{t}, f \equiv 0. \qquad (31)$$

or

$$\rho = (ty)^{-\frac{1}{2}}, \psi = \frac{y}{t}, f \equiv 0. \qquad (32)$$

12) The subalgebra $\{P\ , P_3, B_2\}$

Replacing $x \longleftrightarrow y$ in 11).

3 Conclusions

In this paper, we discuss the existence of partially invariant solutions of a class of 2+1 dimensional nonlinear Schrödinger equations which arise in the propagation of 2 + 1 dimensional surface waves in deep water. It is shown that partially invariant solutions are much more than those of 1+1 dimensional case. Some are easily obtained explicitly, some are not. The constraint on $f(\rho)$ must be imposed in terms of the subgroup of the symmetry group.

It is instructive to compare the partially invariant solutions with solutions obtained by the nonclassical method of Bluman and Cole[2] and the direct method of Clarkson and Kruskal[6].

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(3) 管道内固液混合物运动的基本方程可作为推导固液混合物管道阻力损失计算公式及离心泵输送固液混合物时扬程等计算公式的基本公式。

(4) 由推导过程及结果可知，管道内单相流体运动的计算公式可作为固液混合物运动计算公式的特例。

5 附 录

5.1 物理量说明

A—面积；S—研究对象的外表表面；C—浓度；u—平均速度；E—能量；V—体积；F—外力；W—功；H—水头；Z—标高；K—动量；Δ—变量；M—质量；ρ—密度；N—颗粒总数；α—速度修正系数；P—压力；γ—重度；Q—体积流量。

5.2 下角标说明

a—表面上作用的量；S—固相的量；f—质量力作用的量；W—损失的量；i—管道断面数；x—横坐标；j—颗粒序数；y—纵坐标；L—液相的量；1—1 断面处的量；m—固液混合物的量；2—2 断面处的量；p—压力作用的量。

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The basic equation of solid and liquid-state admixture inside pipeline

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Abstract: By theoretical analysis, it is proved that ① the equation of solid and liquid state admixture movement inside the pipeline has three forms: continuous equation, momentum equation and the mechanical energetic equation. ② The equation can deduce a formula to calculate the loss of resistance, the lift of centrifugal pump in carrying the admixture. It is concluded that the movement of one-way clean water is a special example of the movement of the solidly liquid admixture.

Key words: solid and liquid-state admixture; pipeline; movement equation

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2+1 维非线性 Schrödinger 方程的偏不变解

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O413.1
O175.24

摘 要 推广了描述深水波波幅的 2+1 维非线性 Schrödinger 方程的结果。利用基于子群分类方法上的一般系统化途径，得到一类 2+1 维非线性 Schrödinger 方程的偏不变解。

关键词 对称群；偏不变解；非线性 Schrödinger 方程

薛定谔方程