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一类奇摄动半线性方程组的Robin问题

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摘要: 研究了一类可作为化学反应模型的奇摄动半线性方程组的Robin问题, 在一定条件下利用边界函数法构造了所论问题一致有效的渐近解, 同时讨论了该问题解的存在惟一性, 并给出了余项估计.

关键词: 奇摄动;　边界函数;　稳定流形;　渐近解

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Robin problem for a class of semilinear singularly perturbed ODE systems

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Abstract: The method of boundary function was used to construct uniformly valid asymptotic solutions of the Robin problem of a class of semilinear singularly perturbed equations, which are often used as models for chemical reactions. At the same time the existence and uniqueness of the solution and the estimation of the remainder for the problem were given.

Key words: singular perturbation; boundary functions; stable manifold; asymptotic solution

0　引　　言

奇摄动边值问题是近年来国际上关注的一个热门问题, 对于半线性边值问题的研究已有很多工作[1−3], 但多数研究的是纯量问题[1,2], 也有研究的是Drichlet问题[3], 采用的多是微分不等式方法, 所加的条件较苛刻. 今考虑文献[4]提出的如下半线性系统的Robin问题:

$$\varepsilon^2\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = h(y,t), \qquad 0<\varepsilon\ll 1, \tag{1}$$

$$\boldsymbol{P}y(0,\varepsilon)-\varepsilon y'(0,\varepsilon) = A, \qquad \boldsymbol{Q}y(1,\varepsilon)+\varepsilon y'(1,\varepsilon)=B, \tag{2}$$

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其中 y, A, B 均为 n 维向量, $\boldsymbol{h}$是定义在 $\mathbf{R}^n \times [0,1]$ 上的 n 维向量函数, $\boldsymbol{P}, \boldsymbol{Q}$为 n 阶方阵. 式(1),(2)代表的是有几个反应物的化学反应模型, 其中的每一个部分进行这样的一种反应, 它受各个分量的影响并且也影响各个分量, 其中$y = (y_1, \cdots, y_n)^{\mathrm{T}}$ 是无量纲的向量浓度, h是向量值函数, 代表了非线性动力学, 而 t 是无量纲距离. 本文用边界函数法在较弱的条件下对该类系统解的存在惟一性和渐近性质进行了讨论, 并得到了解的一致有效估计.

引进变换：令$z = \varepsilon \dfrac{\mathrm{d}y}{\mathrm{d}t}$, 则原问题化成等价的 Tihknov 系统的Robin问题：

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}y}{\mathrm{d}t} = z, \\ \varepsilon \dfrac{\mathrm{d}z}{\mathrm{d}t} = h(y,t), \end{cases} \tag{3}$$

$$\boldsymbol{P} y(0,\varepsilon) - z(0,\varepsilon) = A, \qquad \boldsymbol{Q} y(1,\varepsilon) + z(1,\varepsilon) = B. \tag{4}$$

为了讨论方便, 分别记$(x = (y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}, F(x,t) = F(y,z,t) = (z^{\mathrm{T}}, h^{\mathrm{T}}(y,t)))^{\mathrm{T}}$.

1 一些必要的条件和假设

$\boldsymbol{H_1}$ 假设 $h(y,t)$ 对于 y, t 在$\mathbf{R}^n \times [0,1]$上有足够次数的连续可微性. (确切的光滑性条件后面给出.)

$\boldsymbol{H_2}$ 假设 $h(y,t) = 0$ 在$\mathbf{R}^n \times [0,1]$上有惟一解： $y = \varphi(t)$.

于是系统(3)对应的退化方程 (即令 ε =0)

$$\bar{z}(t) = 0, \quad h(\bar{y}(t), t) = 0, \tag{5}$$

有解 $\bar{y}(t) = \varphi(t), \bar{z}(t) = 0$

$\boldsymbol{H_3}$ 假设 n 阶方阵$\bar{\boldsymbol{h}}_{\boldsymbol{y}}(t) = h_y(\bar{y}(t), t)$ 的特征根均不等于负实数和零.

记

$$\bar{\boldsymbol{F}}_{\boldsymbol{x}}(t) = \bar{\boldsymbol{F}}_{\boldsymbol{x}}(\bar{y}(t), \bar{z}(t), t) = \begin{pmatrix} \mathbf{0} & \boldsymbol{I}_n \\ \bar{\boldsymbol{h}}_y(t) & \mathbf{0} \end{pmatrix},$$

其中 $\mathbf{0}$ 为 n 阶零方阵, $\boldsymbol{I_n}$为 n 阶单位阵, $\bar{\boldsymbol{h}}_y(t)$为 n 阶方阵. 由于

$$\det(\bar{\boldsymbol{F}}_{\boldsymbol{x}}(t) - \lambda \boldsymbol{I}_{2n}) = \det \begin{bmatrix} \mathbf{0} - \lambda \boldsymbol{I}_n & \boldsymbol{I}_n \\ \bar{\boldsymbol{h}}_y(t) & \mathbf{0} - \lambda \boldsymbol{I}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{0} & \boldsymbol{I}_n \\ \bar{\boldsymbol{h}}_y(t) - \lambda^2 \boldsymbol{I}_n & -\lambda \boldsymbol{I_n} \end{bmatrix} = 0,$$

即

$$\det(\bar{h}_y(t) - \lambda^2 \boldsymbol{I}_n) = 0,$$

因此从假设$\boldsymbol{H_3}$可知$\bar{\boldsymbol{F}}_{\boldsymbol{x}}(t)$具有 n个正实部和 n 个负实部的特征值, 根据文献[5]的结果,即知存在与$\bar{\boldsymbol{F}}_{\boldsymbol{x}}(t)$有相同光滑性的可逆矩阵$\boldsymbol{B}(t)$, 使得

$$\boldsymbol{B}^{-1}(t) \begin{pmatrix} \mathbf{0} & \boldsymbol{I}_n \\ \bar{\boldsymbol{h}}_{\boldsymbol{y}}(t) & \mathbf{0} \end{pmatrix} \boldsymbol{B}(t) = \begin{pmatrix} \boldsymbol{C}_-(t) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C}_+(t) \end{pmatrix},$$

其中$\boldsymbol{C}_-(t), \boldsymbol{C}_+(t)$均为$n \times n$方阵, $\boldsymbol{C}_-(t)$的特征值均具有负实部, $\boldsymbol{C}_+(t)$的特征值均具有正实部. 由文献[6]可知, Robin问题(3),(4)属于条件稳定情形, 在 $t = 0, 1$ 处均存在边界层.

2 形式渐近解的构造

由边界函数法构造式(3)的形式解. 设

$$x(t,\varepsilon)=\bar{x}(t,\varepsilon)+\Pi x(\tau_0,\varepsilon)+Rx(\tau_1,\varepsilon),\tag{6}$$

其中

$$\bar{x}(t,\varepsilon)=\bar{x}_0(t)+\varepsilon\bar{x}_1(t)+\cdots\tag{7}$$

是正则项级数;

$$\Pi x(\tau_0,\varepsilon)=\Pi_0x(\tau_0)+\varepsilon\Pi_1x(\tau_0)+\cdots\tag{8}$$

是在 $t=0$ 附近的边界层级数, 而

$$Rx(\tau_1,\varepsilon)=R_0x(\tau_1)+\varepsilon R_1x(\tau_1)+\cdots\tag{9}$$

是在 $t=1$ 附近的边界层级数. 此外, $\tau_0=\dfrac{t}{\varepsilon}\geqslant 0,\tau_1=\dfrac{t-1}{\varepsilon}\leqslant 0$.

将式(6)-(9)代入式(3), (4), 并将所得的方程组的左右两端按自变量 t,τ_0,τ_1 分成三组方程, 然后对每组方程两端按 ε 展开成幂级数, 最后比较两边 ε 同次幂的系数, 即可得到一系列的等式, 且根据边界函数法有($\Pi_ix(+\infty)=0,R_ix(-\infty)=0,i=0,1,2,\cdots$).

由 ε 的零次幂的系数可得

$$\bar{z}_0=0,\qquad h(\bar{\boldsymbol{y}}_0,t)=0;\tag{10}$$

$$\begin{cases}\dfrac{\mathrm{d}\Pi_0y}{\mathrm{d}\tau_0}=\Pi_0z,\\ \dfrac{\mathrm{d}\Pi_0z}{\mathrm{d}\tau_0}=h(\bar{y}_0(0)+\Pi_0y,0),\\ \boldsymbol{P}(\bar{y}_0(0)+\Pi_0y(0))-(\bar{z}_0(0)+\Pi_0z(0))=A,\\ \Pi_0y(+\infty)=0,\Pi_0z(+\infty)=0.\end{cases}\tag{11}$$

$$\begin{cases}\dfrac{\mathrm{d}R_0y}{\mathrm{d}\tau_1}=R_0z,\\ \dfrac{\mathrm{d}R_0z}{\mathrm{d}\tau_1}=h(\bar{y}_0(1)+R_0y,1),\\ \boldsymbol{Q}(\bar{y}_0(1)+R_0y(0))+(\bar{z}_0(1)+R_0z(0))=B,\\ R_0y(-\infty)=0,R_0z(-\infty)=0.\end{cases}\tag{12}$$

显然, 方程组(10)的解即为退化方程组(5)的解: $\bar{y}_0(t)=\varphi(t),\bar{z}_0(t)=0$. 由文献[6]知, 如果矩阵$\boldsymbol{B}_{11}(0),\boldsymbol{B}_{22}(1)$可逆, 这里$\boldsymbol{B}_{ij}(i,j=1,2)$是矩阵$\boldsymbol{B}(t)$ 的相应分块矩阵, 在相空间(Π_0y,Π_0z)原点的邻域存在过$(\Pi_0y,\Pi_0z)=(0,0)$ 的n维不变稳定流形$\boldsymbol{S}^-$, 不妨假设为$\Pi_0z=\Phi(\Pi_0y)$. 将此流形沿式(11)中方程组的轨线进行延拓, 并令其定义域为$G^-\subset\mathbf{R}^n$, 假设$\Pi_0y(0)\in G^-$, 则式(11)存在惟一的当 $\tau_0\to+\infty$ 时指数式趋于零的解$(\Pi_0y(\tau_0),\Pi_0z(\tau_0))$.

同样, 对于式(12),在相空间(R_0y,R_0z)原点的邻域存在过$(R_0y,R_0z)=(0,0)$ 的 n 维不稳定流形$\boldsymbol{S}^+$, 不妨假设为$R_0y=\Psi(R_0z)$, 将此流形沿式(12)中方程组的轨线进行延拓, 并令其定义域为$G^+\subset\mathbf{R}^n$, 假设$R_0z(0)\in G^+$, 则系统(12)存在惟一的当 $\tau_1\to-\infty$ 时指数式趋于零的解$(R_0y(\tau_0),R_0z(\tau_0))$. 至此零次近似已完全确定.

为了后面讨论需要, 再作进一步假设:

$\boldsymbol{H_4}$ 假设关于$\Pi_0 y(0)$的方程组$\boldsymbol{P}\Pi_0 y(0)-\Phi(\Pi_0 y(0))+\boldsymbol{P}\varphi(0)-A=0$ 与关于$R_0 z(0)$的方程组$\boldsymbol{Q}\Psi(R_0 z(0))+R_0 z(0)+\boldsymbol{Q}\varphi(1)-B=0$均有惟一解: $\Pi_0 y(0)=\Pi_0^0 y(0)$与$R_0 z(0)=R_0^0 z(0)$, 而且 $\det(\boldsymbol{P}-\boldsymbol{H}(0))\neq 0, \det(\boldsymbol{Q}\tilde{\boldsymbol{H}}(0)+\boldsymbol{I_n})\neq 0$, 其中记$\boldsymbol{H}(\tau_0)=\dfrac{\partial\Phi}{\partial\Pi_0 y}(\Pi_0 y(\tau_0)), \tilde{\boldsymbol{H}}(\tau_1)=\dfrac{\partial\Psi}{\partial R_0 z}(R_0 z(\tau_1))$.

由ε的一次幂的系数可得

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}\bar{y}_0(t)}{\mathrm{d}t}=\bar{z}_1,\\ \dfrac{\mathrm{d}\bar{z}_0(t)}{\mathrm{d}t}=\boldsymbol{h}_y\left(\bar{y}_0(t),t\right)\bar{y}_1,\end{array}\right. \tag{13}$$

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}\Pi_1 y}{\mathrm{d}\tau_0}=\Pi_1 z,\\ \dfrac{\mathrm{d}\Pi_1 z}{\mathrm{d}\tau_0}=h_y(\tau_0)\Pi_1 y+g_1(\tau_0),\\ P(\bar{y}_1(0)+\Pi_1 y(0))-(\bar{z}_1(0)+\Pi_1 z(0))=0,\\ \Pi_1 y(+\infty)=0,\Pi_1 z(+\infty)=0,\end{array}\right. \tag{14}$$

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}R_1 y}{\mathrm{d}\tau_1}=R_1 z,\\ \dfrac{\mathrm{d}R_1 z}{\mathrm{d}\tau_1}=h_y(\tau_1)R_1 y+\tilde{g}_1(\tau_1),\\ Q(\bar{y}_1(1)+R_1 y(0))+(\bar{z}_1(1)+R_1 z(0))=0,\\ R_1 y(-\infty)=0,R_1 z(-\infty)=0,\end{array}\right. \tag{15}$$

其中$\boldsymbol{h}_y(\tau_0)$为h_y在$(\bar{y}_0(0)+\Pi_0 y(\tau_0),0)$处取值的矩阵, $\boldsymbol{h}_y(\tau_1)$ 为h_y在$(\bar{y}_0(1)+R_0 y(\tau_1),1)$处取值的矩阵, $g_1(\tau_0)=h_y(\bar{y}_0(0)+\Pi_0 y,0)(\bar{y}_0'(0)\tau_0+\bar{y}_1(0))+h_t(\bar{y}_0(0)+\Pi_0 y,0)\tau_0-h_y(\bar{y}_0(0),0)(\bar{y}_0'(0)\tau_0+\bar{y}_1(0))-h_t(\bar{y}_0(0,0)\tau_0$,易证$g_1(\tau_0)$满足指数式衰减的估计, 即当$\tau_0\geqslant 0$时, 存在正常数$C,\sigma_0$,使得

$$||g_1(\tau_0)||\leqslant C\exp(-\sigma_0\tau_0), \tag{16}$$

而$\tilde{g}_1(\tau_1)$有类似于$g_1(\tau_0)$的表达式, 并且也满足指数式衰减的估计.

由$\boldsymbol{H_3}$可知式(13)存在惟一解: $\bar{y}_1(t)=0,\bar{z}_1(t)=\varphi'(t)$. 于是有如下引理.

引　理 1　当$\tau_0\to+\infty$时, $\boldsymbol{H}(\tau_0)$极限矩阵的特征值均为负值.

证　明　由式(11)中的方程组得

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}\Pi_0 y}{\mathrm{d}\tau_0}=\Pi_0 z,\\ \dfrac{\mathrm{d}\Pi_0 z}{\mathrm{d}\tau_0}=h_y(\bar{y}_0(0),0)\Pi_0 y+G(\Pi_0 y),\end{array}\right. \tag{17}$$

其中$G(\Pi_0 y)=h(\bar{y}_0(0)+\Pi_0 y,0)-h_y(\bar{y}_0(0),0)\Pi_0 y$, 它有性质: $G(0)=0,\dfrac{\partial G}{\partial\Pi_0 y}(0)=0$. 将式(17)写成矩阵形式:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_0}\begin{pmatrix}\Pi_0 y\\ \Pi_0 z\end{pmatrix}=\begin{pmatrix}0 & \boldsymbol{I}_n\\ \bar{\boldsymbol{h}}_y(0) & 0\end{pmatrix}\begin{pmatrix}\Pi_0 y\\ \Pi_0 z\end{pmatrix}+\begin{pmatrix}0\\ G(\Pi_0 y)\end{pmatrix}. \tag{18}$$

记$\bar{h}_y(0)=h_y(\bar{y}_0(0),0)$,且由矩阵分析和$\boldsymbol{H_3}$知道(见文献[7]), 存在惟一的其特征值全为正实部的 n 阶方阵$\sqrt{\bar{\boldsymbol{h}}_y(0)}$使得$\left(\sqrt{\bar{\boldsymbol{h}}_y(0)}\right)^2=\bar{\boldsymbol{h}}_y(0)$; 对式(18)作变量替换

$$\begin{pmatrix}\Pi_0 y\\ \Pi_0 z\end{pmatrix}=\begin{pmatrix}\boldsymbol{I}_n & \boldsymbol{I}_n\\ -\sqrt{\bar{\boldsymbol{h}}_y(0)} & \sqrt{\bar{\boldsymbol{h}}_y(0)}\end{pmatrix}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \tag{19}$$

且容易验证

$$\begin{pmatrix}\boldsymbol{I}_n & \boldsymbol{I}_n\\ -\sqrt{\bar{\boldsymbol{h}}_y(0)} & \sqrt{\bar{\boldsymbol{h}}_y(0)}\end{pmatrix}^{-1}=\frac{1}{2}\begin{pmatrix}\boldsymbol{I}_n & -\left(\sqrt{\bar{\boldsymbol{h}}_y(0)}\right)^{-1}\\ \boldsymbol{I}_n & \left(\sqrt{\bar{\boldsymbol{h}}_y(0)}\right)^{-1}\end{pmatrix},$$

于是方程组(18)化为

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau_0}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}=&\begin{pmatrix}I_n & I_n\\ -\sqrt{\bar{h}_y(0)} & \sqrt{\bar{h}_y(0)}\end{pmatrix}^{-1}\begin{pmatrix}0 & I_n\\ \bar{h}_y(0) & 0\end{pmatrix}\begin{pmatrix}I_n & I_n\\ -\sqrt{\bar{h}_y(0)} & \sqrt{\bar{h}_y(0)}\end{pmatrix}\\ &\begin{pmatrix}\eta\\ \zeta\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-\left(\sqrt{\bar{h}_y(0)}\right)^{-1}G(\eta+\zeta)\\ \left(\sqrt{\bar{h}_y(0)}\right)^{-1}G(\eta+\zeta)\end{pmatrix}.\end{aligned} \tag{20}$$

可以证明其所有当 $\tau_0\to+\infty$ 时指数式趋于零的解与下面积分方程组的解等价

$$\left\{\begin{aligned}&\eta(\tau_0)=\exp\left(-\sqrt{\bar{h}_y(0)}\tau_0\right)\eta^0-\frac{1}{2}\int_0^{\tau_0}\exp(\left(-\sqrt{\bar{h}_y(0)}\right)(\tau_0-s))\left(\sqrt{\bar{h}_y(0)}\right)^{-1}\\ &\qquad\qquad G(\eta(s)+\zeta(s))\mathrm{d}s,\\ &\zeta(\tau_0)=\frac{1}{2}\int_{+\infty}^{\tau_0}\exp\left(\sqrt{\bar{h}_y(0)}(\tau_0-s)\right)\left(\sqrt{\bar{h}_y(0)}\right)^{-1}G(\eta(s)+\zeta(s))\mathrm{d}s,\end{aligned}\right. \tag{21}$$

其中$\eta(0)=\eta^0,\zeta(0)=\zeta(0,\eta^0)=\dfrac{1}{2}\displaystyle\int_{+\infty}^{0}\exp(-\sqrt{\bar{h}_y(0)}s)\left(\sqrt{\bar{h}_y(0)}\right)^{-1}G(\eta(s)+\zeta(s))\mathrm{d}s$. 由变量替换(19)式得

$$\left\{\begin{aligned}&\Pi_0 y(\tau_0)=\eta(\tau_0)+\zeta(\tau_0),\\ &\Pi_0 z(\tau_0)=\sqrt{\bar{h}_y(0)}(\zeta(\tau_0)-\eta(\tau_0)).\end{aligned}\right.$$

由$G(\Pi_0 y)$性质得出当 $\tau_0\to+\infty$时, $\eta(\tau_0)\to 0,\zeta(\tau_0)\to 0, G(\eta(\tau_0)+\zeta(\tau_0))\to 0$. 因此(21)式中的积分项很小, 当$\tau_0$充分大时可得近似式

$$\left\{\begin{aligned}&\Pi_0 y(\tau_0)\approx\exp\left(-\sqrt{\bar{h}_y(0)}\tau_0\right)\eta^0,\\ &\Pi_0 z(\tau_0)\approx-\sqrt{\bar{h}_y(0)}\exp\left(-\sqrt{\bar{h}_y(0)}\tau_0\right)\eta^0.\end{aligned}\right.$$

从而得到

$$\Pi_0 z(\tau_0)\approx-\sqrt{\bar{h}_y(0)}\Pi_0 y(\tau_0). \tag{22}$$

于是由$\boldsymbol{H}(\tau_0)$定义可得

$$\boldsymbol{H}(\tau_0) \approx -\sqrt{\bar{h}_y(0)}, \tag{23}$$

由此即见其特征值实部均为负值. 引理证毕.

下面求 $\Pi_1 y, \Pi_1 z$, 并对其作出指数式衰减估计. 为此先构造出边值问题(14)的解.

作变量替换：$\Pi_1 y = \delta_1, \Pi_1 z = \boldsymbol{H}(\tau_0)\delta_1 + \delta_2$, 不难证明 n 阶方阵$\boldsymbol{H}(\tau_0)$满足Riccati方程$\boldsymbol{h}_y(\tau_0) - \boldsymbol{H}'(\tau_0) - \boldsymbol{H}^2(\tau_0) = 0$. 于是(14)中的方程组化为

$$\frac{\mathrm{d}\delta_1}{\mathrm{d}\tau_0} = \boldsymbol{H}(\tau_0)\delta_1 + \delta_2, \tag{24}$$

$$\frac{\mathrm{d}\delta_2}{\mathrm{d}\tau_0} = -\boldsymbol{H}(\tau_0)\delta_2 + g_1(\tau_0). \tag{25}$$

由引理1知, $\boldsymbol{H}(\tau_0)$极限矩阵的特征值实部均负, $-\boldsymbol{H}(\tau_0)$极限矩阵的特征值实部均正, 因此若记式(24),(25)对应齐次方程组的基解矩阵分别为$\boldsymbol{\Gamma}_1(\tau_0), \boldsymbol{\Gamma}_2(\tau_0)$, 则必有

$$||\boldsymbol{\Gamma}_1(\tau_0)\boldsymbol{\Gamma}_1^{-1}(s)|| \leqslant C\exp(-\sigma_0(\tau_0 - s)), \quad 0 \leqslant s \leqslant \tau_0, \tag{26}$$

$$||\boldsymbol{\Gamma}_2(\tau_0)\boldsymbol{\Gamma}_2^{-1}(s)|| \leqslant C\exp(-\sigma_0(s - \tau_0)), \quad 0 \leqslant \tau_0 \leqslant s. \tag{27}$$

一般来说, 这里的两个常数C和前面的C并不一样, 但为了简单起见, 约定这种不依赖于ε,其大小本质上不起作用的相似常数以后总用一个符号C来表示.

由于当$\tau_0 \to +\infty$时有$\delta_2(\tau_0) \to 0$,故可得方程组(25)的一个特解

$$\delta_2^0(\tau_0) = \int_{+\infty}^{\tau_0} \boldsymbol{\Gamma}_2(\tau_0)\boldsymbol{\Gamma}_2^{-1}(s) g_1(s)\mathrm{d}s.$$

由式(16),(27)得出

$$||\delta_2^0(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0). \tag{28}$$

把 $\delta_2^0(\tau_0)$ 代入式(24),得到其满足 $\delta_1(0) = 0$ 的特解为

$$\delta_1^0(\tau_0) = \int_0^{\tau_0} \boldsymbol{\Gamma}_1(\tau_0)\boldsymbol{\Gamma}_1^{-1}(s)\delta_2^0(s)\mathrm{d}s.$$

而由式(26),(28)即知

$$||\delta_1^0(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0). \tag{29}$$

于是可得出式(14)中的方程组的一个特解为

$$\Pi_1^0 y(\tau_0) = \int_0^{\tau_0} \boldsymbol{\Gamma}_1(\tau_0)\boldsymbol{\Gamma}_1^{-1}(s)\left[\int_{+\infty}^{s} \boldsymbol{\Gamma}_2(s)\boldsymbol{\Gamma}_2^{-1}(t) g_1(t)\mathrm{d}t\right]\mathrm{d}s, \tag{30}$$

$$\Pi_1^0 z(\tau_0) = H(\tau_0)\int_0^{\tau_0} \boldsymbol{\Gamma}_1(\tau_0)\boldsymbol{\Gamma}_1^{-1}(s)\left[\int_{+\infty}^{s} \boldsymbol{\Gamma}_2(s)\boldsymbol{\Gamma}_2^{-1}(t) g_1(t)\mathrm{d}t\right]\mathrm{d}s + \int_{+\infty}^{\tau_0} \boldsymbol{\Gamma}_2(\tau_0)\boldsymbol{\Gamma}_2^{-1}(s) g_1(s)\mathrm{d}s. \tag{31}$$

这是一组满足初始条件

$$\Pi_1^0 y(0) = 0, \Pi_1^0 z(0) = \int_{+\infty}^{0} \boldsymbol{\Gamma}_2^{-1}(s) g_1(s)\mathrm{d}s,$$

且当$\tau_0 \to +\infty$时指数式趋于零的解.

另一方面, 设$\tilde{\Pi}_1 z(\tau_0), \tilde{\Pi}_1 y(\tau_0)$为(14)中方程组对应齐次方程组的解, 则由文献[6]中的引理4.3-4.4即知, 当$\tau_0 \to +\infty$时, $\tilde{\Pi}_1 y(\tau_0), \tilde{\Pi}_1 z(\tau_0)$指数式趋于零的充要条件是$\tilde{\Pi}_1 z(0) = \boldsymbol{H}(0)\tilde{\Pi}_1 y(0)$, 这里$\boldsymbol{H}(0) = \dfrac{\partial \Phi}{\partial \Pi_0 y}(\Pi_0^0 y(0))$. 于是取 $\tilde{\Pi}_1 y(0) = [\boldsymbol{P} - \boldsymbol{H}(0)]^{-1}(\varphi'(0) + \Pi_1^0 z(0))$, 则有 $\tilde{\Pi}_1 z(0) = \boldsymbol{H}(0)[\boldsymbol{P} - \boldsymbol{H}(0)]^{-1}(\varphi'(0) + \Pi_1^0 z(0))$; 而且解 $\tilde{\Pi}_1 y(\tau_0), \tilde{\Pi}_1 z(\tau_0)$ 满足

$$||\tilde{\Pi}_1 y(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0), \tag{32}$$

$$||\tilde{\Pi}_1 z(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0). \tag{33}$$

由此不难验证(14)中方程组的解

$$\Pi_1 y(\tau_0) = \tilde{\Pi}_1 y(\tau_0) + \Pi_1^0 y(\tau_0), \quad \Pi_1 z(\tau_0) = \tilde{\Pi}_1 z(\tau_0) + \Pi_1^0 z(\tau_0),$$

满足式(14)中的初始条件. 实际上, $\boldsymbol{P}[\Pi_1 y(0) + \bar{y}_1(0)] - [\bar{z}_1(0) + \Pi_1 z(0)] = \boldsymbol{P}\tilde{\Pi}_1 y(0) - \varphi'(0) - \Pi_1^0 z(0) - \tilde{\Pi}_1 z(0) = [\boldsymbol{P} - \boldsymbol{H}(0)]\tilde{\Pi}_1 y(0) - \varphi'(0) - \Pi_1^0 z(0) = 0$. 并由式(28)–(33)可得到

$$||\Pi_1 y(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0), \quad ||\Pi_1 z(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0).$$

因此$\Pi_1 y(\tau_0), \Pi_1 z(\tau_0)$就是边值问题(14)的唯一解.

由于$\tilde{g}_1(\tau_1)$的表达式与$g_1(\tau_0)$很相似, 同样也满足指数式衰减的估计, 因此用同样的方法可得到式(15)存在惟一解$R_1 y(\tau_1), R_1 z(\tau_1)$,且当$\tau_1 \leqslant 0$时, 有

$$||R_1 y(\tau_1)|| \leqslant C\exp(\sigma_0\tau_1), \quad ||R_1 z(\tau_1)|| \leqslant C\exp(\sigma_0\tau_1).$$

至此一次近似已完全确定.

对于$k > 1$,由ε的k次幂的系数可得下列方程组及其定解条件.

$$\begin{cases} \dfrac{\mathrm{d}\bar{y}_{k-1}(t)}{\mathrm{d}t} = \bar{z}_k, \\ \dfrac{\mathrm{d}\bar{z}_{k-1}(t)}{\mathrm{d}t} = h_y(\bar{y}_0(t), t)\bar{y}_k + f_k(t), \end{cases} \tag{34}$$

$$\begin{cases} \dfrac{\mathrm{d}\Pi_k y}{\mathrm{d}\tau_0} = \Pi_k z, \\ \dfrac{\mathrm{d}\Pi_k z}{\mathrm{d}\tau_0} = h_y(\tau_0)\Pi_k y + g_k(\tau_0), \\ \boldsymbol{P}(\bar{y}_k(0) + \Pi_k y(0)) - (\bar{z}_k(0) + \Pi_k z(0)) = 0, \\ \Pi_k y(+\infty) = 0, \Pi_k z(+\infty) = 0, \end{cases} \tag{35}$$

$$\begin{cases} \dfrac{\mathrm{d}R_k y}{\mathrm{d}\tau_1} = R_k z, \\ \dfrac{\mathrm{d}R_k z}{\mathrm{d}\tau_1} = h_y(\tau_1)R_k y + \tilde{g}_k(\tau_1), \\ \boldsymbol{Q}(\bar{y}_k(1) + R_k y(0)) + (\bar{z}_k(1) + R_k z(0)) = 0, \\ R_k y(-\infty) = 0, R_k z(-\infty) = 0, \end{cases} \tag{36}$$

其中$f_k(t), g_k(\tau_0), \tilde{g}_k(\tau_1)$均为按确定方法表示的函数, 而且是只依赖于下标号码小于k的已知函数, 且$g_k(\tau_0), \tilde{g}_k(\tau_1)$同样有估计式

$$||g_k(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0), \quad ||\tilde{g}_k(\tau_1)|| \leqslant C\exp(\sigma_0\tau_1).$$

式(37)是$\bar{y}_k, \bar{z}_k$的代数方程, 其解易求, 而边值问题(35),(36)与边值问题(14),(15)形式上类似, 性质上完全一样, 所以用同样的方法可求解出$\Pi_k y, \Pi_k z, R_k y, R_k Z$, 且

$$\text{当}\tau_0 \geqslant 0\text{时, 有}\quad ||\Pi_k y(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0), \quad ||\Pi_k z(\tau_0)|| \leqslant C\exp(-\sigma_0\tau_0);$$
$$\text{当}\tau_1 \leqslant 0\text{时, 有}\quad ||R_k y(\tau_1)|| \leqslant C\exp(\sigma_0\tau_1), \quad ||R_k z(\tau_1)|| \leqslant C\exp(\sigma_0\tau_1).$$

3 主要结果

为了证明上面构造的形式解是Robin问题(3),(4)在$0 \leqslant t \leqslant 1$的一致有效渐近解, 需进行余项估计, 下面就利用文献[8]中的结果来进行证明.

***H*$_5$** 假设 $U_\delta \subset \mathrm{R}^{2n+1}$ 为(y,z,t)空间中曲线$L = L_1 \cup L_2 \cup L_3$的$\delta$邻域, 其中$\delta > 0$,而

$$L_1 = \{(y,z,t)|y = \bar{y}_0(0) + \Pi_0 z(\tau_0), z = \Pi_0 z(\tau_0), \tau_0 \geqslant 0, t = 0\},$$
$$L_2 = \{(y,z,t)|y = \bar{y}_0(t), z = 0, 0 \leqslant t \leqslant 1\},$$
$$L_3 = \{(y,z,t)|y = \bar{y}_0(1) + R_0 z(\tau_1), z = R_0 z(\tau_1), \tau_1 \leqslant 0, t = 1\}.$$

且函数$h(y,t)$在U_δ上$n+2$次连续可微。

定 理 当满足条件$\boldsymbol{H_1} - \boldsymbol{H_5}$, 则存在$\varepsilon_0 > 0, \delta_0 \in (0,\delta), c > 0$, 使得当$0 < \varepsilon \leqslant \varepsilon_0$时, 在$L$的$\delta$邻域中, 问题(3),(4)存在惟一解$x(t,\varepsilon)$, 且有估计式

$$||x(t,\varepsilon) - x_n(t,\varepsilon)|| \leqslant C\varepsilon^{n+1}, \tag{37}$$

其中 $x_n(t,\varepsilon) = \sum\limits_{k=0}^{n} \varepsilon^k[\bar{x}_k(t) + \Pi_k x(\frac{t}{\varepsilon}) + R_k x(\frac{t-1}{\varepsilon})]$.

证 明 根据文献[6]的定理4.2, 考虑方程组(3)在定解条件

$$y(0,\varepsilon) = y_0^*, z(1,\varepsilon) = z_0^* \tag{38}$$

下的问题, 并记这个问题的解为$x(t, x_0^*, \varepsilon)$, 这里

$$x_0^* = \begin{pmatrix} y_0^* \\ z_0^* \end{pmatrix}. \tag{39}$$

将条件(4)记为

$$\begin{aligned} R(x(0,x_0^*,\varepsilon), x(1,x_0^*,\varepsilon)) &= \begin{pmatrix} Py(0,x_0^*,\varepsilon) - z(0,x_0^*,\varepsilon) - A \\ Qy(1,x_0^*,\varepsilon) + z(1,x_0^*,\varepsilon) - B \end{pmatrix} \\ &= \begin{pmatrix} Py_0^* - \bar{z}(0,x_0^*,\varepsilon) - \Pi z(0,x_0^*,\varepsilon) - A \\ Q\bar{y}(1,x_0^*,x_0^*,\varepsilon) + QRy(0,x_0^*,\varepsilon) + z_0^* - B \end{pmatrix} = 0. \end{aligned} \tag{40}$$

将R对ε进行展开得

$$R = R_0 + \varepsilon R_1 + \cdots, \tag{41}$$

其中

$$\begin{aligned} R_0 &= \begin{pmatrix} P(y_0^*) - (\bar{z}_0(0) + \Pi_0 z(0)) - A \\ Q(\bar{y}_0(1) + R_0 y(0)) + z_0^* - B \end{pmatrix} \\ &= \begin{pmatrix} Py_0^* - \Phi(y_0^* - \varphi(0)) - A \\ Q(\varphi(1) + \Psi(z_0^*)) + z_0^* - B \end{pmatrix}. \end{aligned} \tag{42}$$

由式(40),(41)即知$R_i = 0, i = 0, 1, \cdots$.特别对于$R_0 = 0$, 根据$\boldsymbol{H_4}$, 有惟一解$y_0^* = \varphi(0) + \Pi_0^0 y(0), z_0^* = R_0^0 Z(0)$, 而且有以下关系式成立.

$$\frac{D(R)}{D(x_0^*)} = \frac{D(R_0)}{D(x_0^*)} + O(\varepsilon). \tag{43}$$

具体证明见文献[8,9]. 此外由式(42)有

$$\begin{aligned}\frac{D(R_0)}{D(x_0^*)} &= \det\begin{bmatrix} \boldsymbol{P} - \boldsymbol{H}(0) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}\tilde{\boldsymbol{H}}(0) + \boldsymbol{I}_n \end{bmatrix} \\ &= \det(\boldsymbol{P} - \boldsymbol{H}(0)) \det((\boldsymbol{Q}\tilde{\boldsymbol{H}}(0) + \boldsymbol{I}_n)),\end{aligned} \tag{44}$$

记

$$x_0^0 = \begin{pmatrix} \varphi(0) + \Pi_0^0 y(0) \\ R_0^0 z(0) \end{pmatrix}. \tag{45}$$

由$\boldsymbol{H_4}$可得$\dfrac{D(R_0)}{D(x_0^*)}(x_0^0) \neq 0$. 这就验证了文献[8]中定理的条件$\Delta_0^0 \neq 0$. 至于该定理的另外五个条件, 可直接从$\boldsymbol{H_1}$− $\boldsymbol{H_5}$推出. 因此本文的定理就从文献[8]的定理得到. 定理证毕.

结 论 如果满足条件$\boldsymbol{H_1} - \boldsymbol{H_5}$, 则问题(1),(2)在曲线$L$的$\delta$ 邻域中存在惟一解$x(t, \varepsilon)$, 当$\varepsilon \to 0$时, 对$t \in [0, 1]$一致地有

$$x(t, \varepsilon) = \sum_{i=1}^{n} \varepsilon^k [\bar{x}_k(t) + \Pi_k x(\frac{t}{\varepsilon}) + R_k(\frac{t-1}{\varepsilon})] + O(\varepsilon^{n+1}). \tag{46}$$

用同样的方法可以对拟线性问题$\varepsilon\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} = A(y, t)\dfrac{\mathrm{d}y}{\mathrm{d}t} + B(y, t), y \in \mathbf{R}^n$ 进行讨论.

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