Article ID: 1000-5641(2009)04-0082-10

On cohomology of modular Lie superalgebras

ZHENG Li-sun

(*Department of Mathematics, East China Normal University, Shanghai* 200062, *China*)

Abstract: A result on vanishing cohomology for finite-dimensional modular Lie superalgebras was obtained, with aid of the approach provided by Dzhumadil'daev for modular Lie algebras. Some examples are given as demonstration of the vanishing result, as well as its applications.

Key words: restricted Lie superalgebra; cohomology of Lie superalgebra; p-polynomial

CLC number: O154 **Document code:** A

有限维模李超代数的上同调

郑立笋

(华东师范大学 数学系, 上海 200062)

摘要: 应用 Dzhumadildaev 方法, 研究了有限维模李超代数的上同调问题. 通过研究包络代数的 p-中心对其表示的作用, 得到了有限维模李超代数的一个上同调消失定理. 并作为应用, 计算了一类 Cartan型李超代数的低阶上同调.

关键词: 限制李超代数; 李超代数的上同调; p-多项式

1 Main results

In this paper, all algebras and modules are finite dimensional over a given algebraically closed field k of characteristic $p > 0$. Let $\mathfrak{g}$ be a Lie superalgebra, $U(\mathfrak{g})$ the universal enveloping algebra and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$.

Definition 1.1[1,2] A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called a restricted Lie superalgebra, if the following conditions are satisfied.

(a) $\mathfrak{g}_{\bar{0}}$ is a restricted Lie algebra with p-mapping $[p]$: $\mathfrak{g}_{\bar{0}} \longrightarrow \mathfrak{g}_{\bar{0}}$ in the sense of [3, Chap 4].

(b) $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$-module via the adjoint action, i.e. $\mathrm{ad}(X^{[p]})(X_1) = \mathrm{ad}(X)^p(X_1)$, for $X \in \mathfrak{g}_{\bar{0}}, X_1 \in \mathfrak{g}_{\bar{1}}$.

收稿日期: 2008-09

基金项目: 国家自然科学基金(10871067)

作者简介: 郑立笋, 女, 博士研究生. E-mail: lszhengmath@hotmail.com.

A polynomial of the form $f(t)=\sum\limits_{i\geqslant 0}\lambda_i t^{p^i}\in k[t]$ is called a *p-polynomial*. With every element $l\in\mathfrak{g}_{\bar{0}}$ we can associate a p-polynomial $z(t)$ such that replacing t by l gives a central element $z(l)\in Z(\mathfrak{g})$. Thus we obtain a map $z:\mathfrak{g}_{\bar{0}}\longrightarrow Z(\mathfrak{g})$. Let M be a $\mathfrak{g}$-module and

$$l\longmapsto (l)_M, l\in\mathfrak{g}, (l)_M\in \text{End } M,$$

its associated representation. The main result can be stated as follows.

Theorem 1.2 Let $\mathfrak{g}$ be a Lie superalgebra over k, an algebraically closed field of prime characteristic p, and M an arbitrary $\mathfrak{g}$-module. Suppose for some $l\in\mathfrak{g}_{\bar{0}}$, the endomorphism $z(l)_M$ is not degenerate, with $z(l)$ as above. Then the cohomology $H^*(\mathfrak{g},M)$ is zero.

A direct corollary can be obtained:

Corollary 1.3 Assume $\mathfrak{g}$ is a restricted Lie superalgebra, M is an irreducible $\mathfrak{g}$-module, but not restricted. Then $H^*(\mathfrak{g},M)$ is zero.

2 Cohomology of modular Lie superalgebras with coefficients in a nontrivial module

At first, let us recall the definition of Lie superalgebra cohomology (for more details, the reader is referred to ([4,5]). Let g be a Lie superalgebra and M a $\mathfrak{g}$-module. The $\mathbb{Z}$-graded superspace $C^*(\mathfrak{g},M)$ is defined as

$$C^*(\mathfrak{g},M)=\bigoplus_{q\geqslant 0}C^q(\mathfrak{g},M)\quad\text{with}\quad C^q(\mathfrak{g},M)=\text{Hom}(\bigwedge\nolimits^q(\mathfrak{g}),M),$$

where $\bigwedge^q(\mathfrak{g})=\sum_{i\in\mathbb{Z}_2}\bigwedge^q_i(\mathfrak{g})$ is the superspace of $\mathbb{Z}_2$-graded q-alternating tensors on $\mathfrak{g}$, with $\bigwedge^q_i(\mathfrak{g})$ the k-span of all elements $x_1\wedge x_2\wedge\cdots\wedge x_q, (x_j\in\mathfrak{g})$ satisfying $\sum\limits_{j=1}^q\bar{x}_j=i$ (denote by $\bar{x}_j$ the $\mathbb{Z}_2$ degree of x_j), $i=0,1$, and

$$x_1\wedge\cdots x_j\wedge x_{j+1}\wedge\cdots\wedge x_q=-(-1)^{\bar{x}_j\bar{x}_{j+1}}x_1\wedge\cdots x_{j+1}\wedge x_j\wedge\cdots\wedge x_q.$$

We also set $C^0(\mathfrak{g},M)=M$ and $C^q(\mathfrak{g},M)=0$, if $q<0$.

The coboundary operator is an even linear operator of $\mathbb{Z}$-degree $+1$ on $C^*(\mathfrak{g},M)$ given by $d\phi=\phi'+\phi''$ where $\phi',\phi''\in C^{q+1}(\mathfrak{g},M)$ are the cochains corresponding to $\phi\in C^q(\mathfrak{g},M)$ and determined by the formulas

$$\begin{aligned}\phi'(x_1,\ldots,x_{q+1}) &= \sum_{i<j}(-1)^{\sigma_{i,j}(x_1,\ldots,x_{q+1})}\phi([x_i,x_j],x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_{q+1}),\\ \phi''(x_1,\ldots,x_{q+1}) &= \sum_{i}(-1)^{\gamma_i(x_1,\ldots,x_{q+1},\phi)}x_i\phi(x_1,\ldots,\widehat{x}_i,\ldots,x_{q+1}),\end{aligned}\tag{1}$$

where $x_1,\ldots,x_{q+1}$ and ϕ are taken to be homogeneous and

$$\begin{aligned}&\sigma_{i,j}(x_1,\ldots,x_q):=i+j+\overline{x}_i(\overline{x}_1+\ldots+\overline{x}_{i-1})+\overline{x}_j(\overline{x}_1+\ldots+\overline{x}_{j-1}+\overline{x}_i),\\ &\gamma_i(x_1,\ldots x_q,\phi):=i+1+\overline{x}_i(\overline{x}_1+\ldots+\overline{x}_{i-1}+\overline{\phi}).\end{aligned}$$

We write $d^{(q)}$ for the restriction of d to $C^q(\mathfrak{g}, M)$. It is indeed a cochain complex since $d^2 = 0$, i.e. $d^{(q+1)} \circ d^{(q)} = 0$ for all q in $\mathbb{Z}$. By $B^q(\mathfrak{g}, M)$ we denote the space of q-coboundaries, by $Z^q(\mathfrak{g}, M)$ the space of q-cocycles and by $H^q(\mathfrak{g}, M)$ the space of cohomology classes. By definition, $H^q(\mathfrak{g}, M)$ is the q-th cohomology of g with coefficients in M.

Let θ be a representation of $\mathfrak{g}$ in $C^*(\mathfrak{g}, M)$ of the form

$$(\theta(l)\phi)(x_1, \ldots, x_q) = (l)_M \phi(x_1, \ldots, x_q) + \sum_i (-1)^{i+\overline{x}_i(\overline{x}_1+\ldots+\overline{x}_{i-1})+\bar{l}\bar{\phi}} \phi([l, x_i], \ldots, \widehat{x}_i, \ldots, x_q).$$

This extends to a representation of the universal enveloping algebra $U(\mathfrak{g})$. Every element $l \in \mathfrak{g}$ determines an endomorphism of degree -1 (adjoint endomorphism) $i(l)$ of the cochain complex $C^*(\mathfrak{g}, M)$, if we put

$$(i(l)\phi)(x_1, \ldots, x_{q-1}) = (-1)^{\bar{l}\bar{\phi}} \phi(l, x_1, \ldots, x_{q-1}), \qquad \phi \in C^q(\mathfrak{g}, M). \tag{2}$$

Lemma 2.1 Maintain the notations as above, then

$$d\theta(l) = \theta(l)d, \qquad l \in g, \tag{3}$$

$$di(l) + i(l)d = \theta(l), \qquad l \in g. \tag{4}$$

Proof Let $\phi \in C^q(\mathfrak{g}, M), q \geqslant 0$, we verify that

$$\theta(l)\phi' = (\theta(l)\phi)', \tag{5}$$

$$\theta(l)\phi'' = (\theta(l)\phi)''. \tag{6}$$

$$\begin{aligned}
&\theta(l)\phi'(x_1, \ldots, x_{q+1}) \\
&= (l)_M \phi'(x_1, \ldots, x_{q+1}) + \sum_{i=1}^{q+1} (-1)^{i+\overline{x}_i(\overline{x}_1+\ldots+\overline{x}_{i-1})} (-1)^{\bar{l}\bar{\phi}} \phi'([l, x_i], \ldots, \widehat{x}_i, \ldots, x_{q+1}) \\
&= (l)_M \phi'(x_1, \ldots, x_{q+1}) + \sum_{i=1}^{q+1} (-1)^{1+\bar{l}(\overline{x}_1+\ldots+\overline{x}_{i-1}+\bar{l}\bar{\phi})} \phi'(x_1, \ldots, [l, x_i], \ldots, x_{q+1}) \\
&= (l)_M \sum_{i<j} (-1)^{\sigma_{i,j}(x_1,\ldots,x_q)} \phi([x_i, x_j], \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_{q+1}) + \sum_{i=1}^{q+1} (-1)^{1+\bar{l}(\overline{x}_1+\ldots+\overline{x}_{i-1}+\bar{l}\bar{\phi})} \\
&\quad \sum_{j<s} (-1)^{\sigma_{j,s}(x_1,\ldots,[l,x_i],\ldots,x_{q+1})} \phi([x_j, x_s], \ldots, \widehat{x}_j, \ldots, \widehat{x}_s \ldots, x_{q+1})
\end{aligned}$$

$$\begin{aligned}
&(\theta(l)\phi)'(x_1, \ldots, x_{q+1}) \\
&\quad = \sum_{j<s} (-1)^{\sigma_{j,s}(x_1,\ldots,x_i,\ldots,x_{q+1})} \theta(l)\phi([x_j, x_s], \ldots, \widehat{x}_j, \ldots, \widehat{x}_s, \ldots, x_{q+1}) \\
&\quad = (l)_M \sum_{j<s} (-1)^{\sigma_{j,s}(x_1,\ldots,x_q)} \phi([x_j, x_s], \ldots, \widehat{x}_j, \ldots, \widehat{x}_s, \ldots, x_{q+1}) +
\end{aligned}$$

$$\sum_{j<s}(-1)^{\sigma_{j,s}(x_1,\ldots,x_q)}(-1)^{\bar{l}\bar{\phi}}((-1)\phi([l,[x_j,x_s]]\ldots,\widehat{x}_j,\ldots,\widehat{x}_s,\ldots,x_{q+1})$$
$$+\sum_{i=1}^{j-1}(-1)^{i+1+\overline{x}_i(\overline{x}_j+\overline{x}_s+\overline{x}_1+\ldots+\overline{x}_{i-1})}\phi([l,x_i],[x_j,x_s],\ldots,x_{q+1})$$
$$+\sum_{i=j+1}^{s-1}(-1)^{i-1+1+\overline{x}_i(\overline{x}_j+\overline{x}_s+\overline{x}_1+\ldots+\overline{x}_{i-1}+\overline{x}_j)}\phi([l,x_i],[x_j,x_s],\ldots,x_{q+1})$$
$$+\sum_{i=s+1}^{q+1}(-1)^{i-2+1+\overline{x}_i(\overline{x}_j+\overline{x}_s+\overline{x}_1+\ldots+\overline{x}_{i-1}+\overline{x}_j+\overline{x}_s)}\phi([l,x_i],[x_j,x_s],\ldots,x_{q+1})),$$

according to the relations of i,j,s, and together with the Jacobi identity(super version), it's not difficult to obtain the equation (5).

$$\begin{aligned}
&\theta(l)\phi''(x_1,\ldots,x_{q+1})\\
&\quad=(l)_M\phi''(x_1,\ldots,x_{q+1})+\sum_{i=1}^{q+1}(-1)^{i+\overline{x}_i(\overline{x}_1+\ldots+\overline{x}_{i-1})}(-1)^{\bar{l}\bar{\phi}}\phi''([l,x_i],\ldots,\widehat{x}_i,\ldots,x_{q+1})\\
&\quad=(l)_M(\sum_i(-1)^{\gamma_i(x_1,\ldots x_{q+1},\phi)}x_i\phi(x_1,\ldots,\widehat{x}_i,\ldots,x_{q+1}))+\sum_{i=1}^{q+1}(-1)^{1+\bar{l}(\overline{x}_1+\ldots+\overline{x}_{i-1}+\bar{l}\bar{\phi})}\\
&\qquad\sum_j(-1)^{\gamma_j(x_1,\ldots,[l,x_i],\ldots,x_q,\phi)}x_j\phi(x_1,\ldots,[l,x_i],\ldots,\widehat{x}_i,\ldots,x_{q+1})\\
&(\theta(l)\phi)''(x_1,\ldots,x_{q+1})\\
&\quad=\sum_i(-1)^{\gamma_i(x_1,\ldots x_q,\theta(l)\phi)}x_i(\theta(l)\phi)(x_1,\ldots,\widehat{x}_i,\ldots,x_{q+1})\\
&\quad=\sum_i(-1)^{\gamma_i(x_1,\ldots x_q,\theta(l)\phi)}x_i\big((l)_M\phi(x_1,\ldots,\widehat{x}_i,\ldots,x_{q+1})\\
&\qquad+(-1)^{\bar{l}\bar{\phi}}\sum_{s=1}^{i-1}(-1)^{s+\overline{x}_s(\overline{x}_1+\ldots+\overline{x}_{s-1})}\phi([l,x_s],\ldots,\widehat{x}_s,\ldots,\widehat{x}_i,\ldots,x_{q+1})\\
&\qquad+(-1)^{\bar{l}\bar{\phi}}\sum_{s=i+1}^{q+1}(-1)^{s-1+\overline{x}_s(\overline{x}_1+\ldots+\overline{x}_{s-1}+\overline{x}_i)}\phi([l,x_s],\ldots,\widehat{x}_i,\ldots,\widehat{x}_s,\ldots,x_{q+1})\big).
\end{aligned}$$

Similarly, according to the relations of s,i, (6) can be obtained. Therefore,

$$d\theta(l)=\theta(l)d,\qquad l\in\mathfrak{g}.$$

$$\begin{aligned}
&(di(l)\phi)(x_1,\ldots,x_q)+(i(l)d\phi)(x_1,\ldots,x_q)\\
&\quad=\sum_{i<j}(-1)^{\sigma_{i,j}(x_1,\ldots,x_q)}(i(l)\phi)([x_i,x_j],\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_q)\\
&\qquad+\sum_i(-1)^{\gamma_i(x_1,\ldots,x_q,i(l)\phi)}x_i(i(l)\phi)(x_1,\ldots,\widehat{x}_i,\ldots,x_q)+(-1)^{\bar{l}\bar{\phi}}d\phi(l,x_1,\ldots,x_q).
\end{aligned}$$

Computing each term applying the equations (1) and (2), the above equation reduces to

$$\begin{aligned}&(di(l)\phi)(x_1,\dots,x_q)+(i(l)d\phi)(x_1,\dots,x_q)\\&\quad=(-1)^{\bar{l}\bar{\phi}}\sum_i(-1)^{\sigma_{1,i+1}(x,x_1,\dots,x_q)}\phi([l,x_i],\dots,\widehat{x}_i,\dots,x_q)\\&\qquad+(-1)^{\bar{l}\bar{\phi}}(-1)^{\gamma_1(x,x_1,\dots,x_q,\phi)}(l)_M\phi(x_1,\dots,x_q).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\theta(l)\phi(x_1,\dots,x_q)\\&\quad=(l)_M\phi(x_1,\dots,x_q)+\sum_i(-1)^{i+\overline{x}_i(\overline{x}_1+\dots+\overline{x}_{i-1})}(-1)^{\bar{l}\bar{\phi}}\phi([l,x_i],\dots,\widehat{x}_i,\dots,x_q).\end{aligned}$$

Comparing the indexes of each term in the two equations, (4) is immediately obtained.

Lemma 2.2 Let $f(t)$ be a p-polynomial in $k[t]$and l an element in $\mathfrak{g}_{\bar{0}}$. Then

$$\theta(f(l))=f(\theta(l)).$$

Proof We may restrict ourselves to the case $f(t)=t^{p^s}$. Then it suffices to prove that $(\theta(l))^{p^s}\phi=\theta(l^{p^s})\phi,\quad \phi\in C^q(\mathfrak{g},M),q\geqslant 0$.

We consider the map $\theta_i:\mathfrak{g}\longrightarrow \mathrm{End}C^q(\mathfrak{g},M),\quad 0\leqslant i\leqslant q$, given by

$$\begin{gathered}\theta_0(l)=(l)_M,\\(\theta_i(l)\phi)(x_1,\dots,x_q)=(-1)^{\bar{l}\bar{\phi}}(-1)^{i+\overline{x}_i(\overline{x}_1+\dots+\overline{x}_{i-1})}\phi([l,x_i],x_1,\dots,\widehat{x}_i,x_q).\end{gathered}$$

First, θ_i is a representation of $\mathfrak{g}$ on $C^q(\mathfrak{g},M)$.

$$\begin{aligned}&\theta_i(l_1)\theta_i(l_2)\phi(x_1,\dots,x_q)\\&\quad=(-1)^{\bar{l}_2\bar{\phi}}(-1)^{i+\overline{x}_i(\overline{x}_1+\dots+\overline{x}_{i-1})}\theta_i(l_1)\phi([l_2,x_i],x_1,\dots,\widehat{x}_i,x_q)\\&\quad=(-1)^{\bar{l}_2\bar{\phi}-1+\bar{l}_2(\overline{x}_1+\dots+\overline{x}_{i-1})}\theta_i(l_1)\phi(x_1,\dots,[l_2,x_i],\dots,x_q)\\&\quad=(-1)^{(\bar{l}_1+\bar{l}_2)(\bar{\phi}+(\overline{x}_1+\dots+\overline{x}_{i-1}))}\phi(x_1,\dots,[l_1,[l_2,x_i]],\dots,x_q)).\end{aligned}$$

Similarly,

$$\theta_i(l_2)\theta_i(l_1)\phi(x_1,\dots,x_q)=(-1)^{(\bar{l}_1+\bar{l}_2)(\bar{\phi}+(\overline{x}_1+\dots+\overline{x}_{i-1}))}\phi(x_1,\dots,[l_2,[l_1,x_i]],\dots,x_q).$$

Then

$$\begin{aligned}&(\theta_i(l_1)\theta_i(l_2)-(-1)^{\bar{l}_1\bar{l}_2}\theta_i(l_2)\theta_i(l_1))\phi(x_1,\dots,x_q)\\&\quad=(-1)^{(\bar{l}_1+\bar{l}_2)(\bar{\phi}+(\overline{x}_1+\dots+\overline{x}_{i-1}))}[\phi(x_1,\dots,[l_1,[l_2,x_i]],\dots,x_q)\\&\qquad-(-1)^{\bar{l}_1\bar{l}_2}\phi(x_1,\dots,[l_2,[l_1,x_i]],\dots,x_q)]\\&\quad=\theta_i([l_1,l_2])\phi(x_1,\dots,x_q),\end{aligned}$$

that is to say, $\theta_i([l_1,l_2])=[\theta_i(l_1),\theta_i(l_2)]$.

For $0 < i < j < q$, we also can get the following equations:

$$\theta_i(l)\theta_j(l)\phi(x_1,\dots,x_q) = (-1)^{\bar{l}\bar{l}}\theta_j(l)\theta_i(l)\phi(x_1,\dots,x_q).$$

So $\theta_i(l), \theta_j(l)$ are supercommutative.

Clearly, $\theta(l) = \sum_{i=0}^{q}\theta_i(l)$. Then $(\theta(l))^{p^s} = \sum_{i=0}^{q}(\theta_i(l))^{p^s}$, for $l \in \mathfrak{g}_{\bar{0}}$.

Corollary 2.3 Let $z(l)$ be a central element associated with an element $l \in \mathfrak{g}_{\bar{0}}$. Then

$$\theta(z(l)) = z(l)_M.$$

Proof Recall that $z(\theta_i(l)) = \theta_i(z(l)), 0 < i \leqslant q$, by Lemma 2.2,

$$z(\theta(l) = z\big(\sum_{i=0}^{q}\theta_i(l)\big) = z((l)_M) = z(l)_M.$$

Lemma 2.4 Suppose the endomorphism $z(l)_M, l \in \mathfrak{g}_{\bar{0}}$ be invertible and denote the inverse by $\tilde{l}$. Then

$$\tilde{l}d = d\tilde{l}.$$

Proof Let $\phi \in C^q(\mathfrak{g}, M)$. It suffices to verify $(\tilde{l}\phi)' = \tilde{l}(\phi)'$, $(\tilde{l}\phi)'' = \tilde{l}(\phi)''$.

$$\begin{aligned}&\tilde{l}(\phi)'(x_1,\dots,x_{q+1})\\&\quad= \tilde{l}\big(\sum_{i<j}(-1)^{\sigma_{i,j}(x_1,\dots,x_{q+1})}\phi([x_i,x_j],\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_{q+1})\big)\\&\quad= (\tilde{l}\phi)',\end{aligned}$$

$l \in \mathfrak{g}_{\bar{0}}$, for any $x \in \mathfrak{g}$, $z(l)x = xz(l)$, then $(z(l))_M(x)_M = (x)_M(z(l))_M$, therefore, $\tilde{l}(x)_M = (x)_M\tilde{l}$.

$$\begin{aligned}&(\tilde{l}\phi)''(x_1,\dots,x_{q+1})\\&\quad= \sum_i(-1)^{\gamma_i(x_1,\dots,x_{q+1},\tilde{l}\phi)}(x_i)_M(\tilde{l}\phi)(x_1,\dots,\widehat{x}_i,\dots,x_{q+1})\\&\quad= \sum_i(-1)^{i+1+\overline{x}_i(\overline{x}_1+\overline{x}_{i-1}+\overline{\tilde{l}\phi})}\tilde{l}(x_i)_M\phi(x_1,\dots,\widehat{x}_i,\dots,x_{q+1})\\&\quad= \tilde{l}\phi''(x_1,\dots,x_{q+1}).\end{aligned}$$

With the above Lemmas, we complete the proof of Theorem 1.2 now.

Proof of Theorem 1.2

Multiplying both sides of (4) by an element of the form $\theta(l)^q$ on the right and considering (3) gives that

$$di(l)\theta(l)^q + i(l)\theta(l)^q d = \theta(l)^{q+1}.$$

Passing to linear combinations of such relations , we derive that for any p-polynomial $f(t) \in k[t]$ there exists an endomorphism of degree -1 such that

$$di_{f'}(l) + i_{f'}(l)d = f(\theta(l)),$$

where $f^{'}(t) = t^{-1}f(t)$. In particular, for $z(l)$, making use of corollary 2.3, we find that

$$d\tilde{li}_{z^{'}(l)} + \tilde{li}_{z^{'}(l)}d = \mathrm{id}_M.$$

Thus the theorem is proved.

Corollary 2.5 Let M be an irreducible g-module. The cohomology is nonzero only if all the endomorphisms of the form $(z(l))_M$ are zero.

Proof By Schur Lemma for Lie superalgebra, $\mathrm{End}_{\mathfrak{g}}(M)$ is spanned by id_M if M is absolutely irreducible, or by $\{\mathrm{id}_M, \sigma\}$ if $\dim M_0 = \dim M_1$, where σ is a non-singular operator in $\mathfrak{g}$ permuting M_0 and M_1. That is $\bar{\sigma} = 1$. Since $z(l) \in \mathfrak{g}_{\bar{0}}$, the endomorphism $(z(l))_M$ is invertible if and only if it is nonzero.

As a special case, Corollary 1.3 follows from the implication of the above corollary.

Example 2.6 Let $\mathfrak{g} = osp(1|2)$. It contains three bosonic generators E^+, E^-, H which form the Lie algebra $sl(2)$ and two ferminic generators F^+, F^-.

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$F^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The non-vanishing commutation relation in the generators read as

$$\begin{array}{lll} [H, E^{\pm}] = \pm 2E^{\pm} & [E^+, E^-] = H & [H, F^{\pm}] = \pm F^{\pm}, \\ [F^+, F^-] = H & [E^{\pm}, F^{\mp}] = -F^{\pm} & [F^{\pm}, F^{\pm}] = \pm 2E^{\pm}. \end{array}$$

It has p-restricted structure. Furthermore, $(E^{\pm})^{[p]} = 0$, and $H^{[p]} = H$. According to Corollary 2.5, the cohomology $H^*(\mathfrak{g}, M)$ of the Lie superalgebra $osp(1|2)$ with coefficients in an irreducible module M is nontrivial only in the case where

$$((E^+)^p)_M = 0, \quad ((E^-)^p)_M = 0, \quad (H^p)_M = (H)_M.$$

3 Cohomology of $W(n)$ with trivial coefficient

In the following, we consider the cohomology of Cartan type Lie superalgebra $W(n)$ with trivial coefficient in k.

We recall that $W(n) = \mathrm{Der}\ \Lambda(n)$ is the derivation superalgebra of the Grassmann superalgebra $\Lambda(n)$ (cf. [6]). Any derivation $D \in W(n)$ is written as

$$D = \sum_i P_i \partial_i,$$

where $P_i \in \Lambda(n)$, ∂_i is the derivation defined by $\partial_i(\xi_j) = \delta_{ij}$.

Letting $\deg \xi_i = \bar{1}, i = 1, \cdots, n$, we obtain a consistent $\mathbb{Z}$-grading of $\Lambda(n)$, which induces the grading of $W(n) = \bigoplus\limits_{l=-1}^{n-1} W(n)_l$, where

$$W(n)_l = \{\Sigma P_i \partial_i \mid \deg P_i = l + 1\}.$$

In particular, $[W(n)_{l_i}, W(n)_{l_j}] \subseteq W(n)_{l_i+l_j}$.

$W(n)$ is a restriced Lie superalgebra (cf. [7]). Furthermore, $(\xi_i\partial_i)^{[p]} = \xi_i\partial_i$, $(\xi_i\partial_j)^{[p]} = 0, i \neq j$, and $D^{[p]} = 0, D \in W(n)_l, l \neq 0$.

Let $g = W(n)$. Denote by the complex $C^*(g) := C^*(g,k)$, where for each $q \geqslant 0$, $C^q(\mathfrak{g}) := C^q(\mathfrak{g}, k) = \bigoplus\limits_{q_0+q_1=q} \mathrm{Hom}(\Lambda^{q_0}(\mathfrak{g}_{\bar{0}}) \otimes S^{q_1}(\mathfrak{g}_{\bar{1}}), k)$. Then the coboundary operator defined in (1) is simplified, i.e.

$$d\phi(g_1, g_2, ...g_{q+1}) = \sum_{i<j} (-1)^{\sigma_{i,j}(g_1,\ldots,g_q)} \phi([g_i, g_j], g_1, \ldots, \widehat{g}_i, \ldots, \widehat{g}_j, \ldots, g_{q+1}).$$

Define $C^q_{[l]}(\mathfrak{g}) = \{\phi \in C^q(\mathfrak{g}) \mid \phi(g_1, g_2, ...g_q) = 0, g_i \in \mathfrak{g}_{l_i}, \sum\limits_{i=1}^{q} l_i = l\}$. For $\phi \in C^q_{[l]}(\mathfrak{g})$, we have $d\phi \in C^{q+1}_{[l]}(g)$.

Set $C^*_{[\bar{l}]}(g) = \bigoplus\limits_{s\geqslant 0} C^*_{[sp+l]}(g)$, $0 \leqslant l < p$, which is a complex. So we have the cohomology: $H^*_{[\bar{l}]}(g)$. Here $[\bar{l}]$ is regarded to be the residue class of l in $\mathbb{Z}/p\mathbb{Z}$.

Lemma 3.1 If $q > 0, l(\mathrm{mod}\, p) \not\equiv 0$, then $H^q_{[l]}(\mathfrak{g}) = 0$.

Proof Let $\mathfrak{g}_0 = \sum\limits_{i=1}^{n} \xi_i\partial_i$, for any $\xi_{i_1}\xi_{i_2}...\xi_{i_s}\partial_j \in g_{s-1}$. It has

$$[g_0, \xi_{i_1}\xi_{i_2}...\xi_{i_s}\partial_j] = (s-1)\xi_{i_1}\xi_{i_2}...\xi_{i_s}\partial_j.$$

Define the map:

$$\begin{aligned} i^{(q)}(g_0) : C^q_{[l]}(\mathfrak{g}) &\longrightarrow C^{q-1}_{[l]}(\mathfrak{g}), \\ \phi &\longmapsto i^{(q)}(g_0)(\phi), \end{aligned}$$

where $i^{(q)}(g_0)(\phi)(g_1, g_2, ..., g_{q-1}) = \phi(g_0, g_1, g_2, ..., g_{q-1})$. Then for a homogenous element $\phi \in C^q_{[l]}(\mathfrak{g}), g_1 \in \mathfrak{g}_{l_1}, g_2 \in \mathfrak{g}_{l_2}, ..., g_q \in \mathfrak{g}_{l_q}, \sum\limits_{i=1}^{q} l_i \equiv l(\mathrm{mod}\, p)$, we have

$$\begin{aligned} &i^{(q+1)}(g_0)(d(\phi))(g_1, g_2, ..., g_q) \\ &\quad = d(\phi)(g_0, g_1, g_2, ..., g_q) \\ &\quad = \sum_{s=1}^{q} (-1)^{\sigma_{1,s+1}(g_0,g_1,g_2,\ldots,g_q)} \phi([g_0, g_s], g_1, ..., \widehat{g_s}, ..., g_q) \\ &\qquad + \sum_{1\leqslant s\leqslant q} (-1)^{\sigma_{s+1,t+1}(g_0,g_1,g_2,\ldots,g_q)} \phi([g_s, g_t], g_0, ..., \widehat{g_s}, ..., \widehat{g_t} ..., g_q) \\ &\quad = \sum_{s=1}^{q} l_s(-1)\phi(g_1, ..., g_s, ..., g_q) \\ &\qquad + \sum_{1\leqslant s\leqslant q} (-1)^{\sigma_{s,t}(g_1,g_2,\ldots,g_q)}(-1)(i^q(g_0)(\phi))([g_s, g_t], ..., \widehat{g_s}, ..., \widehat{g_t}, ..., g_q) \\ &\quad = (-1)(l\phi + d^{(q-1)} i^q(g_0)(\phi))(g_1, ..., g_s, ..., g_q). \end{aligned}$$

Therefore, $\frac{1}{-\bar{l}}(i^{(q+1)}(g_0)(d^{(q)})+d^{(q-1)}i^q(g_0))=\text{id}$. Thus, we construct a contracting homotopy in the case when $l(\text{mod}\, p)\not\equiv 0, q>0$.

Example 3.2 Let $n=2$, $g=W(2)$.

We have $\mathfrak{g}=\bigoplus_{i=-1}^{1}\mathfrak{g}_i$, where $\mathfrak{g}_0=k\text{-span}\{\xi_i\partial_j, i,j=1,2\}$, $\mathfrak{g}_{-1}=k\text{-span}\{\partial_i, i=1,2\}$, $\mathfrak{g}_1=k\text{-span}\{\xi_2\xi_1\partial_i, i=1,2\}$. And it's clear that $\mathfrak{g}_{[\bar{0}]}=\mathfrak{g}_0$ and $\mathfrak{g}_{\bar{1}}=\mathfrak{g}_{-1}+\mathfrak{g}_1$.

When $q<p$,

$$C^q(\mathfrak{g})=\bigoplus_{l=-q}^{q}C^q_{[l]}(\mathfrak{g}).$$

The cochain complex is the following:

$$0\to C^0(\mathfrak{g})\xrightarrow{d^{(0)}}C^1(\mathfrak{g})\xrightarrow{d^{(1)}}C^2(\mathfrak{g})\xrightarrow{d^{(2)}}C^3(\mathfrak{g})\to\cdots$$

Since $d^{(0)}(k)(x)=x\cdot k=0$, for any $x\in\mathfrak{g}$, so $\text{Ker}d^{(0)}=k$, then $H^0(g)\cong k$.

To compute the $H^q(\mathfrak{g})$, for $0<q<p$. We only need to consider the subcomplex $C^*_{[0]}(\mathfrak{g})$ by Lemma 3.1.

$$C^q_{[0]}(\mathfrak{g})=\bigoplus_{\substack{q_0+q_1+q_2=q\\ q_1=q_2}}\text{Hom}(\wedge^{q_0}(\mathfrak{g}_0)\otimes S^{q_1}(\mathfrak{g}_{-1})\otimes S^{q_1}(\mathfrak{g}_1),k). \tag{7}$$

In particular, if $q=2m$,

$$C^q_{[0]}(\mathfrak{g})=\bigoplus_{i=0}^{2}\text{Hom}(\wedge^{2i}(\mathfrak{g}_0)\otimes S^{m-i}(\mathfrak{g}_{-1})\otimes S^{m-i}(\mathfrak{g}_1),k).$$

If $q=2m+1$,

$$C^q_{[0]}(\mathfrak{g})=\bigoplus_{i=0}^{1}\text{Hom}(\wedge^{2i+1})(\mathfrak{g}_0\otimes S^{m-i}(\mathfrak{g}_{-1})\otimes S^{m-i}(\mathfrak{g}_1),k).$$

Now let us begin to compute the dimensions of lower components of complex and cohomology. The main computing result is that

$$\text{dim}H^0(\mathfrak{g})=1=\text{dim}H^3(\mathfrak{g}),\quad \text{dim}H^1(\mathfrak{g})=0=\text{dim}H^2(\mathfrak{g}).$$

First, by (7) one has

$$\text{dim}C^1_{[0]}(\mathfrak{g})=4, \text{dim}C^2_{[0]}(\mathfrak{g})=10, \text{dim}C^3_{[0]}(\mathfrak{g})=20, \text{dim}C^4_{[0]}(\mathfrak{g})=34.$$

When $q=1$,

$$\text{Ker}\ d^{(1)}_{[0]}=\phi\in C^1_{[0]}(g)\mid \phi([g_1,g_2])=0=0,$$

since $[\mathfrak{g}_{-1},\mathfrak{g}_1]=\mathfrak{g}_0$. Therefore $H^1(\mathfrak{g})=0$, and dim Im $d^{(1)}_{[0]}=\text{dim}C^1_{[0]}(\mathfrak{g})=4$.

When $q=2$, choose the bases of $\bigwedge^2(\mathfrak{g}_0)$ and $S(\mathfrak{g}_{-1})\otimes S(\mathfrak{g}_1)$ respectively,

$$X_1=\xi_1\partial_1\wedge\xi_1\partial_2,\ X_2=\xi_1\partial_1\wedge\xi_2\partial_1,\ X_3=\xi_1\partial_1\wedge\xi_2\partial_2,\ X_4=\xi_1\partial_2\wedge\xi_2\partial_1,$$
$$X_5=\xi_1\partial_2\wedge\xi_2\partial_2,\ X_6=\xi_2\partial_1\wedge\xi_2\partial_2,\ X_7=\partial_1\otimes\xi_2\xi_1\partial_1,$$
$$X_8=\partial_1\otimes\xi_2\xi_1\partial_2,\ X_9=\partial_2\otimes\xi_2\xi_1\partial_1,\ X_{10}=\partial_2\otimes\xi_2\xi_1\partial_2.$$

For $\phi \in \text{Ker}\ d_{[0]}^{(2)}$, i.e.

$$\begin{aligned} d_{[0]}^{(2)}\phi(g_1,g_2,g_3) &= \sum_{i,j,k}(-1)^{\sigma_{i,j}(g_1,g_2,g_3)}\phi([g_i,g_j],g_k) \\ &= (-1)^3[\phi([g_1,g_2],g_3)-(-1)^{\bar{g_3}\bar{g_2}}\phi([g_1,g_3],g_2)+(-1)^{\bar{g_2}\bar{g_1}}\phi([g_2,g_3],g_1)] \\ &= 0. \end{aligned}$$

Assume $\phi = \sum\limits_{i=1}^{10} a_i\phi_i, a_i \in k$. where $\{\phi_i\}_{i=1}^{10}$ is the dual basis of $C_{[0]}^2(\mathfrak{g})$ such that $\phi_i(X_j) = \delta_{ij}, i,j = 1,2,...,10$. It can be computed that

$\phi = a_1\phi_1 + a_2\phi_2 + (a_8+a_9)\phi_4 + a_1\phi_5 + a_2\phi_6 + a_2\phi_7 + a_8\phi_8 + a_9\phi_9 + a_1\phi_{10}$.

So dim Ker $d_{[0]}^{(2)} = 4 =$ dim Im $d_{[0]}^{(1)}$, and dim $H^2(g) = 0$.

When $q = 3$, let $\phi \in C_{[0]}^3(\mathfrak{g}), \phi = \sum\limits_{i=1}^{20} b_i\phi_i$, $\{\phi_i, i = 1,2,...,20\}$ is the basis of $C_{[0]}^3(\mathfrak{g})$ $b_i \in k$.

For $\phi \in \text{Ker}\ d_{[0]}^{(3)}$, we have

$$d^{(3)}\phi(g_1,g_2,g_3,g_4) = \sum_{1\leqslant i<j\leqslant 4}(-1)^{\sigma_{i,j}(g_1,g_2,g_3,g_4)}\phi([g_i,g_j],g_1,...,g_4) = 0.$$

By similar computation and discussion, we have

$$\dim\ \text{Ker}\ d_{[0]}^{(3)} = 7, \text{ and } \dim H^3(g) = 1,$$

since dim Im $d_{[0]}^{(2)} = \dim C_{[0]}^2(g) - \dim \text{Ker}\ d_{[0]}^{(2)} = 6$.

Acknowledgements The author would like to thank Professor Bin Shu for his helpful comments and suggestions.

[References]

[1] DZHUMADIL'DAEV A S. On the cohomology of modular Lie superalgebras[J]. Math USSR Sbornik, 1984, 47: 127-143.

[2] SHU B, WANG W Q. Modular representations of the ortho-symplectic supergroups[J]. Proceedings of London Math Soc, 2008, 96: 251-271.

[3] JACOBSON N. Lie Algebra[M]. New York: Interscience, 1962.

[4] BOE B, KUJAWA J R, NAKANO D K. Cohomology and support varieties for Lie superalgebras[J]. Proceedings of LMS, 2009, 98(1):19-44.

[5] FUKS D B. Cohomology for Infinite-Dimensional Lie Algebras[M]. New York: Contemporary Soviet mathematics, Consultants Bureau, 1986.

[6] KAC V G. Lie superalgebras [J]. Advances in Math, 1977, 26, 8-96.

[7] ZHANG Y Z, LIU W D. Modular Lie Superalgebras[M]. Beijing: Science Press, 2004.