Article ID: 1000-5641(2009)05-0107-11

Pricing forward starting call option in a jump diffusion model

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Abstract: By the way of change of measure, a closed solution of pricing formula of European forward starting call option was given in a double exponential jump diffusion model. Moreover, a problem of pricing forward call option when the log jump size has a general distribution was also considered.

Key words: jump diffusion model; change of measure; Girsanov's theorem

CLC number: F224.7; F224.9 **Document code:** A

跳扩散模型中远期生效看涨期权的定价

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摘要: 通过测度变换的方法给出了双指数跳扩散模型中远期生效看涨期权的价格公式. 此外, 还考虑了当股票跳的大小的对数满足一般分布时远期生效看涨期权的定价问题. 所得结果可推广到随机利率和随机波动的情况.

关键词: 跳扩散模型; 测度变换; Girsanov 定理

0 Introduction

Despite the success of the Black-Scholes model based on Brownian motion and normal distribution, two empirical phenomena have received much attention recently. Firstly, stock returns are leptokurtic: Relative to the normal distribution, there are too many observations around the mean and too many extreme observations in the tails of the distribution. Secondly, the volatility of stock returns changes randomly over time, and on occasions there are large, rapid price movements resembling jumps. Moveover, numerous empirical studies show that the interest rate risk should not be ignored. The effect of jumps have been examined by Merton[1], Naik and Lee[2], Ahn et al[3]. The effects of stochastic volatility have been examined by Hull

收稿日期: 2009-02

基金项目: 国家自然科学基金(10771070);教育部高等学校博士学科总基金(20060269016)

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and White[4], Heston[5], Scott[6], Duffie[7], Kurse and Nögel[8]. As to the effects of interest rates, see, for example, Biger and Hull[9], Cox etc[10]. Even though forward starting call option seems to be quite simple exotic derivatives, depending on the underlying model, their pricing can be demanding. For example, in Kurse and Nögel[8], it gives a pricing formula of forward starting call option in Heston's model on stochastic volatility. Motivatied by their papers, we consider the problem of pricing European forward starting call option in a jump diffusion model.

This paper is structured as follows. In Section 1, we give a brief introduce about forward starting call option and assumptions. In Section 2, we contain the derivation of a closed form solution for forward starting call option in a double exponential jump-diffusion model. In Section 3, we consider a problem of pricing forward call option when the log jump size has a general distribution. Brief conclusion is in Section 4.

1 Forward starting call option

We consider a financial market with a stock and a bank account. $S(t)$ denote the price of stock, $B(t)$ is the bank account satisfying $B(t) = \mathrm{e}^{\int_0^t r_s \mathrm{d}s}$. A forward starting call option is an exotic option whose strike price is not fully determined until an intermediate date t^* before maturity T, called the determination time of the strike or starting date of the option. The payoff structure of a forward starting call option is as follow.

$$\psi(T) = (S(T) - KS(t^*))^+ , \tag{1.1}$$

where K is the percentage or proportion strike.

Since the risk of jump process exists in market, the market is not complete, there are infinitely many equivalent probability measure. However, in this paper, we make use of the assumption in Merton[1]. According to Merton, the systematic risk can be hedged but the unsystematic risk, being unique to the particular risky asset, cannot be hedged. Merton identified the Brownian motion component as the component the systematic risk. The compensated jump component contributes to the unsystematic risk. Under a suitable risk-neutral measure, the parameters of the jump component should not change. Denote $(\Omega, \mathcal{F}, Q)$ be a probability space with filtration $(\mathcal{F}_t)_{t\geqslant 0}$ of market information and the equivalent martingale measure Q such that the asset price is equal to

$$\frac{\mathrm{d}S(t)}{S(t-)} = r_t \mathrm{d}t + \sigma_t \mathrm{d}W_t - \lambda\beta \mathrm{d}t + \mathrm{d}\sum_{i=1}^{N(t)}(Y_i - 1). \tag{1.2}$$

By the Itó-Doeblin formula and equation (1.2), we can have that

$$S(t) = \exp\left\{\int_0^t r_s \mathrm{d}s - \frac{1}{2}\int_0^t \sigma_s^2 \mathrm{d}s - \lambda\beta t + \int_0^t \sigma_s \mathrm{d}W_s\right\}\prod_{i=1}^{N(t)} Y_i, \tag{1.3}$$

where $\beta = \mathbb{E}_Q[Y_i - 1]$, r_t, σ_t are time-dependent and deterministic, which represent free interest rate and the volatility of the stock return respectively, W_t is a standard Brownian motion under risk neutral probability Q, Y_i is a sequence of independent identically distributed (i.i.d) nonnegative random variables, nonnegative condition ensures that $S(t)$ is nonnegative. $f(y)$ represents the probability density function of random variables $\ln Y_i$. $N(t)$ is a poisson process with intensity λ. In the model, all sources of randomness, $N(t)$, $W(t)$, and Y_i, are assumed to be independent.

In terms of the fundamental theorem of asset pricing, we have that the arbitrage free price of the forward starting call option at time t as follow

$$C(t,T)=\mathbb{E}_Q\left[\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\left(S(T)-KS(t^*)\right)^+ \mid \mathcal{F}_t\right] \tag{1.4}$$

2 Pricing forward starting call option under a double exponential jump diffusion model

Merton[1] assumes that the jump component of the assets return characterizes non-systematic risk, the log jump size $\ln Y_i$ has a standard normal distribution. Kou[11] provide another example of the jump-diffusion model, in which $\ln Y_i$ has a double exponential distribution. In this section, we consider a problem of pricing forward call option under a double exponential jump diffusion model. We assume interest rates and the volatility of the stock return in (1.2) are constant r and σ for simplicity. Y_i is a sequence of independent identically distributed nonnegative random such that $\ln Y$ has an asymmetric double exponential distribution with the density

$$f(y)=p\eta_1\mathrm{e}^{-\eta_1 y}I_{y\geqslant 0}+q\eta_2\mathrm{e}^{\eta_2 y}I_{y<0},\quad \eta_1>1,\quad \eta_2>0, \tag{2.1}$$

where $p,q\geqslant 0, p+q=1$, represent the probabilities of upward and downward jumps, and the condition $\eta_1>1$ is imposed to ensure that the underlying asset price has finite expectation. In other words,

$$\ln Y\overset{d}{=}\left\{\begin{array}{l}\xi^+,\text{with probability p}\\ \xi^-,\text{with probability q}\end{array}\right\},$$

where ξ^+ and ξ^- are exponential random variables with means $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$ respectively, and the notation $\overset{d}{=}$ means equal in distribution. Hence, we have that $\beta=\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1$.

Below, is Proposition B.1 of Kou[11], which will be used to prove Theorem 2.1.

Proposition 2.1 For every $n\geqslant 1$, we have the following decomposition

$$\sum_{i=1}^n \ln Y_i\overset{d}{=}\left\{\begin{array}{l}\sum\limits_{i=1}^m \xi_i^+,\text{with probability}P_{n,m},\quad m=1,2,...n\\ -\sum\limits_{i=1}^m \xi_i^-,\text{with probability}Q_{n,m},\quad m=1,2,...n\end{array}\right\},$$

where $P_{n,m}$ and $Q_{n,m}$ are given by

$$P_{n,m}=\sum_{i=m}^{n-1}\left(\begin{array}{c}n-m-1\\ i-m\end{array}\right)\left(\begin{array}{c}n\\ i\end{array}\right)\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-m}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i}p^iq^{n-i}, 1\leqslant m\leqslant n-1,$$

$$Q_{n,m}=\sum_{i=m}^{n-1}\left(\begin{array}{c}n-m-1\\ i-m\end{array}\right)\left(\begin{array}{c}n\\ i\end{array}\right)\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i}\left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-m}p^{n-i}q^i, 1\leqslant m\leqslant n-1,$$

$P_{n,n} = p^n$, $Q_{n,n} = q^n$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is defined to be one. Here ξ_i^+ and ξ_i^- are i,i,d. exponential random variables with rates η_1 and η_2, respectively.

In this section, for pricing forward starting call option, we have to study the distribution of the sum of the double exponential random variables and normal random variables. As in Kou[8], we know this distribution can be obtained in closed form in terms of the Hh function. We restate the definition of Hh function as follows.

Definition 2.1 For each $m \geqslant 0$, the Hh function is a nonincreasing function which defined by as follows

$$\begin{aligned} Hh_m(x) &= \int_x^\infty Hh_{m-1}(y)\mathrm{d}y \\ &= \frac{1}{m!}\int_x^\infty (t-x)^n \mathrm{e}^{-\frac{t^2}{2}}\mathrm{d}t, \end{aligned}$$

$Hh_{-1}(x) = \mathrm{e}^{-\frac{x^2}{2}} = \sqrt{2\pi}\varphi(x)$, $Hh_0(x) = \sqrt{2\pi}\int_{-\infty}^{-x}\varphi(y)\mathrm{d}y$.

Moreover, a three-term recursion is also available for the Hh function:

$$mHh_m(x) = Hh_{m-2}(x) - xHh_{m-1}(x), \quad m \geqslant 1.$$

Hence, we can calculate all $Hh_m(x)$ using the normal density function and normal distribution function. For the simplify of notation, we denote

$$I_m(c;\alpha,\beta,\delta) = \int_c^\infty \mathrm{e}^{\alpha x} Hh_m(\beta x - \delta)\mathrm{d}x, \quad m \geqslant 0.$$

Theorem 2.1 Let $0 \leqslant t < t^* < T$, The price of the forward starting call option under double exponential jump-diffusion model at time t is equal to

$$\begin{aligned} C(t,T) &= \mathrm{e}^{-\tilde{\lambda}(T-t^*)}S(t)\mathbb{N}(d_1) + \Upsilon(\delta_1,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2,T-t^*) \\ &- K\mathrm{e}^{-(\lambda+r)(T-t^*)}S(t)\mathbb{N}(d_2) - K\mathrm{e}^{-r(T-t^*)}\Upsilon(\delta_2,\sigma,\lambda,p,\eta_1,\eta_2,T-t^*), \end{aligned}$$

Where $\mathbb{N}(\cdot)$ denotes the cumulative distribution of normal random variable, and

$$\begin{aligned} &\Upsilon(x,\sigma,\lambda,p,\eta_1,\eta_2,T-t^*) \\ = \ & S(t)\sum_{n=1}^\infty \pi_n \sum_{m=1}^n P_{n,m}\frac{(\sigma\sqrt{T-t^*}\eta_1)^m e^{\frac{(\sigma\eta_1)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}} I_{m-1}\left(x;-\eta_1,-\frac{1}{\sigma\sqrt{T-t^*}},-\sigma\sqrt{T-t^*}\eta_1\right) \\ + \ & S(t)\sum_{n=1}^\infty \pi_n \sum_{m=1}^n Q_{n,m}\frac{(\sigma\sqrt{T-t^*}\eta_2)^m e^{\frac{(\sigma\eta_2)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}} I_{m-1}\left(x;\eta_2,\frac{1}{\sigma\sqrt{T-t^*}},-\sigma\sqrt{T-t^*}\eta_2\right), \end{aligned}$$

$$\delta_1 = \ln K - r(T-t^*) - \tfrac{1}{2}\sigma^2(T-t^*) + \lambda\beta(T-t^*), \quad \delta_2 = \delta_1 + \sigma^2(T-t^*),$$

$$d_1 = \frac{(r+\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*) - \ln K}{\sigma\sqrt{T-t^*}}, \quad d_2 = d_1 - \sigma\sqrt{T-t^*},$$

$$\pi_n = \mathrm{e}^{-\lambda(T-t^*)}\frac{(\lambda(T-t^*))^n}{n!}, \quad \tilde{p} = \frac{p\eta_1}{(1+\beta)(\eta_1-1)}, \quad \tilde{\lambda} = \lambda(\beta+1),$$

$$\tilde{\eta}_1=\eta_1-1,\quad \tilde{\eta}_2=\eta_2+1,\quad \beta=\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1.$$

Before prove Theorem 2.1, we prepare the following two lemmas. Denote

$$Z(T)=\frac{\mathrm{d}Q_S}{\mathrm{d}Q}\Big|_{\mathcal{F}_T}=\frac{S(T)}{B(T)S(0)}. \tag{2.2}$$

It obvious that the measure Q_S is a probability measure.

Lemma 2.1 Under the probability measure Q_S, then $\sum\limits_{j=1}^{N(t)}\ln Y_j$ is a compound Poisson process with intensity $\tilde{\lambda}=\lambda(\beta+1)$. Furthermore, the $\ln Y_j$ in compound Poisson process $\sum\limits_{j=1}^{N(t)}\ln Y_j$ are independent and identically distributed with density function $\tilde{f}(y)=\dfrac{e^y f(y)}{\beta+1}$.

Proof We need to show that, under the probability measure Q_S has the characteristic function corresponding to a compound Poisson process with intensity $\tilde{\lambda}$ and density $\tilde{f}(y)$. Since under the probability measure Q the characteristic function for the compound Poisson process $\sum\limits_{j=1}^{N(t)}\ln Y_j$ is

$$\mathbb{E}_Q[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j)]=\mathrm{e}^{\lambda t(\varphi_{\ln Y}(u)-1)},$$

Where $\varphi_{\ln Y}(u)=\mathbb{E}_Q\left[\mathrm{e}^{iu\ln Y_j}\right]=\int_{-\infty}^{\infty}\mathrm{e}^{iuy}f(y)\mathrm{d}y$.

Thus, we must show that

$$\mathbb{E}_{Q_S}[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j)]=\mathrm{e}^{\tilde{\lambda}t(\tilde{\varphi}_{\ln Y}(u)-1)},$$

Where $\tilde{\varphi}_{\ln Y}(u)=\int_{-\infty}^{\infty}\mathrm{e}^{iuy}\tilde{f}(y)\mathrm{d}y=\int_{-\infty}^{\infty}\mathrm{e}^{iuy}\frac{\mathrm{e}^y f(y)}{\beta+1}\mathrm{d}y$.

Note that (2.2), we have

$$\mathbb{E}_{Q_S}\left[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j)\right] = \mathbb{E}_Q\left[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j)Z(T)\right],$$

Since $N(t)$, $W(t)$, and Y_j, are assumed to be independent, imply

$$\begin{aligned}\mathbb{E}_{Q_S}[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j)] &= \mathbb{E}_Q\left[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j-\lambda\beta t)\prod_{j=1}^{N(t)}Y_j\right]\\ &\times \mathbb{E}_Q\left[\mathrm{e}^{-\lambda\beta(T-t)}\prod_{j=N(t)+1}^{N(T)}Y_j\right].\end{aligned}$$

Because when $N(t)=0$, it means the jump is not occurrence, $\prod_{j=1}^{N(t)} Y_j = 1$, thus we have that

$$\begin{aligned}
&\mathbb{E}_Q\left[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j - \lambda\beta t)\prod_{j=1}^{N(t)} Y_j\right]\\
= \;& Q(N(t)=0)e^{-\lambda\beta t} + \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda t-\lambda\beta t}\frac{(\lambda t)^k}{k!}\left(\mathbb{E}_Q\left[\mathrm{e}^{iu\ln Y_j}Y_j\right]\right)^k\\
= \;& \mathrm{e}^{-\lambda(\beta+1)t} + \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda(\beta+1)t}\frac{(\lambda t)^k}{k!}\left(\mathbb{E}_Q\left[\mathrm{e}^{(iu+1)\ln Y_j}\right]\right)^k\\
= \;& \mathrm{e}^{-\lambda(\beta+1)t}\left(\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}\left(\mathbb{E}_Q\left[\mathrm{e}^{(iu+1)\ln Y_j}\right]\right)^k\right)\\
= \;& \mathrm{e}^{-\lambda(\beta+1)t}\left(\sum_{k=0}^{\infty}\frac{(\lambda t(\beta+1))^k}{k!}\left(\mathbb{E}_Q\left[\frac{\mathrm{e}^{(iu+1)\ln Y_j}}{\beta+1}\right]\right)^k\right)\\
= \;& \mathrm{e}^{-\lambda(\beta+1)t}e^{\lambda(\beta+1)t\tilde{\varphi}_{\ln Y}(y)}\\
= \;& \mathrm{e}^{\lambda(\beta+1)t(\tilde{\varphi}_{\ln Y}(y)-1)}.
\end{aligned}$$

By the same way, we can obtain that $\mathbb{E}_Q\left[\mathrm{e}^{-\lambda\beta(T-t)}\prod_{j=N(t)+1}^{N(T)} Y_j\right] = 1$, Hence, we have

$$\mathbb{E}_{Q_S}[\exp(iu\sum_{j=1}^{N(t)}\ln Y_j)] = \mathrm{e}^{\tilde{\lambda}t(\tilde{\varphi}_Y(u)-1)},$$

Where $\tilde{\lambda}=\lambda(\beta+1)$, $\quad \tilde{f}(y)=\frac{\mathrm{e}^y f(y)}{\beta+1}$.

Hence, we complete the proof.

Lemma 2.2 Under the probability measure Q_S, the process

$$\tilde{W}_t = W_t - \int_0^t \sigma_s \mathrm{d}s \tag{2.3}$$

is a standard Brownian motion, the processes $\tilde{W}_t$ and $\sum_{j=1}^{N(t)}\ln Y_j$ are independent.

Proof See Theorem 11.6.9 in Shreve[13], we can prove in the same way.

Proof of Theorem 2.1 In terms of the fundamental theorem of asset pricing, we have that the arbitrage free price of the forward starting call option at time t as follow.

$$\begin{aligned} C(t,T) &= \mathbb{E}_Q\left[\mathrm{e}^{-r(T-t)}\left(S(T)-KS(t^*)\right)^+\middle|\mathbb{F}_t\right] \\ &= S(t)\mathbb{E}_{Q_S}\left[\frac{(S(T)-KS(t^*))^+}{S(T)}\middle|\mathbb{F}_t\right] \\ &= S(t)\mathbb{E}_{Q_S}\left[\mathbb{E}_{Q_S}\left[\frac{(S(T)-KS(t^*))^+}{S(T)}\middle|\mathbb{F}_{t^*}\right]\middle|\mathbb{F}_t\right] \\ &= S(t)\mathbb{E}_{Q_S}\left[\mathbb{E}_{Q_S}\left[I_{[\frac{S(T)}{S(t^*)}\geqslant K]}\middle|\mathbb{F}_{t^*}\right]\middle|\mathbb{F}_t\right] \\ &- KS(t)\mathrm{e}^{-r(T-t^*)}\mathbb{E}_{Q_S}\left[\mathbb{E}_Q\left[I_{[\frac{S(T)}{S(t^*)}\geqslant K]}\middle|\mathbb{F}_{t^*}\right]\middle|\mathbb{F}_t\right], \end{aligned} \tag{2.4}$$

Note (1.3), $\frac{S(T)}{S(t^*)}$ is independent $\mathbb{F}_{t^*}$. Hence, the Eqn. (2.4) can be rewritten as follow

$$\begin{aligned} & S(t)Q_S\left((r+\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*)+\sigma(\tilde{W}(T)-\tilde{W}(t^*))+\sum_{j=N(t^*)+1}^{N(T)}\ln Y_j\geqslant\ \ln K\right) \\ - \ & KS(t)e^{-r(T-t^*)}Q\left((r-\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*)+\sigma(W(T)-W(t^*))+\sum_{j=N(t^*)+1}^{N(T)}\ln Y_j\geqslant\ \ln K\right) \\ = \ & \Pi_1-\Pi_2. \end{aligned} \tag{2.5}$$

By Lemma 2.1 and (2.3), Π_1 can be rewritten as follows.

$$S(t)\sum_{n=0}^{\infty}\tilde{\pi}_nQ_S\left((r+\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*)+\sigma(\tilde{W}(T)-\tilde{W}(t^*))+\sum_{j=1}^{n}\ln Y_j\geqslant\ln K\right), \tag{2.6}$$

where $\tilde{\pi}_n=Q_S(N(T)-N(t^*)=n)=\mathrm{e}^{-\tilde{\lambda}(T-t^*)}\frac{(\tilde{\lambda}(T-t^*))^n}{n!}$.

By Proposition 2.1, Equation (2.6) is equal to

$$\begin{aligned} & S(t)\tilde{\pi}_0Q_S\left(\sigma(\tilde{W}(T)-\tilde{W}(t^*))\geqslant\ln K-(r+\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*)\right) \\ & +S(t)\sum_{n=1}^{\infty}\tilde{\pi}_n\sum_{m=1}^{n}P_{n,m}Q_S\Bigg(\sigma(\tilde{W}(T)-\tilde{W}(t^*))+\sum_{j=1}^{m}\xi_j^+\geqslant\ln K \\ & -(r+\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*)\Bigg) \\ & +S(t)\sum_{n=1}^{\infty}\tilde{\pi}_n\sum_{m=1}^{n}Q_{n,m}Q_S\Bigg(\sigma(\tilde{W}(T)-\tilde{W}(t^*))-\sum_{j=1}^{m}\xi_j^-\geqslant\ln K \\ & -(\frac{1}{2}\sigma^2+r-\lambda\beta)(T-t^*)\Bigg), \end{aligned}$$

The distribution of the sum of exponential random variables and normal random variables provided by Kou[11] can be acquired in closed-form of the Hh functions. Hence, we have that

$$\begin{aligned} & Q_S\left(\sigma(\tilde{W}(T)-\tilde{W}(t^*))+\sum_{j=1}^{m}\xi_j^+\geqslant \ln K-(\frac{1}{2}\sigma^2+r-\lambda\beta)(T-t^*)\right) \\ = & \frac{(\sigma\sqrt{T-t^*}\tilde{\eta}_1)^m \mathrm{e}^{\frac{(\sigma\tilde{\eta}_1)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}}\int_{\delta_1}^{\infty}e^{-x\tilde{\eta}_1}Hh_{m-1}\left(\frac{-x}{\sigma\sqrt{T-t^*}}+\sigma\sqrt{T-t^*}\tilde{\eta}_1\right)\mathrm{d}x, \quad (2.7)\end{aligned}$$

and

$$\begin{aligned} & Q_S\left(\sigma(\tilde{W}(T)-\tilde{W}(t^*))-\sum_{j=1}^{m}\xi_j^-\geqslant \ln K-(\frac{1}{2}\sigma^2+r-\lambda\beta)(T-t^*)\right) \\ = & \frac{(\sigma\sqrt{T-t^*}\tilde{\eta}_2)^m e^{\frac{(\sigma\tilde{\eta}_2)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}}\int_{\delta_1}^{\infty}\mathrm{e}^{x\tilde{\eta}_2}Hh_{m-1}\left(\frac{x}{\sigma\sqrt{T-t^*}}+\sigma\sqrt{T-t^*}\tilde{\eta}_2\right)\mathrm{d}x, \quad (2.8)\end{aligned}$$

where $\delta_1=\ln K-(r+\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*)$. Since

$$I_m(c;\alpha,\beta,\delta)=\int_c^{\infty}e^{\alpha x}Hh_m(\beta x-\delta)\mathrm{d}x, \qquad m\geqslant 0$$

and Definition 2.1, we have that equations (2.7) and (2.8) can be written as follow

$$\frac{(\sigma\sqrt{T-t^*}\tilde{\eta}_1)^m e^{\frac{(\sigma\tilde{\eta}_1)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}}I_{m-1}\left(\delta_1;-\tilde{\eta}_1,-\frac{1}{\sigma\sqrt{T-t^*}},-\sigma\sqrt{T-t^*}\tilde{\eta}_1\right),$$

and

$$\frac{(\sigma\sqrt{T-t^*}\tilde{\eta}_2)^m e^{\frac{(\sigma\tilde{\eta}_2)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}}I_{m-1}\left(\delta_1;\tilde{\eta}_1,\frac{1}{\sigma\sqrt{T-t^*}},-\sigma\sqrt{T-t^*}\tilde{\eta}_2\right).$$

Denote

$$\begin{aligned} & \Upsilon(\delta_1,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2,T-t^*) \\ = & S(t)\sum_{n=1}^{\infty}\tilde{\pi}_n\sum_{m=1}^{n}P_{n,m}\frac{(\sigma\sqrt{T-t^*}\tilde{\eta}_1)^m e^{\frac{(\sigma\tilde{\eta}_1)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}}I_{m-1}\left(\delta_1;-\tilde{\eta}_1,-\frac{1}{\sigma\sqrt{T-t^*}},-\sigma\sqrt{T-t^*}\tilde{\eta}_1\right) \\ + & S(t)\sum_{n=1}^{\infty}\tilde{\pi}_n\sum_{m=1}^{n}Q_{n,m}\frac{(\sigma\sqrt{T-t^*}\tilde{\eta}_2)^m e^{\frac{(\sigma\tilde{\eta}_2)^2(T-t^*)}{2}}}{\sigma\sqrt{T-t^*}}I_{m-1}\left(\delta_1;\tilde{\eta}_2,\frac{1}{\sigma\sqrt{T-t^*}},-\sigma\sqrt{T-t^*}\tilde{\eta}_2\right).\end{aligned}$$

Moreover, we can easily obtain that

$$S(t)\tilde{\pi}_0 Q_S\left(\sigma(\tilde{W}(T)-\tilde{W}(t^*))\geqslant \ln K-(r+\frac{1}{2}\sigma^2-\lambda\beta)(T-t^*)\right)=S(t)\mathrm{e}^{-\tilde{\lambda}(T-t^*)}\mathbb{N}(d_1).$$

Hence, we have that

$$\Pi_1=S(t)\mathrm{e}^{-\tilde{\lambda}(T-t^*)}\mathbb{N}(d_1)+\Upsilon(\delta_1,\sigma,\tilde{\lambda},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2,T-t^*). \quad (2.9)$$

By the same way, we can obtain that as follow.

$$\Pi_2 \;=\; K\mathrm{e}^{-(\lambda+r)(T-t^*)}S(t)\mathbb{N}(d_2)+K\mathrm{e}^{-r(T-t^*)}\Upsilon(\delta_2,\sigma,\lambda,p,\eta_1,\eta_2,T-t^*). \quad (2.10)$$

Note that $\beta = \mathbb{E}_Q[Y_i - 1]$ and (2.1), we can obtain that $\beta = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2+1} - 1$.

By lemma 2.1 , we have that

$$\tilde{p} = \frac{p\eta_1}{(1+\beta)(\eta_1 - 1)} \quad \tilde{\eta}_1 = \eta_1 - 1, \quad \tilde{\eta}_2 = \eta_2 + 1.$$

The proof of this result is simple and it was listed in the Theorem 2 of Kou[11].

From (2.5), (2.9) and (2.10), we complete the proof.

3 On the valuation of forward starting call option when the log jump size has a general distribution

In this section, we assume the log jump size $\ln Y_i$ has a general distribution, its density function is $f(y)$, the stock price satisfy (1.3).

Theorem 3.1 Let $0 \leqslant t < t^* < T$, The price of the forward starting call option at time t is equal to

$$C(t,T) = S(t)\mathrm{e}^{-\tilde{\lambda}(T-t^*)} \left(\sum_{m=1}^{\infty} \frac{\left[\tilde{\lambda}(T-t^*)\right]^m}{m!} \int_{-\infty}^{\infty} \mathbb{N}\left(d_1(t^*,T,y)\right) \tilde{f}^m(y)\mathrm{d}y + \mathbb{N}\left(d_1(t^*,T,0)\right) \right)$$
$$-KS(t)\mathrm{e}^{-\int_{t^*}^{T} r_s \mathrm{d}s - \lambda(T-t^*)} \left(\sum_{m=1}^{\infty} \frac{[\lambda(T-t^*)]^m}{m!} \int_{-\infty}^{\infty} \mathbb{N}\left(d_2(t^*,T,y)\right) f^m(y)\mathrm{d}y + \mathbb{N}\left(d_2(t^*,T,0)\right) \right),$$

where $\mathbb{N}(\cdot)$ denotes the cumulative distribution of normal random variable, and

$$d_1(t^*,T,y) = \frac{\int_{t^*}^{T} r_s \mathrm{d}s - \lambda\beta(T-t^*) + \frac{1}{2}\int_{t^*}^{T}\sigma_s^2\mathrm{d}s - \ln K + y}{\sqrt{\int_{t^*}^{T}\sigma_s^2\mathrm{d}s}}, \tag{3.1}$$

$$d_2(t^*,T,y) = d_1(t^*,T,y) - \sqrt{\int_{t^*}^{T}\sigma_s^2\mathrm{d}s}. \tag{3.2}$$

$\tilde{f}^m(y)$ denotes the m-fold convolution of density function $\tilde{f}(y)$ of $\ln Y_i$ under the probability measure Q_S. $f^m(y)$ denotes the m-fold convolution of density function $f(y)$ of $\ln Y_i$ under the probability measure Q.

Proof By(2.2), we have that

$$\begin{aligned} &\mathbb{E}_Q\left[\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\left(S(T) - KS(t^*)\right)^+ \mid \mathcal{F}_t\right] \\ &= S(t)\mathbb{E}_{Q_S}\left[I_{[S(T)\geqslant KS(t^*)]}\middle|\mathcal{F}_t\right] - K\mathrm{e}^{-\int_t^T r_s\mathrm{d}s}\mathbb{E}_Q\left[S(t^*)I_{[S(T)\geqslant KS(t^*)]} \mid \mathcal{F}_t\right]. \end{aligned} \tag{3.3}$$

We first consider the first part of above formula, note that

$$\mathbb{E}_{Q_S}[I_{[S(T)\geqslant KS(t^*)]} \mid \mathcal{F}_t] = \mathbb{E}_{Q_S}[\mathbb{E}_{Q_S}[I_{[S(T)\geqslant KS(t^*)]} \mid \mathcal{F}_{t^*} \bigvee \sigma(\prod_{i=N(t^*)+1}^{N(T)} Y_i)] \mid \mathcal{F}_t]. \tag{3.4}$$

In this paper, we set $A=\{S(T)\geqslant KS(t^*)\}$. Hence, by (1.3), and Lemma 2.2, we have the event A is equal to

$$\int_{t^*}^{T}(r_s+\frac{1}{2}\sigma_s^2-\lambda\beta)\mathrm{d}s+\int_{t^*}^{T}\sigma_s\mathrm{d}\tilde{W}_s+\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i\geqslant\ln K. \tag{3.5}$$

We note that the dependence of increment property of Brownian motion and

$$\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i \quad is \quad \sigma(\prod_{i=N(t^*)+1}^{N(T)}Y_i)-\text{measurable}. \tag{3.6}$$

Thus, by direct calculation, (3.4) can be rewritten as

$$\begin{aligned}
&\mathbb{E}_{Q_S}\left[\mathbb{N}\left(d_1(t^*,T,\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i)\right)\Big|\mathcal{F}_t\right]\\
&=\mathbb{E}_{Q_S}\left[\mathbb{N}\left(d_1(t^*,T,\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i)\right)\right]\\
&=\sum_{m=1}^{\infty}\mathbb{E}_{Q_S}\left[\mathbb{N}\left(d_1(t^*,T,\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i)\right)\Big|N(T)-N(t^*)=m\right]\\
&\times Q_S\left(N(T)-N(t^*)=m\right)+Q_S\left(N(T)-N(t^*)=0\right)\mathbb{N}\left(d_1(t^*,T,0)\right)\\
&=e^{-\tilde{\lambda}(T-t^*)}\left(\sum_{m=1}^{\infty}\frac{\left[\tilde{\lambda}(T-t^*)\right]^m}{m!}\int_{-\infty}^{\infty}\mathbb{N}\left(d_1(t^*,T,y)\right)\tilde{f}^m(y)dy\right.\\
&\left.+\mathbb{N}\left(d_1(t^*,T,0)\right)\right).
\end{aligned} \tag{3.7}$$

Eqn.(3.7) is obtained because of Lemmas 2.1 and 2.2. Under the measure Q, the event A is equal to

$$\int_{t^*}^{T}(r_s-\frac{1}{2}\sigma_s^2-\lambda\beta)\mathrm{d}s+\int_{t^*}^{T}\sigma_s\mathrm{d}W_s+\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i\geqslant\ln K. \tag{3.8}$$

Thus, we have that

$$\begin{aligned}
&\mathbb{E}_Q\left[S(t^*)I_{[S(T)\geqslant KS(t^*)]}\mid\mathcal{F}_t\right]\\
&=\mathbb{E}_Q[\mathbb{E}_Q[S(t^*)I_{[S(T)\geqslant KS(t^*)]}\mid\mathcal{F}_{t^*}\bigvee\sigma(\prod_{i=N(t^*)+1}^{N(T)}Y_i)]\mid\mathcal{F}_t]\\
&=\mathbb{E}_Q[S(t^*)\mathbb{E}_Q[I_{[S(T)\geqslant KS(t^*)]}\mid\mathcal{F}_{t^*}\bigvee\sigma(\prod_{i=N(t^*)+1}^{N(T)}Y_i)]\mid\mathcal{F}_t]\\
&=\mathbb{E}_Q\left[S(t^*)\mathbb{N}\left(d_2(t^*,T,\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i)\right)\Big|\mathcal{F}_t\right]\\
&=S(t)\mathbb{E}_Q\left[\frac{S(t^*)}{S(t)}\mathbb{N}\left(d_2(t^*,T,\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i)\Big|\mathcal{F}_t\right)\right].
\end{aligned} \tag{3.9}$$

From (1.3), we have $\frac{S(t^*)}{S(t)}$ is independent of $\mathcal{F}_t$, moreover, we note that Y_i is i.i.d. Hence, the above formula can be rewritten as follow

$$
\begin{aligned}
&S(t)\mathbb{E}_Q[\frac{S(t^*)}{S(t)}]\mathbb{E}_Q\left[\mathbb{N}\left(d_2(t^*,T,\sum_{i=N(t^*)+1}^{N(T)}\ln Y_i)\right)\right]\\
&=S(t)\mathrm{e}^{\int_t^{t^*}r_s\mathrm{d}s-\lambda(T-t^*)}\sum_{m=1}^{\infty}\frac{[\lambda(T-t^*)]^m}{m!}\int_{-\infty}^{\infty}\mathbb{N}\left(d_2(t^*,T,y)\right)f^m(y)\mathrm{d}y\\
&\quad+S(t)\mathrm{e}^{\int_t^{t^*}r_s\mathrm{d}s}Q\left(N(T)-N(t^*)=0\right)\mathbb{N}\left(d_2(t^*,T,0)\right)\\
&=S(t)\mathrm{e}^{\int_t^{t^*}r_s\mathrm{d}s-\lambda(T-t^*)}\left(\sum_{m=1}^{\infty}\frac{[\lambda(T-t^*)]^m}{m!}\int_{-\infty}^{\infty}\mathbb{N}\left(d_2(t^*,T,y)\right)f^m(y)\mathrm{d}y\right.\\
&\quad\left.+\mathbb{N}\left(d_2(t^*,T,0)\right)\right). \qquad (3.10)
\end{aligned}
$$

As a direct consequence from (3.3), (3.9), (3.10), we complete the proof.

4 Conclusion

We have considered the problem of pricing forward starting call option in jump-diffusion model. It can be extended to the case of stochastic interest rate and stochastic volatility. These problems are very significant.

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