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H 值相关定理及证明

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摘要: 进一步研究如何利用 H 值来筛选证券组. 推导了关于 H 值组成部分相关量的具体表达式, 并得到了证券组中去掉最"差"的一种证券后 H 值改变量满足的一个不等式.

关键词: H 值; Treynor 比率; 市场指数模型; 证券数目

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Some theorems about the H-value rule

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Abstract: Further study was given in this paper to the H value as a standard for evaluation of the quality of a security set. The H value in terms of its related parameters was characterized. An inequality which the changed H value satisfies after removing the "worst" security from the set was proved.

Key words: H-value; Treynor ratio; market index model; number of the securities

0 引 言

含有无风险证券的证券组的前沿曲线是两条斜率绝对值相等的射线, 绝对值越大, 前沿曲线张开角度越大, 前沿上的组合就越好, 文献[1]用 H 值表示该斜率的平方, 并利用证券组的 H 值越大越好的特性, 对证券组（并非证券组合）进行筛选, 得到了一些有意义的结论（有关 H 值还可见文献[2-4]）. H 值是风险证券前沿曲线切点组合的夏普率的平方, 但由于考虑问题的出发点不同, 两者的作用还是有区别的：夏普率是单位风险的报酬, 现在常用来作为评价基金业绩的一种标准. 我们在这里是利用 H 值来进行证券组的筛选, 以筛选出好的证券组(注意, 不是证券组合）. 在夏普的经典著作[5]中也有类似的工作, 如在已有的证券组合中再增加一种零投资组合来实现合适的风险下组合预期收益最优的问题. 在评价组合好坏的方面, Treynor, Jack L. and Fischer Black[6]也采用了夏普率的平方, 但不是考虑切点组合夏普率的平方, 而是考虑积极性组合和被动型组合的整个组合的夏普率平方. Treynor比率也是评价基金业绩好坏的一个常用标准.

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1 定理及证明

以下定理均在市场指数模型

$$r_i - r_f = \alpha_i + \beta_i (r_M - r_f) + \varepsilon_i, \forall i \in \{1, 2, \cdots, n\}$$

中考虑; 并假定在本文中, $\beta_i > 0$, $\forall i \in \{1, 2, \cdots, n\}$.

引理 1(切点组合引理[7]104) 若 $r_f \neq \frac{A}{C}$, 则

(1) 含无风险证券的证券组合前沿曲线与全部无风险证券的前沿曲线相切;

(2) 切点组合 $W_e = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, $w_i = y_i \Big/ \sum\limits_{i=1}^{n} y_i$, 其中

$$\begin{aligned} y_i &= (\beta_i/\sigma_{\varepsilon_i}^2)\left((\overline{r}_i - r_f)/\beta_i - C_n\right), \\ C_n &= \sigma_M^2 \sum_{i=1}^{n} \left((\overline{r}_i - r_f)\beta_i/\sigma_{\varepsilon_i}^2\right) \Bigg/ \left(1 + \sigma_M^2 \sum_{i=1}^{n} \left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)\right). \end{aligned}$$

由于 $\beta_i > 0$, $\forall i \in \{1, 2, \cdots, n\}$, 则 $(\overline{r}_i - r_f)/\beta_i - C_n$ 的符号直接决定了第 i 种证券的卖空与否.

定理 1 $(\overline{r}_j - r_f)/\beta_j - C_n$ 是关于 $\overline{r}_j - r_f$ 的增函数.

证明 因为

$$\begin{aligned} (\overline{r}_j - r_f)/\beta_j - C_n &= (\overline{r}_j - r_f)/\beta_j - \sigma_M^2 \sum_{i=1}^{n} \left((\overline{r}_i - r_f)\beta_i/\sigma_{\varepsilon_i}^2\right) \Big/ \left(1 + \sigma_M^2 \sum_{i=1}^{n} \left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)\right) \\ &= \frac{(\overline{r}_j - r_f)}{\beta_j}\left(1 + \sigma_M^2 \sum_{i=1, i\neq j}^{n} \left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)\right) \Big/ \left(1 + \sigma_M^2 \sum_{i=1}^{n} \left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)\right) \\ &\quad -\sigma_M^2 \sum_{i=1, i\neq j}^{n} \left((\overline{r}_i - r_f)\beta_i/\sigma_{\varepsilon_i}^2\right) \Big/ \left(1 + \sigma_M^2 \sum_{i=1}^{n} \left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)\right). \end{aligned}$$

而 $\frac{1}{\beta_j}(1 + \sigma_M^2 \sum\limits_{i=1, i\neq j}^{n} \left(\beta_i^2/\sigma_{\varepsilon_i}^2\right))$ 为正数. 所以 $(\overline{r}_j - r_f)/\beta_j - C_n$ 关于 $\overline{r}_j - r_f$ 为增函数.

定理 2 记含 n 种证券的 H 值为 H_n, 去掉第 j 种证券后的 H 值为 $H_n(j)$, 则 H 值的改量

$$\begin{aligned} H_n - H_n(j) &= \left((\overline{r}_j - r_f)/\beta_j - C_n\right)^2 \Bigg/ \left(\sigma_M^2\left(1 + \sigma_M^2\left(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2) - \beta_j^2/\sigma_{\varepsilon_j}^2\right)\right) \Bigg/ \right. \\ &\quad \left. \left((\sigma_M^2\beta_j^2/\sigma_{\varepsilon_j}^2)\left(1 + \sigma_M^2 \sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right). \end{aligned}$$

证明 由文献[1]知, 含 n 种证券的证券组的 H 值为

$$H_n = \sum_{i=1}^{n}\left((\overline{r}_i - r_f)/\sigma_{\varepsilon_i}\right)^2 - \sigma_M^2\left(\sum_{i=1}^{n}((\overline{r}_i - r_f)\beta_i/\sigma_{\varepsilon_i}^2)\right)^2 \Bigg/ \left(1 + \sigma_M^2 \sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right).$$

记 $b = \sigma_M^2(\overline{r}_j - r_f)\beta_j \Big/ \sigma_{\varepsilon_j}^2, a = \sigma_M^2\beta_j^2 \Big/ \sigma_{\varepsilon_j}^2, M = \sigma_M^2 \sum\limits_{i=1}^{n}\left((\overline{r}_i - r_f)\beta_i/\sigma_{\varepsilon_i}^2\right)$, $N = 1 + \sigma_M^2 \sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)$. 则 $C_n = M/N$. 于是有

$$
\begin{aligned}
H_n-H_n(j) &= \sum_{i=1}^{n}\big((\overline{r}_i-r_f)/\sigma_{\varepsilon_i}\big)^2-\sigma_M^2\left(\sum_{i=1}^{n}((\overline{r}_i-r_f)\beta_i/\sigma_{\varepsilon_i}^2)\right)^2\Big/\left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\
&\quad-\left(\sum_{i=1}^{n}((\overline{r}_i-r_f)/\sigma_{\varepsilon_i})^2-((\overline{r}_j-r_f)/\sigma_{\varepsilon_j})^2\right)\\
&\quad+\sigma_M^2\left(\sum_{i=1}^{n}((\overline{r}_i-r_f)\beta_i/\sigma_{\varepsilon_i}^2)-(\overline{r}_j-r_f)\beta_j\big/\sigma_{\varepsilon_j}^2\right)^2\Big/\left(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2\big/\sigma_{\varepsilon_j}^2)\right)\\
&= \big((\overline{r}_j-r_f)/\sigma_{\varepsilon_j}\big)^2-\sigma_M^2\left(\sum_{i=1}^{n}((\overline{r}_i-r_f)\beta_i/\sigma_{\varepsilon_i}^2)\right)^2\Big/\left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\
&\quad+\sigma_M^2\left(\sum_{i=1}^{n}((\overline{r}_i-r_f)\beta_i/\sigma_{\varepsilon_i}^2)-(\overline{r}_j-r_f)\beta_j\big/\sigma_{\varepsilon_j}^2\right)^2\Big/\left(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2\big/\sigma_{\varepsilon_j}^2)\right)\\
&= \frac{1}{\sigma_M^2}\Big(b^2/a-M^2/N+(M-b)^2/(N-a)\Big)=(b/a-M/N)^2/\big(\sigma_M^2(N-a)/aN\big)\\
&= \big((\overline{r}_j-r_f)/\beta_j-C_n\big)^2\Big/\left(\sigma_M^2\Big(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2/\sigma_{\varepsilon_j}^2)\Big)\Big/\right.\\
&\quad\left.\Big((\sigma_M^2\beta_j^2/\sigma_{\varepsilon_j}^2)\Big(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\Big)\Big)\right).
\end{aligned}
$$

在市场指数模型 $r_i-r_f=\alpha_i+\beta_i(r_M-r_f)+\varepsilon_i$ 下, 如果第 j 种证券只有 α_j 变化, 系统风险 $\sigma_M^2\beta_j^2$ 与自身风险 $\sigma_{\varepsilon_j}^2$ 不变. 于是去掉第 j 种风险证券后的 H 值改变量 $H_n-H_n(j)$ 的值直接由 $((\overline{r}_j-r_f)/\beta_j-C_n)^2$ 决定. 根据定理1, $(\overline{r}_j-r_f)/\beta_j-C_n$ 关于 $\overline{r}_j-r_f$ 为增函数. 所以若 $(\overline{r}_j-r_f)/\beta_j-C_n>0$, 则 H 值改变量 $H_n-H_n(j)$ 为 $\overline{r}_j-r_f$ 的增函数. 也就是说在剩下的 $n-1$ 种证券中加上第 j 种证券后的 H 值会随着 $\overline{r}_j-r_f$ 的值的增大而增大. 因此有定理3.

定理 3　设在市场指数模型 $r_i-r_f=\alpha_i+\beta_i(r_M-r_f)+\varepsilon_i$ 下, 含 n 种证券的证券组的第 j 种证券仅 α_j 变化, 系统风险 $\sigma_M^2\beta_j^2$ 与自身风险 $\sigma_{\varepsilon_j}^2$ 等不变, 如果第 j 种证券在 α_j 未改变时是持有的, 即 $(\overline{r}_j-r_f)/\beta_j-C_n>0$, 并且 α_j 改变后也有 $(\overline{r}_j-r_f)/\beta_j-C_n>0$, 则整个证券组的 H 值是 $\overline{r}_j-r_f$ 的增函数, 或者是 $(\overline{r}_j-r_f)\big/\beta_j$ 的增函数.

下面用 $H_n(i_1,i_2,\cdots,i_{s-1},i_s)$ 表示 n 种证券中去掉第 $i_1,i_2,\cdots,i_{s-1},i_s$ 种证券后的 H 值.

定理 4　含 n 种证券的 H 值

$$
\begin{aligned}
H_n &= \sum_{j=1}^{n}\big((\overline{r}_j-r_f)/\beta_j-C_j\big)^2\Big/\left(\sigma_M^2\Big(1+\sigma_M^2(\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2/\sigma_{\varepsilon_j}^2)\Big)\Big/\right.\\
&\quad\left.\Big((\sigma_M^2\beta_j^2/\sigma_{\varepsilon_j}^2)\Big(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\Big)\Big)\right).
\end{aligned}
$$

证明　由定理2知含 n 种证券的 H 值

$$
\begin{aligned}
H_n &= \big(H_n-H_n(n)\big)+\big(H_n(n)-H_n(n,n-1)\big)+\cdots+\big(H_n(n,n-1,\cdots,3)\\
&\quad-H_n(n,n-1,\cdots,3,2)\big)+H_n(n,n-1,\cdots,3,2)\\
&= \big((\overline{r}_n-r_f)/\beta_n-C_n\big)^2\Big/\left(\sigma_M^2\Big(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_n^2/\sigma_{\varepsilon_n}^2)\Big)\Big/\right.\\
&\quad\left.\Big(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2\Big(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\Big)\Big)\right)
\end{aligned}
$$

$$+\left((\overline{r}_{n-1}-r_f)/\beta_{n-1}-C_{n-1}\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{n-1}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_{n-1}^2/\sigma_{\varepsilon_{n-1}}^2\big)\right)\Big/\right.$$
$$\left.\left(\sigma_M^2\beta_{n-1}^2/\sigma_{\varepsilon_{n-1}}^2\left(1+\sigma_M^2\sum_{i=1}^{n-1}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)$$
$$+\cdots+\left((\overline{r}_2-r_f)/\beta_2-C_2\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{2}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_2^2/\sigma_{\varepsilon_2}^2\big)\right)\Big/\right.$$
$$\left.\left(\sigma_M^2\beta_2^2/\sigma_{\varepsilon_2}^2\left(1+\sigma_M^2\sum_{i=1}^{2}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)+\left(\overline{r}_1-r_f\right)^2\Big/\mathrm{var}(r_1).$$

注意到

$$\left((\overline{r}_1-r_f)/\beta_1-C_1\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{1}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_1^2/\sigma_{\varepsilon_1}^2\big)\right)\Big/\left((\sigma_M^2\beta_1^2/\sigma_{\varepsilon_1}^2)\left(1+\sigma_M^2\sum_{i=1}^{1}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)$$

$$=\frac{\left((\overline{r}_1-r_f)/\beta_1-\sigma_M^2\left((\overline{r}_1-r_f)\beta_1/\sigma_{\varepsilon_1}^2\right)/\left(1+\sigma_M^2\left(\beta_1^2/\sigma_{\varepsilon_1}^2\right)\right)\right)^2}{\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum\limits_{i=1}^{1}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_1^2/\sigma_{\varepsilon_1}^2\big)\right)\Big/\left((\sigma_M^2\beta_1^2/\sigma_{\varepsilon_1}^2)\left(1+\sigma_M^2\sum\limits_{i=1}^{1}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)}$$
$$=\left(\overline{r}_1-r_f\right)^2\Big/(\sigma_M^2\beta_1^2+\sigma_{\varepsilon_1}^2)=\left(\overline{r}_1-r_f\right)^2\Big/\mathrm{var}(r_1)=H_n(n,n-1,\cdots,3,2).$$

所以

$$H_n=\left((\overline{r}_n-r_f)/\beta_n-C_n\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_n^2/\sigma_{\varepsilon_n}^2\big)\right)\Big/\left((\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)\big(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\big)\right)\right)$$
$$+\left((\overline{r}_{n-1}-r_f)/\beta_{n-1}-C_{n-1}\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{n-1}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_{n-1}^2/\sigma_{\varepsilon_{n-1}}^2\big)\right)\Big/\right.$$
$$\left.\left((\sigma_M^2\beta_{n-1}^2/\sigma_{\varepsilon_{n-1}}^2)\big(1+\sigma_M^2\sum_{i=1}^{n-1}(\beta_i^2/\sigma_{\varepsilon_i}^2)\big)\right)\right)$$
$$+\cdots+\left((\overline{r}_2-r_f)/\beta_2-C_2\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{2}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_2^2/\sigma_{\varepsilon_2}^2\big)\right)\Big/\left((\sigma_M^2\beta_2^2/\sigma_{\varepsilon_2}^2)\big(1+\sigma_M^2\sum_{i=1}^{2}(\beta_i^2/\sigma_{\varepsilon_i}^2)\big)\right)\right)$$
$$\left((\overline{r}_1-r_f)/\beta_1-C_1\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{1}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_1^2/\sigma_{\varepsilon_1}^2\big)\right)\Big/\left((\sigma_M^2\beta_1^2/\sigma_{\varepsilon_1}^2)\big(1+\sigma_M^2\sum_{i=1}^{1}(\beta_i^2/\sigma_{\varepsilon_i}^2)\big)\right)\right)$$
$$=\sum_{j=1}^{n}\left((\overline{r}_j-r_f)/\beta_j-C_j\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2\big(\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2/\sigma_{\varepsilon_j}^2\big)\right)\Big/\right.$$
$$\left.\left((\sigma_M^2\beta_j^2/\sigma_{\varepsilon_j}^2)\big(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\big)\right)\right).$$

定理 5 设市场指数模型 $r_i-r_f=\alpha_i+\beta_i\left(r_M-r_f\right)+\varepsilon_i$ 中, $\alpha_i=0,\forall i\in\{1,2,\cdots,n\}$, 相应的 H 值记为 H_n^0, 则

$$H_n^0=\frac{\left(\overline{r}_M-r_f\right)^2\sum\limits_{i=1}^{n}\left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)}{1+\sigma_M^2\sum\limits_{i=1}^{n}\left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)}.$$

证明 若 $\alpha_i=0,\ \forall i\in\{1,2,\cdots,n\}$, 则 $r_i-r_f=\beta_i\left(r_M-r_f\right)+\varepsilon_i$; $\overline{r}_i-r_f=\beta_i\left(\overline{r}_M-r_f\right)$. 于是

$$\begin{aligned}
H_n^0 &= \sum_{i=1}^{n}((\overline{r}_i - r_f)/\sigma_{\varepsilon_i})^2 - \sigma_M^2\left(\sum_{i=1}^{n}((\overline{r}_i - r_f)\beta_i/\sigma_{\varepsilon_i}^2)\right)^2 \Bigg/ \left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right) \\
&= \sum_{i=1}^{n}(\beta_i(\overline{r}_M - r_f)/\sigma_{\varepsilon_i})^2 - \sigma_M^2\left(\sum_{i=1}^{n}(\beta_i(\overline{r}_M - r_f)\beta_i/\sigma_{\varepsilon_i}^2)\right)^2 \Bigg/ \left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right) \\
&= (\overline{r}_M - r_f)^2\left(\sum_{i=1}^{n}(\beta_i/\sigma_{\varepsilon_i})^2 - \sigma_M^2\left(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)^2 \Bigg/ \left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right) \\
&= \frac{(\overline{r}_M - r_f)^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)}{1+\sigma_M^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)}.
\end{aligned}$$

定理6 若$\beta_1^2/\sigma_{\varepsilon_1}^2 \geqslant \beta_2^2/\sigma_{\varepsilon_2}^2 \geqslant \cdots \geqslant \beta_{n-1}^2\Big/\sigma_{\varepsilon_{n-1}}^2 \geqslant \beta_n^2/\sigma_{\varepsilon_n}^2$, 当$\alpha_i = 0$, $\forall i \in \{1,2\cdots,n\}$时, n种证券中去掉第k种证券后的H值记为$H_n^0(k)$, 则H值的改变量$H_n^0 - H_n^0(k) \leqslant \frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}H_n^0$.

证明

$$\begin{aligned}
H_n^0 - H_n^0(k) &= \frac{(\overline{r}_M - r_f)^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)}{1+\sigma_M^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)} - \frac{(\overline{r}_M - r_f)^2\sum\limits_{i=1,i\neq k}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)}{1+\sigma_M^2\sum\limits_{i=1,i\neq k}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)} \\
&= \frac{(\overline{r}_M - r_f)^2(\beta_k^2/\sigma_{\varepsilon_k}^2)}{(1+\sigma_M^2\sum\limits_{i=1,i\neq k}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2))(1+\sigma_M^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2))} \\
&= \frac{(\overline{r}_M - r_f)^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)}{1+\sigma_M^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)}\frac{\beta_k^2/\sigma_{\varepsilon_k}^2}{(1+\sigma_M^2\sum\limits_{i=1,i\neq k}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2))\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)} \\
&= \frac{\beta_k^2/\sigma_{\varepsilon_k}^2}{(1+\sigma_M^2\sum\limits_{i=1,i\neq k}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2))\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)}H_n^0 \\
&\leqslant \frac{\beta_k^2/\sigma_{\varepsilon_k}^2}{(1+\sigma_M^2\sum\limits_{i=1,i\neq k}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2))n(\beta_n^2/\sigma_{\varepsilon_n}^2)}H_n^0 \leq \frac{\beta_k^2/\sigma_{\varepsilon_k}^2}{(n-1)(\beta_n^2/\sigma_{\varepsilon_n}^2)n(\beta_n^2/\sigma_{\varepsilon_n}^2)}H_n^0 \\
&= \frac{\beta_k^2/\sigma_{\varepsilon_k}^2}{(\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}H_n^\circ.
\end{aligned}$$

定理7 设含n种证券的证券组满足$\beta_1^2/\sigma_{\varepsilon_1}^2 \geqslant \beta_2^2/\sigma_{\varepsilon_2}^2 \geqslant \cdots \geqslant \beta_{n-1}^2\Big/\sigma_{\varepsilon_{n-1}}^2 \geqslant \beta_n^2/\sigma_{\varepsilon_n}^2$, $C_n < (\overline{r}_k - r_f)/\beta_k = \min\limits_{i\in\{1,2,\cdots,n\}}((\overline{r}_i - r_f)/\beta_i)$. 并且这$n$种证券去掉最"差"的一种后的$H$值记为$H_{n-1}$, 则$H$值的改变量$H_n - H_{n-1} \leqslant \dfrac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}H_n$.

证明 先假设 $\alpha_i=0,\forall i\in\{1,2,\cdots,n\}$, 则对任意的 $1\leqslant i\leqslant j\leqslant n$, 有

$$\begin{aligned}(\overline{r}_i-r_f)/\beta_i-C_j&=\beta_i(\overline{r}_M-r_f)/\beta_i-\sigma_M^2\sum_{i=1}^{j}\left(\beta_i(\overline{r}_M-r_f)\beta_i/\sigma_{\varepsilon_i}^2\right)\Big/\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\&=(\overline{r}_M-r_f)-(\overline{r}_M-r_f)\sigma_M^2\sum_{i=1}^{j}\beta_i^2/\sigma_{\varepsilon_i}^2\Big/\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\&=(\overline{r}_M-r_f)\Big/\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\&\triangleq x_j\end{aligned}$$

只与 j 有关, 与 i 无关. 由定理6, 当 $\alpha_i=0,\forall i\in\{1,2,\cdots,n\}$ 时, H 值的改变量

$$H_n^0-H_n^0(k)\quad\leqslant\frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2n(n-1)}H_n^0.$$

结合定理2, 定理4知

$$\begin{aligned}&x_n^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_k^2/\sigma_{\varepsilon_k}^2)\right)\Big/\left((\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2)\left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)\\&\qquad\leqslant\frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2n(n-1)}\sum_{j=1}^{n}\left(x_j^2\Big/\frac{\sigma_M^2\left(1+\sigma_M^2(\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2/\sigma_{\varepsilon_j}^2)\right)}{\left((\sigma_M^2\beta_j^2/\sigma_{\varepsilon_j}^2)\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)}\right).\qquad(1)\end{aligned}$$

下面讨论 $\alpha_i,\forall i\in\{1,2,\cdots,n\}$ 不全为0的情况. 为了证明的需要分两步操作.

(1) 将 $\alpha_i,\forall i\in\{1,2,\cdots,n\},i\neq k$ 适当调整, 使得 $(\overline{r}_i-r_f)/\beta_i$ 都等于最小值 $(\overline{r_k}-r_f)/\beta_k$.

(2) 将 $\alpha_i,\forall i\in\{1,2,\cdots,n\},i\neq k$ 调整回原状, 使 $(\overline{r}_i-r_f)/\beta_i$ 增大到原来的值.

考虑第1步操作. 此时对于任意的 $1\leqslant i\leqslant j\leqslant n$ 有

$$\begin{aligned}(\overline{r}_i-r_f)/\beta_i-C_j&=(\overline{r}_k-r_f)/\beta_k-\sigma_M^2\sum_{i=1}^{j}\left(((\overline{r}_k-r_f)/\beta_k)\beta_i^2/\sigma_{\varepsilon_i}^2\right)\Big/\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\&=(\overline{r}_k-r_f)/\beta_k-((\overline{r}_k-r_f)/\beta_k)\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\Big/\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\&=((\overline{r}_k-r_f)/\beta_k)\Big/\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\\&\triangleq y_j,\end{aligned}$$

只与 j 有关, 与 i 无关. 容易看出 $y_j=\dfrac{((\overline{r}_k-r_f)/\beta_k)}{\overline{r}_M-r_f}x_j\triangleq Ax_j$, 其中 A 为常数. (1)式两边同时乘上 A^2 得

$$\begin{aligned}&(Ax_n)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_k^2/\sigma_{\varepsilon_k}^2)\right)\Big/\left((\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2)\left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)\\&\qquad\leqslant\frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2n(n-1)}\sum_{j=1}^{n}\left((Ax_j)^2\Big/\frac{\sigma_M^2\left(1+\sigma_M^2(\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2/\sigma_{\varepsilon_j}^2)\right)}{\left((\sigma_M^2\beta_j^2/\sigma_{\varepsilon_j}^2)\left(1+\sigma_M^2\sum_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)}\right),\end{aligned}$$

即 $y_n^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2(\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_k^2/\sigma_{\varepsilon_k}^2)\right)\Big/\left((\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2)\left(1+\sigma_M^2\sum\limits_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)$

$$\leqslant \frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}\sum_{j=1}^{n}\left(y_j^2\Big/\frac{\sigma_M^2\left(1+\sigma_M^2(\sum\limits_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2\big/\sigma_{\varepsilon_j}^2)\right)}{\left((\sigma_M^2\beta_j^2\big/\sigma_{\varepsilon_j}^2)\left(1+\sigma_M^2\sum\limits_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)}\right).$$

也就是

$$((\overline{r}_k-r_f)/\beta_k-C_n)^2\Big/\left(\sigma_M^2(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_k^2\big/\sigma_{\varepsilon_k}^2)\Big/\left((\sigma_M^2\beta_k^2\big/\sigma_{\varepsilon_k}^2)\left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)$$

$$\leqslant \frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2\big/\sigma_{\varepsilon_n}^2)^2 n(n-1)}\sum_{j=1}^{n}\left(((\overline{r}_k-r_f)/\beta_k-C_j)^2\Big/\frac{\sigma_M^2\left(1+\sigma_M^2(\sum\limits_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2\big/\sigma_{\varepsilon_j}^2)\right)}{\left((\sigma_M^2\beta_j^2\big/\sigma_{\varepsilon_j}^2)\left(1+\sigma_M^2\sum\limits_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)}\right). \quad (2)$$

接下来考虑第2步操作, 将 $(\overline{r}_i-r_f)/\beta_i$, $\forall i\in\{1,2,\cdots,n\}, i\neq k$ 恢复到原状, 则恢复后 H 值会增大. 原因是 $C_n=\sigma_M^2\sum\limits_{i=1}^{n}\left((\overline{r}_i-r_f)\beta_i/\sigma_{\varepsilon_i}^2\right)\Big/\left(1+\sigma_M^2\sum\limits_{i=1}^{n}\left(\beta_i^2/\sigma_{\varepsilon_i}^2\right)\right)$ 为 $(\overline{r}_j-r_f)/\beta_j$, $\forall j\in\{1,2,\cdots,n\}$ 的增函数, 所以将 $(\overline{r}_i-r_f)/\beta_i$, $\forall i\in\{1,2,\cdots,n\}, i\neq k$ 增大到原状时 C_n 达到最大. 又由题设 $C_n<(\overline{r_k}-r_f)/\beta_k=\min\limits_{i\in\{1,2,\cdots,n\}}((\overline{r_i}-r_f)/\beta_i)$, 所以 $(\overline{r}_i-r_f)/\beta_i-C_n>0$ 在 $(\overline{r}_i-r_f)/\beta_i$, $\forall i\in\{1,2,\cdots,n\}, i\neq k$ 增大到原状的过程中始终成立. 结合定理3知, 当 $(\overline{r}_i-r_f)/\beta_i$, $\forall i\in\{1,2,\cdots,n\}, i\neq k$ 依次增大到原状时, H 值会逐渐增大.

(2)式右边为 n 种证券的证券组 H 值的 $\dfrac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}$ 倍, 于是(2)式右边

$$\begin{aligned}&\leqslant \frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}\sum_{j=1}^{n}\left(((\overline{r}_j-r_f)/\beta_j-C_j)^2\Big/\frac{\sigma_M^2\left(1+\sigma_M^2(\sum\limits_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2\big/\sigma_{\varepsilon_j}^2)\right)}{\left((\sigma_M^2\beta_j^2\big/\sigma_{\varepsilon_j}^2)\left(1+\sigma_M^2\sum\limits_{i=1}^{j}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)}\right) \quad (3)\\ &= \frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}H_n.\end{aligned}$$

(注 (3)式中的 C_j 与 (2) 式中的 C_j 是不一样的. 因 $(\overline{r}_j-r_f)/\beta_j$ 变大了.)

从而

$$\begin{aligned}H_n-H_{n-1} &= \min_{j\in\{1,2,\cdots,n\}}\left\{\left((\overline{r}_j-r_f)/\beta_j-C_n\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_j^2/\sigma_{\varepsilon_j}^2\right)\Big/\right.\right.\\ &\quad\left.\left.\left((\sigma_M^2\beta_j^2/\sigma_{\varepsilon_j}^2)\left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)\right\}\\ &\leqslant \left((\overline{r}_k-r_f)/\beta_k-C_n\right)^2\Big/\left(\sigma_M^2\left(1+\sigma_M^2(\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)-\beta_k^2/\sigma_{\varepsilon_k}^2\right)\Big/\right.\\ &\quad\left.\left((\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2)\left(1+\sigma_M^2\sum_{i=1}^{n}(\beta_i^2/\sigma_{\varepsilon_i}^2)\right)\right)\right)\\ &\leqslant \frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}H_n.\end{aligned}$$

补充一点：当$C_n < (\overline{r}_k - r_f)/\beta_k = \min\limits_{i\in\{1,2,\cdots,n\}}((\overline{r}_i - r_f)/\beta_i)$时，去掉任一种证券后的$C_n$会变小. 容易证明此时$C_n = M/N > (M-b)/(N-a) = C_n^{\text{new}}$. 其中$b = \sigma_M^2(\overline{r}_j - r_f)\beta_j\Big/\sigma_{\varepsilon_j}^2$，$a=\sigma_M^2\beta_j^2\Big/\sigma_{\varepsilon_j}^2$，$b/a = (\overline{r}_j - r_f)/\beta_j$，由于$C_n^{\text{new}} < \min\big((\overline{r_i} - r_f)/\beta_i\big)$仍成立, 所以$n$变小时$H$值改变量不等式也仍然成立.

2 结 论

定理7说明持有$C_n < (\overline{r}_k - r_f)/\beta_k = \min\limits_{i\in\{1,2,\cdots,n\}}((\overline{r}_i - r_f)/\beta_i)$表明持有证券）很大数量的证券时去掉一种最“差”的证券, H值的改变量$H_n - H_{n-1}$一般是很不明显的. 因为如果对一切$i \in \{1,2,\cdots,n\}$, 系统风险$\sigma_M^2\beta_i^2$与自身风险$\sigma_{\varepsilon_i}^2$, 则$\frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2}$将是一个不会很大的正数, 假定取为2, 则$H_n - H_{n-1} \leqslant \frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2 n(n-1)}H_n = \frac{2}{n(n-1)}H_n$, 当证券数目为$n$ =10时, 十种证券中去掉一种最“差”证券后的H值的改变倍数将比$\frac{2}{10\times 9} = 0.0222\cdots$还要小, 即十种证券和九种证券的$H$值相差不多, 如果选十种以上变化就会更小. 所以定理7告诉我们, 在选择证券组合时证券数目并不需要很多, 只要选择适量的H值意义下“好”的证券就可以了. 定理7同时也给出了如何筛选“好”的证券组的方法：只要依次去掉Treynor比率$(\overline{r}_i - r_f)/\beta_i$最小的证券, 虽不能保证得到的$H$值最大, 但可以保证风险控制在一定的范围内. 所以在选股时要尽量选择Treynor比率$(\overline{r}_i - r_f)/\beta_i$大的证券组成证券组, 再计算出该证券组的切点组合, 并与无风险证券进行组合就可以了. 如果将H值及H值改变量不等式编入行情软件中, 构造移动Treynor比率, 作为股票的技术指标, 将对选股有一定的指导意义.

定理7的不足有两点. 第一, 定理是在不能卖空的情况下得到的, 虽适合我国的证券市场, 但对可以卖空的市场, 还需进一步研究; 其次, 证券数目的多少取决于$\frac{\sigma_M^2\beta_k^2/\sigma_{\varepsilon_k}^2}{(\sigma_M^2\beta_n^2/\sigma_{\varepsilon_n}^2)^2}$的大小, 这个比值的范围需要实证数据的检验.

[参 考 文 献]

[1] 梁亦孔, 柴俊. H值意义下优良证券组合的筛选[J]. 华东师范大学学报(自然科学版), 2007(5): 92-97.

[2] 王一鸣. 数理金融经济学[M]. 北京：北京大学出版社, 2000.

[3] HUANG C F, LITZENBERGER R H. Foundations for Financial Economics[M]. Englewood Cliffs: Prentice-Hall, 1988.

[4] 宋逢明. 金融经济学基础[M]. 北京：清华大学出版社, 2003.

[5] SHARPE W F. The Sharpe ratio[J]. The Journal of Portfolio Management, 1994, 20(3): 49-58.

[6] TREYNOR J L, FISCHER B. How to use security analysis to improve portfolio selection[J]. Journal of Business, 1973(1): 66-85.

[7] 欧阳光中, 李敬湖. 证券组合与投资分析[M]. 北京：高等教育出版社, 1997.