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非线性奇摄动方程的脉冲解

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摘要: 对一类比较一般的施图姆-刘维尔型奇摄动问题, 揭示了其脉冲解的存在性, 并相应地给出了脉冲解的存在性条件. 利用边界层函数法构造其一致有效渐近解, 以及进行余项估计. 最后通过一个例子来进一步验证前面的结论.

关键词: 奇摄动; 边界层函数; 脉冲解

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Spike-like solutions of nonlinear singularly perturbed equations

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Abstract: The existence of spike-like solutions of a class of Sturm-Liouville singularly perturbed problem was investigated and its sufficient condition was given. Meanwhile, the uniformly valid asymptotic solutions were constructed by the method of boundary layer functions and remainder estimation was presented. Finally, an example was given to illustrate the obtained results.

Key words: singular perturbation; boundary layer functions; spike-like solutions

0 引 言

非线性奇摄动方程激波解的研究是当前国内外学术界十分关注的问题. 近年来国内很多学者做了很多这一方面的研究工作 [1-4], 他们主要是利用匹配法和微分不等式方法来处理一些比较特殊的问题. 其中莫嘉琪 [1]考虑了如下一类问题

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}x}\big(g(u)h(x)\frac{\mathrm{d}u}{\mathrm{d}x}\big)+\frac{f(u,\varepsilon)}{h(x)}=0\,,\quad x\in(-1,1)\,, \tag{0.1}$$

$$u(-1)=u(1)=0\,, \tag{0.2}$$

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其中 $0<\varepsilon\ll 1$, 并给出了脉冲解存在的条件.

现在我们讨论下列一类更为一般的施图姆-刘维尔型问题,

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}x}\big(p(u,x)\frac{\mathrm{d}u}{\mathrm{d}x}\big)+q(u,x)=0\,,\quad x\in(-1,1)\,, \tag{0.3}$$

$$u(-1)=u(1)=0\,. \tag{0.4}$$

令 $\varepsilon=\mu^2$, $v=\mu p(u,x)\dfrac{\mathrm{d}u}{\mathrm{d}x}$, 上述方程就转化为下列等价的吉洪诺夫方程组

$$\mu\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{v}{p(u,x)}\,,\qquad \mu\frac{\mathrm{d}v}{\mathrm{d}x}=-q(u,x)\,. \tag{0.5}$$

并作如下假设

[H_1] 函数 $p(u,x)$, $q(u,x)$ 在区域 D=$\{-1\leqslant x\leqslant 1,|u|\leqslant B\}$ 上二阶连续, 且 $p(u,x)>0$.

[H_2] 方程 $q(u,x)=0$ 有两个孤立解 $u_1=\varphi(x)$, $u_2=\phi(x)$, 其中 $\varphi(x)<\phi(x)$, 且 $q_u(\varphi(x),x)<0$, $q_u(\phi(x),x)>0$.

令 $z=(u,v)^{\mathrm{T}}$, 由文献 [5] 知

$$\overline{F}_z=\begin{pmatrix} 0 & \dfrac{1}{p(u,x)} \\ -q_u(u,x) & 0 \end{pmatrix}\,, \tag{0.6}$$

则可知坐标点 $A_1(\varphi(x),0)$, $A_2(\phi(x),0)$ 在 (u,v) 相平面上分别是鞍点和中心. 问题在 $t=-1$, $t=1$ 处均有边界层 [5]. 除此之外还在区间 (-1,1) 中的某个区域内产生脉冲解.

不妨假设在点 $x^*\in[-1,1]$ 的某个邻域内产生脉冲解, 即

$$u'(x^*,\mu)=0\,,\quad u(x^*,\mu)=\chi(x^*)\,. \tag{0.7}$$

其中点 x^* 暂时未知, 设其渐近展开式为

$$x^*=x_0+\mu x_1+\cdots\,, \tag{0.8}$$

接着利用边界层函数法分别在区间 $[0,x^*]$, $[x^*,1]$ 上构造一致有效的形式渐近解, 并根据点 $x=x^*$ 处的光滑性条件来确定转移点的位置.

1 形式渐近解构造

根据边界层函数法 [6] 构造如下形式渐近解,

$$z(x,\mu)=\bar{z}(x,\mu)+\Pi z(\tau_{-1},\mu)+Qz(\tau,\mu)+Rz(\tau_1,\mu)\,, \tag{1.1}$$

其中 $\bar{z}(x,\mu)=\bar{z}_0(x)+\mu\bar{z}_1(x)+\mu^2\bar{z}_2(x)+\cdots$, 称为正则级数部分;

$$\Pi z(\tau_{-1},\mu)=\Pi_0 z(\tau_{-1})+\mu\Pi_1 z(\tau_{-1})+\mu^2\Pi_2 z(\tau_{-1})+\cdots\,,$$

称为左边界层级数部分, $\tau_{-1}=(x+1)/\mu$;

$$Q^{(\pm)}z(\tau,\mu)=Q_0^{(\pm)}z(\tau)+\mu Q_1^{(\pm)}z(\tau)+\mu^2 Q_2^{(\pm)}z(\tau)+\cdots\,,$$

称为内部层级数部分, $\tau = (x - x^*)/\mu$;

$$Rz(\tau_1, \mu) = R_0 z(\tau_1) + \mu R_1 z(\tau_1) + \mu^2 R_2 z(\tau_1) + \cdots,$$

称为右边界层级数部分, $\tau_1 = (x-1)/\mu$. 且满足 $\Pi z(+\infty) = 0$, $Rz(-\infty) = 0$, $Q^{(\pm)} z(\pm\infty) = 0$. 在此我们主要考虑内部层的级数构造, 左右边界层级数的构造参见文献 [6].

形式解 (1.1) 代入方程组 (0.5) 得到

$$\mu \frac{\mathrm{d}\bar{u}}{\mathrm{d}x} + \frac{\mathrm{d}Q^{(\pm)}u}{\mathrm{d}\tau} = \frac{\bar{v} + Q^{(\pm)}v}{p(\bar{u} + Q^{(\pm)}u, x)}, \tag{1.2}$$

$$\mu \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} + \frac{\mathrm{d}Q^{(\pm)}v}{\mathrm{d}\tau} = -q(\bar{u} + Q^{(\pm)}u, x). \tag{1.3}$$

按快慢变量 t, τ 进行分离, 其正则级数部分为

$$\mu \frac{\mathrm{d}\bar{u}}{\mathrm{d}x} = \frac{\bar{v}}{p(\bar{u}, x)}, \quad \mu \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} = -q(\bar{u}, x). \tag{1.4}$$

零次近似

$$\bar{v}_0(x) = 0, \quad q(\bar{u}_0, x) = 0. \tag{1.5}$$

由 $[\mathrm{H}_2]$ 知 $\bar{u}_0(x) = \varphi(x)$, $\bar{v}_0(x) = 0$.

内部层零次近似 $Q_0^{(\pm)}u$ 由下面方程确定,

$$\frac{\mathrm{d}Q_0^{(\pm)}u}{\mathrm{d}\tau} = \frac{Q_0^{(\pm)}v}{p(\varphi(x_0) + Q_0^{(\pm)}u, x_0)}, \quad \frac{\mathrm{d}Q_0^{(\pm)}v}{\mathrm{d}\tau} = -q(\varphi(x_0) + Q_0^{(\pm)}u, x_0), \tag{1.6}$$

$$Q_0^{(\pm)}u(\pm\infty) = Q_0^{(\pm)}v(\pm\infty) = 0, \tag{1.7}$$

$$Q_0^{(\pm)}u(0) = \chi(x_0) - \varphi(x_0), \tag{1.8}$$

$$\frac{\mathrm{d}Q_0^{(\pm)}u}{\mathrm{d}\tau}\Big|_{\tau=0} = 0. \tag{1.9}$$

得到 $Q_0^{(\pm)}v(0) = 0$, 故 $Q_0^{(\pm)}v(\tau) \equiv 0$. 由 (1.6) 式可知

$$\frac{\mathrm{d}Q_0^{(\pm)}u}{\mathrm{d}Q_0^{(\pm)}v} = \frac{-Q_0^{(\pm)}v}{p(\varphi(x_0) + Q_0^{(\pm)}u, x_0)q(\varphi(x_0) + Q_0^{(\pm)}u, x_0)}. \tag{1.10}$$

其中 $Q_0^{(\pm)}u$, $Q_0^{(\pm)}v$ 都是含有 x_0 的未知表达式, 对 $Q_0^{(-)}u$ 上式两边同时从 $-\infty$ 积分到 0, 对 $Q_0^{(+)}u$ 上式两边从 0 积分到 $+\infty$, 结合条件 $\varphi(x_0) + Q_0^{(\pm)}u(0) = \chi(x_0)$ 得到

$[\mathrm{H}_3]$　存在 $\chi(x_0)$, 使得

$$\int_{\varphi(x_0)}^{\chi(x_0)} p(s, x_0)q(s, x_0)\mathrm{d}s = 0, \tag{1.11}$$

这里的 x_0 可以在内部层的一次近似中确定.

为此考虑一次近似

$$\frac{\mathrm{d}Q_1^{(\pm)}u}{\mathrm{d}\tau} = \frac{Q_1^{(\pm)}v}{p(\varphi(x_0) + Q_0^{(\pm)}u, x_0)} + F_1(\tau) = \frac{Q_1^{(\pm)}v}{p(\chi(x_0), x_0)} + F_1(\tau), \tag{1.12}$$

$$\frac{\mathrm{d}Q_1 v}{\mathrm{d}\tau}=-q_u(\varphi(x_0)+Q_0^{(\pm)}u,x_0)Q_1^{(\pm)}u+F_2(\tau)=-q_u(\chi(x_0),x_0)Q_1^{(\pm)}u+F_2(\tau)\,, \tag{1.13}$$

$$Q_1^{(\pm)}u(\pm\infty)=Q_1^{(\pm)}v(\pm\infty)=0\,, \tag{1.14}$$

$$\frac{\mathrm{d}Q_1^{(\pm)}u}{\mathrm{d}\tau}|_{\tau=0}=-\varphi'(x_0)\,. \tag{1.15}$$

得到

$$\frac{\mathrm{d}^2 Q_1^{(\pm)}u}{\mathrm{d}\tau^2}=\frac{1}{p(\chi(x_0),x_0)}\frac{\mathrm{d}Q_1^{(\pm)}v}{\mathrm{d}\tau}+\frac{\mathrm{d}F_1}{\mathrm{d}\tau}=-\frac{q_u(\chi(x_0),x_0)}{p(\chi(x_0),x_0)}Q_1^{(\pm)}u+F_1'+F_2\,, \tag{1.16}$$

其中 $F_1(\tau)$, $F_2(\tau)$ 是含有 $Q_0^{(\pm)}u$, $Q_0^{(\pm)}v$, $\bar{u}_i$ 和 $\bar{u}_i(i=0,1)$ 的已知函数项, 结合初始条件有

$$Q_1^{(\pm)}u(-0)=-\frac{1}{p(\chi(x_0),x_0)q(\chi(x_0),x_0)}\int_{-\infty}^{0}(F_1'(\tau)+F_2(\tau))Z(\tau)\mathrm{d}\tau\,, \tag{1.17}$$

$$Q_1^{(\pm)}u(+0)=-\frac{1}{p(\chi(x_0),x_0)q(\chi(x_0),x_0)}\int_{\infty}^{0}(F_1'(\tau)+F_2(\tau))Z(\tau)\mathrm{d}\tau\,. \tag{1.18}$$

其中 $Z(\tau)=\dfrac{\mathrm{d}Q_0^{(\pm)}u(\tau)}{\mathrm{d}\tau}$. 因 $Q_1^{(-)}u(-0)=Q_1^{(+)}u(+0)$, 故有

$$\Phi(x_0)=\int_{-\infty}^{+\infty}(F_1'(\tau)+F_2(\tau))Z(\tau)\mathrm{d}\tau=0\,. \tag{1.19}$$

[H_4] 假设由方程 (1.19) 可以确定 x_0 , 且 $\Phi'(x_0)\neq 0$.

至此, 转移点的零次近似 x_0 确定, 故内部层级数的零次近似 $Q_0^{(\pm)}u(\tau)$ 确定, 一次近似 $Q_1^{(\pm)}u(\tau)$ 可以根据相类似的过程确定, 并有下列结论:

定理 若条件 [H_1] – [H_4] 下, 对充分小的 $\mu>0$, 问题 (0.3), (0.4) 必存在唯一解 $u(x,\mu)$, 且具有下列形式:

$$u(x,\mu)=\begin{cases}\varphi(x)+\Pi_0 u(\tau_{-1})+Q_0^{(\pm)}u(\tau)+R_0 u(\tau_1)\,, & x\in[0,x_0)\cup(x_0,1]\,,\\ \chi(x_0)\,, & x=x_0\,.\end{cases} \tag{1.20}$$

注 在这里我们不给出证明 (具体过程可见文献 [6]). 通过一个具体的算例[7]来验证前面定理.

2 例 子

考虑如下问题

$$\varepsilon\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}+u^2-1=0\,,\quad u(-1)=u(1)=0\,. \tag{2.1}$$

经过变化后, 原问题转化为下列等价的 Tikhnov 系统形式

$$\mu\frac{\mathrm{d}u}{\mathrm{d}x}=v\,,\quad \mu\frac{\mathrm{d}v}{\mathrm{d}x}=1-u^2\,, \tag{2.2}$$

$$u(-1)=u(1)=0\,. \tag{2.3}$$

令

$$u(x,\mu)=\begin{cases}\bar{u}_0(x,\mu)+\Pi_0 u(\tau_{-1})+Q_0^{(-)}u(\tau)\,, & x\in[-1,x_0]\,,\\ \bar{u}_0(x,\mu)+R_0 u(\tau_1)+Q_0^{(+)}u(\tau)\,, & x\in[x_0,1]\,.\end{cases} \tag{2.4}$$

$$v(x,\mu)=\begin{cases}\bar{v}_0(x,\mu)+\Pi_0 v(\tau_{-1})+Q_0^{(-)}v(\tau)\,, & x\in[-1,x_0]\,,\\ \bar{v}_0(x,\mu)+R_0 v(\tau_1)+Q_0^{(+)}v(\tau)\,, & x\in[x_0,1]\,.\end{cases} \tag{2.5}$$

先考虑正则部分

$$\mu\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}=\bar{v}\,,\quad \mu\frac{\mathrm{d}\bar{v}}{\mathrm{d}x}=1-\bar{u}^2\,. \tag{2.6}$$

其零次近似

$$\bar{v}_0=0\,,\quad 1-\bar{u}_0^2=0\,. \tag{2.7}$$

可得

$$\bar{v}_0(x)=0\,,\quad \varphi(x)=-1,\phi(x)=1\,. \tag{2.8}$$

一次近似可由下列方程确定,

$$\frac{\mathrm{d}\bar{u}_0(x)}{\mathrm{d}x}=\bar{v}_1\,,\quad \frac{\mathrm{d}\bar{v}_0(x)}{\mathrm{d}x}=-2\bar{u}_0(x)\bar{u}_1\,. \tag{2.9}$$

解得

$$\bar{u}_1(x)=0\,,\quad \bar{v}_1(x)=0\,. \tag{2.10}$$

类似地, 我们可以得到 k 阶近似.

对于左边界层 $\Pi u(\tau_{-1})$ 和 $\Pi v(\tau_{-1})$

$$\frac{\mathrm{d}\Pi u}{\mathrm{d}\tau_{-1}}=\Pi v,\quad \frac{\mathrm{d}\Pi v}{\mathrm{d}\tau_{-1}}=-2\bar{u}\Pi u-\Pi^2 u\,. \tag{2.11}$$

零次近似为

$$\frac{\mathrm{d}\Pi_0 u}{\mathrm{d}\tau_{-1}}=\Pi_0 v\,,\quad \frac{\mathrm{d}\Pi_0 v}{\mathrm{d}\tau_{-1}}=-2\bar{u}_0(-1)\Pi_0 u-\Pi_0^2 u=2\Pi_0 u-\Pi_0^2 u\,. \tag{2.12}$$

利用条件 $\Pi_0 u(0)=1$, $\Pi_0 u(+\infty)=0$, $\Pi_0 v(+\infty)=0$. 可得 $\Pi_0 u(\tau_{-1})$, $\Pi_0 v(\tau_{-1})$(见文献[9]).

类似地可到右边界层 $R_0 u(\tau_1)$ 和 $R_0 v(\tau_1)$.

内部层 $Q_0^{(\pm)}u(\tau)$ 和 $Q_0^{(\pm)}v(\tau)$ 可由下列方程确定

$$\frac{\mathrm{d}Q^{(\pm)}u}{\mathrm{d}\tau}=Q^{(\pm)}v\,,\quad \frac{\mathrm{d}Q^{(\pm)}v}{\mathrm{d}\tau}=-2\bar{u}Q^{(\pm)}u-(Q^{(\pm)}u)^2\,. \tag{2.13}$$

零次近似

$$\frac{\mathrm{d}Q_0^{(\pm)}u}{\mathrm{d}\tau}=Q_0^{(\pm)}v\,,\frac{\mathrm{d}Q_0^{(\pm)}v}{\mathrm{d}\tau}=2Q_0^{(\pm)}u-(Q_0^{(\pm)}u)^2\,. \tag{2.14}$$

根据 $[\mathrm{H}_3]$ 有

$$\int_{-1}^{\chi(x_0)}(1-s^2)\mathrm{d}s=0\,, \tag{2.15}$$

可知 $\chi(x_0)=2$, 故 $Q_0^{(\pm)}u(0)=3$, 则可根据边界层函数的性质确定 $Q_0^{(\pm)}u(\tau)$, $Q_0^{(\pm)}v(\tau)$. 其中 x_0 可由条件 $[\mathrm{H}_4]$ 确定的, 具体可见文献[8]. 至此问题 (0.3), (0.4) 的零次渐近表达式确定, 更高阶的渐近表达式可以类似得到.

[参 考 文 献]

[1] 莫嘉琪. 一个非线性方程的渐近激波解 [J]. 数学物理学报, 2004, 24A(2): 164-167.
MO J Q. Asymptotic skock solution for a nonlinear equation [J]. Acta Mathematica Scientia, 2004, 24A(2): 164-167.

[2] 韩祥临. 一类非线性方程的渐近激波解 [J]. 数学物理学报, 2003, 23A(2): 253-256.
HAN X L. The skock solution for a class of nonlinear equations [J]. Acta Mathematica Scientia, 2003, 23A(2): 253-256.

[3] 韩祥临, 莫嘉琪. 一类拟线性边值问题的激波解 [J]. 应用数学学报, 2003, 16(2): 130-132.
HAN X L, MO J Q. A class of shock solution for quasilinear boundary value problem[J]. Mathematica Applicata, 2003, 16(2): 130-132.

[4] TANG R. The shock problems for a class of nonlinear singularly perturbed equations [J]. Advances Mathematics. 2005, 34(2): 20-22.

[5] VASIL'EVA A B, BUTUZOV V F. Asymptotic Expansions of Singularly Perturbed Differential Equations [M]. Moscow: Nauka, 1973.

[6] VASIL'EVA A B, BUTUZOV V F, KALACHEV L V. The Boundary Funcation Method For Singular Perturbation Problem [M]. Philadelphia: SIAM studies in Applied Mathematics, 1995.

[7] O'MALLEY R E Jr. Phase-plane solution to singular perturbation problems [J]. J Math Anal Appl, 1976: 54; 449-466.

[8] 王爱峰. 二阶非线性奇摄动方程脉冲状空间对照结构 [D]. 上海: 华东师范大学数学系, 2005.
WANG A F. Contrast space structure of spike-like solutions for the second order nonlinear singular perturbed equations [D]. Shanghai: East China Normal University, 2005.

[9] CODDINGTON E A, LEVINSON N. Theory of Ordinary Differential Equation [M], New York, McGraw-Hill Book Company, Inc, 1955.

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[References]

[1] RANDIC M. On dissection of acyclic graphs[J]. MATCH Commun Math Comput Chem, 1979(5): 135-148.

[2] RANDIC M, GUO X F, CALKINS P. Graph dissetion revisited: application to smaller alikanes[J]. Acta Chim Slov, 2000, 47: 489-506.

[3] HU C, XU L. Developing molecular identification numbers by an all-paths method[J]. J Chem Inf Comput Sci, 1997, 37(2): 311-315.

[4] RANDIC M, WOODWORTH W L. Characterization of acyclic graphs by successive dissection[J]. MATCH Commun Math Comput Chem, 1982, 13: 291-313.

[5] XU Z X, WU B, GUO X F. On dissection of graphs[J]. MATCH Commun. Math Comput Chem, 2006, 56: 519-526.

[6] BONDY J A, MURTY U S R. Graph Theory with Applications[M]. London: Macmillan, 1976.