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Comparison theorems of stochastic difference equations

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Abstract: This paper studied a kind of nonlinear stochastic difference equations, whose randomness is driven by a stochastic series. Two comparison theorems were obtained. At last, p-moment stability and p-moment boundedness of solutions to stochastic difference equations were presented as applications of the comparison theorems.

Key words: stochastic difference equation; comparison theorem; p-moment stability; p-moment boundedness

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随机差分方程的比较定理

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摘要: 研究一类由随机序列驱动的非线性随机差分方程; 给出了该类方程的两个比较定理; 并作为比较定理的应用, 给出了随机差分方程解的 p-阶矩稳定和 p-阶矩有界的判别条件.

关键词: 随机差分方程; 比较定理; p-阶矩稳定性; p-阶矩有界性

0 Introduction

Stochastic differential equations play an important role in areas such as option pricing, forecast of the growth of population, etc[1]. When using stochastic differential equations to solve problems, we generally change them into stochastic difference equations in discrete forms. Actually, there are some results on stochastic difference equations[2−9], most of which are on the stability of the equations[2−6]. This paper first gave two comparison theorems of stochastic difference equations. Then p-moment stability and p-moment boundedness of solutions to stochastic difference equations were presented as applications of the comparison theorems.

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The paper was organized as follows: Notations and definitions were given in Section 1, then two comparison theorems of stochastic difference equations were obtained in Section 2, And some applications were presented in Section 3.

1 Notations and definitions

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\geqslant 0}$. A general form of stochastic difference equations follows as

$$x_{n+1}(\omega)=f(n,x_n(\omega),\xi_n(\omega)),\qquad x_{n_0}(\omega)=x_0(\omega),\tag{1.1}$$

where $\xi_n:\mathbf{N}^+\times\Omega\to\mathbf{R}^\ell$ is an ℓ-dimensional random sequence, $f:\mathbf{N}^+\times\mathbf{R}^m\times\mathbf{R}^\ell\to\mathbf{R}^m$, $\mathbf{N}^+=\{n_0,n_0+1,n_0+2,\cdots\}$, and f is continuous on the second variable.

It is obvious that there exists a unique random process satisfying systems (1.1) if $f(n,\cdot,\xi_n(\omega))$ is reasonable for all $n\in\mathbf{N}^+$. Furthermore, the random process determined by system (1.1) is a Markov chain if ξ_n is an ℓ-dimensional Markov chain independent of $x_0(\omega)$.

For the sake of simplification, we use the following notations.

Notation 1 Let $\boldsymbol{B}$ be a vector or matrix. Then

(i) By $\boldsymbol{B}\leqslant 0$ we mean each element of $\boldsymbol{B}$ is *non-positive*.

(ii) By $\boldsymbol{B}<0$ we mean $\boldsymbol{B}\leqslant 0$ and at least one element of $\boldsymbol{B}$ is *negative*.

(iii) By $\boldsymbol{B}\ll 0$ we mean all elements of $\boldsymbol{B}$ are *negative*.

Notation 2 Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be two vectors or matrixes with same dimensions. Denote by $\boldsymbol{B}_1\leqslant\boldsymbol{B}_2$, $\boldsymbol{B}_1<\boldsymbol{B}_2$ and $\boldsymbol{B}_1\ll\boldsymbol{B}_2$ if and only if $\boldsymbol{B}_1-\boldsymbol{B}_2\leqslant 0$, $\boldsymbol{B}_1-\boldsymbol{B}_2<0$ and $\boldsymbol{B}_1-\mathbf{B_2}\ll\mathbf{0}$, respectively. We can note reverse relation similar to above notations.

Now we give three definitions.

Definition 1 A function $g(n,u,v):\mathbf{N}^+\times\mathbf{R}^k\times\mathbf{R}^\ell\to\mathbf{R}^k$ is said to be

(i) *quasi-nondecreasing* on u if for any $j\in\{1,2,\cdots,k\}$, $g_j(n,u,v)$ is nondecreasing on u_j;

(ii) *quasi-increasing* on u if for any $j\in\{1,2,\cdots,k\}$, $g_j(n,u,v)$ is increasing on u_j.

Definition 2 Assume that $f(n,0,\cdot)\equiv 0$ for all $n\in\mathbf{N}$ and $p>0$. The zero solution of system (1.1) is

(i) *p-moment stable* if for any $\varepsilon>0$, there exists a $\delta=\delta(n_0)>0$ such that

$$E|x_n(\omega)|^p<\varepsilon\qquad\text{for all } n\geqslant n_0 \text{ and } E|x_0(\omega)|^p<\delta;$$

(ii) *uniformly p-moment stable* if the δ in (i) is independent of n_0;

(iii) *asymptotically p-moment stable* if it is p-moment stable and there exists a $\delta_0>0$ and $N=N(n_0,\varepsilon)\in\mathbf{N}$ such that

$$E|x_n(\omega)|^p<\varepsilon\qquad\text{for all } n\geqslant n_0+N \text{ and } E|x_0(\omega)|^p<\delta_0;$$

(iv) *uniformly asymptotically p-moment stable* if it is uniformly p-moment stable and the N in (iii) is independent of n_0.

Definition 3 Let $p>0$. System (1.1) is called

(i) *p-moment boundedness* if for any $B_1 > 0$ and $n_0 \in \mathbf{N}^+$, there exists a $B_2 = B_2(B_1, n_0) > 0$ satisfying that $E|x_n(\omega)|^p < B_2$ for all $n \geqslant n_0$ and $|x_{n_0}|^p < B_1$, where and in the sequel, (n_0, x_{n_0}) is the initial value of system (1.1);

(ii) *uniform p-moment boundedness* if the B_2 in (i) is independent of n_0;

(iii) *ultimate p-moment boundedness* if there exists a $B > 0$ satisfying that, for any $B_3 > 0$ and $n_0 \in \mathbf{N}^+$, there exists a $N = N(n_0, B_3) > 0$ such that $E|x_n(\omega)|^p < B$ for all $n \geqslant n_0 + N$ and $|x_{n_0}|^p < B_3$;

(iv) *uniformly ultimate p-moment boundedness* if N in (iii) is independent of n_0.

2 Main results

To establish main results, we introduce a comparison equation

$$y_{n+1}(\omega) = g(n, y_n(\omega), \xi_n(\omega)), \qquad y_{n_0}(\omega) = y_0(\omega), \tag{2.1}$$

where $\xi_n : \Omega \to \mathbf{R}^\ell$ is an ℓ-dimensional random sequence, $g : \mathbf{N}^+ \times \mathbf{R}^k \times \mathbf{R}^\ell \to \mathbf{R}^k$, $\mathbf{N}^+ = \{n_0, n_0+1, n_0+2, \cdots\}$, and g is continuous on the second variable.

Theorem 1 Let $\varphi(\cdot) : \mathbf{R}^m \to \mathbf{R}^k$, $x_n(\omega)$ and $y_n(\omega)$ respectively satisfy stochastic difference equations (1.1) and (2.1), we have the results that

(i) if $g(n,\cdot,\cdot)$ is quasi-nondecreasing on the second variable and

$$\varphi(f(n,u,v)) \leqslant g(n,\varphi(u),v) \qquad \text{for all } n \in \mathbf{N}^+, u \in \mathbf{R}^m, v \in \mathbf{R}^k,$$

then $\varphi(x_0(\omega)) \leqslant y_0(\omega)$ a.s. implies $\varphi(x_n(\omega)) \leqslant y_n(\omega)$ a.s. for all $n \geqslant n_0$;

(ii) if $g(n,\cdot,\cdot)$ is quasi-nondecreasing on the second variable and

$$\varphi(f(n,u,v)) < g(n,\varphi(u),v) \qquad \text{for all } n \in \mathbf{N}^+, u \in \mathbf{R}^m, v \in \mathbf{R}^k,$$

then $\varphi(x_0(\omega)) < y_0(\omega)$ a.s. implies $\varphi(x_n(\omega)) < y_n(\omega)$ a.s. for all $n \geqslant n_0$;

(iii) if $g(n,\cdot,\cdot)$ is quasi-increasing on the second variable and

$$\varphi(f(n,u,v)) \leqslant g(n,\varphi(u),v) \qquad \text{for all } n \in \mathbf{N}^+, u \in \mathbf{R}^m, v \in \mathbf{R}^k,$$

then $\varphi(x_0(\omega)) \ll y_0(\omega)$ a.s. implies $\varphi(x_n(\omega)) \ll y_n(\omega)$ a.s. for all $n \geqslant n_0$.

Proof Suppose conclusion (i) is not true. Considering $\varphi(x_0(\omega)) \leqslant y_0(\omega)$ for all $\omega \in \Omega \setminus N$, where N is the union set of all sets of zero-measure in Ω, there exists at least a $\tilde{n} \in \mathbf{N}^+$, a $j \in \{1, 2, \cdots, k\}$ and $\Omega_j \subset \Omega$ such that

(a) $P(\Omega_j) > 0$; (b) $\varphi_j(x_{\tilde{n}+1}(\omega)) > y^j_{\tilde{n}+1}(\omega)$, $\omega \in \Omega_j$;

(c) $\varphi(x_n(\omega)) \leqslant y_n(\omega)$ for all $n_0 \leqslant n \leqslant \tilde{n}$, $\omega \in \Omega_j$.

From (c), we have

$$\varphi_i(x_{\tilde{n}}(\omega)) \leqslant y^i_{\tilde{n}}(\omega) \qquad \text{for all } i = 1, 2, \cdots, k.$$

Let vectors $w_i = (w_i^1, w_i^2, \cdots, w_i^k)^{\mathrm{T}}$ with each element as

$$\boldsymbol{w}_i^s = \begin{cases} y^s_{\tilde{n}}(\omega), & 1 \leqslant s \leqslant i \\ \varphi^s_{\tilde{n}}(x_{\tilde{n}}(\omega)), & i+1 \leqslant s \leqslant k \end{cases}, \ i = 1, \cdots, k,$$

then

$$\varphi(x_{\tilde{n}}(\omega)) \leqslant \boldsymbol{w}_1 \leqslant \boldsymbol{w}_2 \leqslant \cdots \leqslant \boldsymbol{w}_k = y_{\tilde{n}}(\omega).$$

By the definition of quasi-nondecreasing and mathematical induction, we have

$$g(\tilde{n}, \varphi(x_{\tilde{n}}(\omega)), \xi_{\tilde{n}}(\omega)) \leqslant g(\tilde{n}, w_1, \xi_{\tilde{n}}(\omega)) \leqslant \cdots \leqslant g(\tilde{n}, y_{\tilde{n}}(\omega), \xi_{\tilde{n}}(\omega)),$$

i.e.

$$g(\tilde{n}, \varphi(x_{\tilde{n}}(\omega)), \xi_{\tilde{n}}(\omega)) \leqslant g(\tilde{n}, y_{\tilde{n}}(\omega), \xi_{\tilde{n}}(\omega)). \tag{2.2}$$

Because of the conditions (i) of Theorem 1 and (2.2), then we have

$$\begin{aligned} \varphi_j(x_{n_1}(\omega)) &= \varphi_j(f(n_1-1, x_{n_1-1}(\omega), \xi_{n_1-1}(\omega))) \\ &\leqslant g_j(n_1-1, \varphi(x_{n_1-1}(\omega)), \xi_{n_1-1}(\omega)) \\ &\leqslant g_j(n_1-1, y_{n_1-1}(\omega), \xi_{n_1-1}(\omega)) \\ &\leqslant y^j_{n_1}(\omega) \qquad \text{for all } \omega \in \Omega_j \setminus N, \quad n_1 = \tilde{n}+1, \end{aligned}$$

which is a contradiction with (b), so conclusion (i) is true.

If conclusion (iii) is not true, considering $\varphi(x_0(\omega)) \ll y_0(\omega)$ for all $\omega \in \Omega \setminus N$, where N is the union set of all sets of zero-measure in Ω, there at least exists a $\tilde{n} \in \mathbf{N}^+$, a $j \in \{1, 2, \cdots, k\}$ and $\Omega_j \subset \Omega$ such that

(a) $P(\Omega_j) > 0$; (b) $\varphi_j(x_{\tilde{n}+1}(\omega)) \geqslant y^j_{\tilde{n}+1}(\omega),\ \omega \in \Omega_j$;

(c) $\varphi(x_n(\omega)) \ll y_n(\omega)$ for all $n_0 \leqslant n \leqslant \tilde{n},\ \omega \in \Omega_j$.

From (c), we have

$$\varphi_i(x_{\tilde{n}}(\omega)) < y^i_{\tilde{n}}(\omega) \qquad \text{for all } i = 1, 2, \cdots, k.$$

Let vectors $w_i = (w_i^1, w_i^2, \cdots, w_i^k)^{\mathrm{T}}$ with each element as

$$\boldsymbol{w}_i^s = \begin{cases} y^s_{\tilde{n}}(\omega), & 1 \leqslant s \leqslant i \\ \varphi^s_{\tilde{n}}(x_{\tilde{n}}(\omega)), & i+1 \leqslant s \leqslant k \end{cases}, \ i = 1, \cdots, k,$$

then

$$\varphi(x_{\tilde{n}}(\omega)) < \boldsymbol{w}_1 < \boldsymbol{w}_2 < \cdots < \boldsymbol{w}_k = y_{\tilde{n}}(\omega).$$

By the definition of quasi-increasing and mathematical induction, we have

$$g(\tilde{n}, \varphi(x_{\tilde{n}}(\omega), \xi_{\tilde{n}}(\omega)) \ll g(\tilde{n}, w_1, \xi_{\tilde{n}}(\omega)) \ll \cdots \ll g(\tilde{n}, y_{\tilde{n}}(\omega), \xi_{\tilde{n}}(\omega)),$$

i.e.

$$g(\tilde{n}, \varphi(x_{\tilde{n}}(\omega), \xi_{\tilde{n}}(\omega)) \ll g(\tilde{n}, y_{\tilde{n}}(\omega), \xi_{\tilde{n}}(\omega)). \tag{2.3}$$

Because of the conditions (i) of Theorem 1 and (2.3), then we have

$$\begin{aligned} \varphi_j(x_{n_1}(\omega)) &= \varphi_j(f(n_1-1, x_{n_1-1}(\omega), \xi_{n_1-1}(\omega))) \\ &\leqslant g_j(n_1-1, \varphi(x_{n_1-1}(\omega)), \xi_{n_1-1}(\omega)) \\ &< g_j(n_1-1, y_{n_1-1}(\omega), \xi_{n_1-1}(\omega)) \\ &\leqslant y^j_{n_1}(\omega), \qquad \text{for all } \omega \in \Omega_j \setminus N, \quad n_1 = \tilde{n}+1, \end{aligned}$$

which is a contradiction with (b), so conclusion (iii) is true.

The proof of conclusion (ii) is easy to obtain in a way similar to those of (i) and (iii), so we omit it. The proof is completed.

Theorem 2 Assume that $x_n(\omega)$and $y_n(\omega)$ are random processes respectively determined by stochastic difference equations (1.1) and (2.1), $f, g : \mathbf{N}^+ \times \mathbf{R}^m \times \mathbf{R}^\ell \to \mathbf{R}^m$, then we have

(i) if $f(n,u,v)$ (or $g(n,u,v)$) is quasi-nondecreasing on u and

$$f(n,u,v) \leqslant g(n,u,v) \qquad \text{for all } n \in \mathbf{N}^+, u \in \mathbf{R}^m, v \in \mathbf{R}^\ell,$$

then $x_0(\omega) \leqslant y_0(\omega)$ a.s. implies $x_n(\omega) \leqslant y_n(\omega)$ a.s. for all $n \geqslant n_0$;

(ii) if $f(n,u,v)$ (or $g(n,u,v)$) is quasi-nondecreasing on u and

$$f(n,u,v) < g(n,u,v) \qquad \text{for all } n \in \mathbf{N}^+, u \in \mathbf{R}^m, v \in \mathbf{R}^\ell,$$

then $x_0(\omega) < y_0(\omega)$ a.s. implies $x_n(\omega) < y_n(\omega)$ a.s. for all $n \geqslant n_0$;

(iii) if $f(n,u,v)$ (or $g(n,u,v)$) is quasi-increasing on u and

$$f(n,u,v) \ll g(n,u,v) \qquad \text{for all } n \in \mathbf{N}^+, u \in \mathbf{R}^m, v \in \mathbf{R}^\ell,$$

then $x_0(\omega) \ll y_0(\omega)$ a.s. implies $x_n(\omega) \ll y_n(\omega)$ a.s. for all $n \geqslant n_0$.

The proof is similar to that of Theorem 1, so we omit it.

3 Applications

In this section, we gave some examples to show application of obtained results in researching stability and boundedness of stochastic difference equations.

Consider stochastic difference equations

$$y_{n+1}(\omega) = g(n, y_n(\omega), \xi_n(\omega)), \tag{3.1}$$

where $g(n,u,v) : \mathbf{N}^+ \times [0,+\infty) \times \mathbf{R}^\ell \to \mathbf{R}$ and ξ_n be a random sequence from Ω to $\mathbf{R}^\ell$.

Theorem 3 Let $p > 0$ and $g(n,u,v)$ be nondecreasing on u, $g(n,0,\cdot) \equiv 0$ for all $n \in \mathbf{N}$, and satisfy

$$|f(n,u,\xi_n(\omega))| \leqslant g(n,|u|,\xi_n(\omega)) \text{ a.s.},$$

where $\{\xi_n\}_{n=0,1,2,\cdots}$ is a stochastic sequence and $|\cdot|$ is some norm in $\mathbf{R}^m$, then

(i) (uniform) p-moment stability of system (3.1) implies (uniform) p-moment stability of system (1.1).

(ii) (uniformly) asymptotic p-moment stability of system (3.1) implies (uniformly) asymptotic p-moment stability of system (1.1).

Proof (i) Let $\varphi(\cdot) = |\cdot|$ and $y_0(\omega) = |x_0(\omega)|$, by Theorem 1, we have

$$|x_n(\omega)| \leqslant y_n(\omega) \qquad \text{for all } n \geqslant n_0. \tag{3.2}$$

Because system (3.1) is p-moment stable, for any $\varepsilon > 0$, there exists a $\delta = \delta(n_0) > 0$ such that

$$E|y_n(\omega)|^p < \varepsilon \qquad \text{for all } n \geqslant n_0 \text{ and } E|y_0(\omega)|^p < \delta.$$

From (3.2), we know

$$|x_n(\omega)|^p \leqslant |y_n(\omega)|^p \qquad \text{for all } n \geqslant n_0,$$

and $E|x_0(\omega)|^p = E|y_0(\omega)|^p$. Thus $E|x_n(\omega)|^p \leqslant E|y_n(\omega)|^p$ for all $n \geqslant n_0$, and that is

$$E|x_n(\omega)|^p < \varepsilon \qquad \text{for all } n \geqslant n_0 \text{ and } E|x_0(\omega)|^p < \delta.$$

Thus, system (1.1) is p-moment stable.

If the system (3.1) is uniformly p-moment stable, then δ is independent of n_0, so the system (1.1) is uniformly p-moment stable, too.

(ii) If system (3.1) is asymptotically p-moment stable, from (i), we know system (1.1) is p-moment stable.

Furthermore, because system (3.1) is asymptotically p-moment stable, there exists a $\delta_0 > 0$ satisfying that for any $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that

$$E|y_n(\omega)|^p < \varepsilon \qquad \text{for all } n \geqslant N \text{ and } E|y_0(\omega)|^p < \delta_0.$$

From (3.2), we know $|x_n(\omega)|^p \leqslant |y_n(\omega)|^p$ for all $n \geqslant n_0$. Thus $E|x_n(\omega)|^p \leqslant E|y_n(\omega)|^p$ for all $n \geqslant n_0$, and $E|x_0(\omega)|^p = E|y_0(\omega)|^p$. That is

$$E|x_n(\omega)|^p < \varepsilon \qquad \text{for all } n \geqslant N \text{ and } E|x_0(\omega)|^p < \delta_0.$$

Thus, system (1.1) is asymptotically p-moment stable.

If the system (3.1) is uniformly asymptotically p-moment stable, then N is independent of n_0, so the system (1.1) is uniformly asymptotically p-moment stable, too. The proof is completed.

Theorem 4 Let $p > 0$ and $g(n, u, v)$ be nondecreasing on u and satisfy

$$|f(n, u, z_n(\omega))| \leqslant g(n, |u|, z_n(\omega)) \text{ a.s.},$$

where $\{z_n\}_{n=0,1,2,\cdots}$ is a stochastic sequence and $|\cdot|$ is some norm in R^m, then

(i) (uniform) p-moment boundedness of system (3.1) implies (uniform) p-moment boundedness of system (1.1);

(ii) (uniformly) ultimate p-moment boundedness of system (3.1) implies (uniformly) ultimate p-moment boundedness of system (1.1).

Proof Let $\varphi(\cdot) = |\cdot|$ and $y_0(\omega) = |x_0(\omega)|$, by Theorem 1, we have

$$|x_n(\omega)| \leqslant y_n(\omega) \qquad \text{for all } n \geqslant n_0. \tag{3.3}$$

(i) Because system (3.1) is p-moment bounded, we have that for any $B_1 > 0$ and $n_0 \in N^+$, there exists a $B_2 = B_2(B_1, n_0) > 0$ satisfying that

$$E|y_n(\omega)|^p < B_2 \qquad \text{for all } n \geqslant n_0.$$

From (3.3), we know

$$E|x_n(\omega)|^p \leqslant B_2 \qquad \text{for all } n \geqslant n_0.$$

Thus, system (1.1) is p-moment bounded.

If the system (3.1) is uniformly p-moment bounded, then B_2 is independent of n_0, so the system (1.1) is uniformly p-moment bounded, too.

(ii) Because system (3.1) is ultimately p-moment bounded, there exists a $B>0$ satisfying that, for any $B_3>0$ and $n_0\in\mathbf{R}^+$, there exists a $N=N(n_0,\ B_3)>0$ such that

$$E|y_n(\omega)|^p<\ B \qquad \text{for all } n\geqslant n_0+N,$$

when $|x_{n_0}|^p<\ B_3$. From (3.2), we know

$$E|x_n(\omega)|^p\leqslant\ B \qquad \text{for all } n\geqslant n_0+N.$$

Thus, system (1.1) is ultimate p-moment bounded.

If the system (3.1) is uniformly ultimately p-moment bounded, then N is independent of n_0, so the system (1.1) is uniformly ultimately p-moment bounded, too.

In the following, we presented an example to show the application of Theorem 1.

Example 1 Consider the following stochastic difference equations

$$\left\{\begin{aligned} x_{n+1}^{(1)} &= f(n,x_n,\xi_n)=\frac{nx_n^{(1)}}{2n+1+\xi_n^2}+\frac{nx_n^{(2)}}{2n+2+\xi_n^2},\\ x_{n+1}^{(2)} &= f(n,x_n,\xi_n)=\frac{nx_n^{(2)}}{2n+2+\xi_n^2}+\frac{nx_n^{(3)}}{2n+3+\xi_n^2},\\ &\vdots\\ x_{n+1}^{(m-1)} &= f(n,x_n,\xi_n)=\frac{nx_n^{(m-1)}}{2n+m-1+\xi_n^2}+\frac{nx_n^{(m)}}{2n+m+\xi_n^2},\\ x_{n+1}^{(m)} &= f(n,x_n,\xi_n)=\frac{nx_n^{(m)}}{2n+m+\xi_n^2}+\frac{nx_n^{(1)}}{2n+1+\xi_n^2}, \end{aligned}\right. \tag{3.4}$$

and

$$y_{n+1}=g(n,y_n,\xi_n)=\left[\frac{2n}{2n+\xi_n^2}\right]^2\cdot y_n, \tag{3.5}$$

where $\xi_n:\Omega\to\mathbf{R}$ is a random sequence.

Put $\varphi(u)=[u^{(1)}]^2+[u^{(2)}]^2+\cdots+[u^{(m)}]^2$ in Theorem 1, we obtain that

$$\begin{aligned} &\varphi(f(n,u,v))\\ &\quad=\left[\frac{nu^{(1)}}{2n+1+v^2}+\frac{nu^{(2)}}{2n+2+v^2}\right]^2+\left[\frac{nu^{(2)}}{2n+2+v^2}+\frac{nu^{(3)}}{2n+3+v^2}\right]^2+\cdots\\ &\qquad+\left[\frac{nu^{(m-1)}}{2n+m-1+v^2}+\frac{nu^{(m)}}{2n+m+v^2}\right]^2+\left[\frac{nu^{(m)}}{2n+m+v^2}+\frac{nu^{(1)}}{2n+1+v^2}\right]^2\\ &\quad\leqslant\left[\frac{2n}{2n+v^2}\right]^2\cdot\sum_{s=1}^m[u^{(s)}]^2\\ &\quad=g(n,\varphi(u),v). \end{aligned} \tag{3.6}$$

Further, from (3.5) we know that

$$y_{n+1}=\prod_{s=1}^{n+1}\left[\frac{2s}{2s+\xi_s^2}\right]^2\cdot y_0,$$

then

$$E[y_{n+1}^2] = E\left(\prod_{s=1}^{n+1}\left[\frac{2s}{2s+\xi_s^2}\right]^2 \cdot y_0\right)^2 \leqslant E[y_0^2]. \tag{3.7}$$

It follows from (3.7) that system (3.5) is uniformly stable in mean square. Thus, System (3.4) is uniformly stable in mean square by (3.6), Theorem 1 and Definition 2.

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