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Admissibility of Linear Estimators in Multivariate Linear Models with Respect to an Incomplete Ellipsoidal Restriction *

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Abstract

This paper studies the admissibility of linear estimators in multivariate linear models with respect to an incomplete ellipsoidal restriction $\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)\le\sigma^2$. Specifically, we study the influence of the matrix N and Θ_1 which is the center of a restricted set to the admissibility of linear estimators in multivariate linear models with respect to the incomplete ellipsoidal restriction $\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)\le\sigma^2$. The main results show that the class of admissible linear estimators with the restriction $\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)\le\sigma^2$ is the same as the one with the restriction $\mathrm{tr}(\Theta-\Theta_2)'N(\Theta-\Theta_2)\le\sigma^2$ for Θ_1 and Θ_2 with certain relationship.

Keywords: Admissibility, incomplete ellipsoidal restriction, linear estimators, multivariate linear models.

AMS Subject Classification: 62C05, 62C15, 62F10.

§1. Introduction and Notations

We consider the following multivariate linear model

$$Y=X\Theta+\varepsilon, \tag{1.1}$$

where Y is an $n\times p$ random matrix, $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_p)$, $\varepsilon_i\sim\mathrm{IID}(0,\sigma^2V)$, where V is a known $n\times n$ error matrix, X is a known $n\times q$ design matrix. The unknown parameters $(\Theta,\sigma^2)\in\mathrm{T}$ where $\mathrm{T}\subseteq\mathbf{R}^{q\times p}\times\mathrm{R}^+$. For simplicity, we denotes by $(Y,X\Theta,\sigma^2\,|\,(\Theta,\sigma^2)\in\mathrm{T})$ the above model. The reader is referred to the literatures: for example [1–7] for more details on the admissibility of linear estimators in multivariate linear models.

For the case with $p=1$, the admissibility of linear estimators has received great attention, see [2]. In recent, it has made some progress on the admissibility of linear estimators in linear models with respect to an incomplete ellipsoidal restriction or an

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inequality restriction, see [7–9]. This paper studies the admissibility of linear estimators for model (1.1) with $p > 1$ when the unknown parameters satisfy the following restriction:

$$H(N, \Theta_1) = \{(\Theta, \sigma^2) : \mathrm{tr}(\Theta - \Theta_1)' N(\Theta - \Theta_1) \le \sigma^2,\ N \ge 0,\ \Theta_1 \text{ known}\}.$$

For simplicity, the model (1.1) with this above restriction can be denoted by $(Y, X\Theta, \sigma^2 \,|\, (\Theta, \sigma^2) \in H(N, \Theta_1))$. Specifically, we study the influence of the matrix N and Θ_1 which is the center of a restricted set to the admissibility of linear estimators in multivariate linear models with respect to the incomplete ellipsoidal restriction $\mathrm{tr}(\Theta - \Theta_1)' N(\Theta - \Theta_1) \le \sigma^2$. The main results in this paper show that the class of admissible linear estimators with the restriction $\mathrm{tr}(\Theta - \Theta_1)' N(\Theta - \Theta_1) \le \sigma^2$ is the same as the one with the restriction $\mathrm{tr}(\Theta - \Theta_2)' N(\Theta - \Theta_2) \le \sigma^2$ for Θ_1 and Θ_2 with certain relationship.

We close this section with some notations. For a matrix A, A' and $\mathrm{tr}A$ denote respectively the transpose and trace of A. $A \ge 0$ means that A is nonnegative definite. For matrices A and B, $A - B \ge 0$ means that A, B and $A - B$ are all nonnegative definite. Define $LH = \{AY : A \text{ is an } s \times n \text{ matrix}\}$, the class of all homogeneous linear estimators. Define $L(S\Theta, \sigma^2, AY) = \mathrm{tr}(AY - S\Theta)'(AY - S\Theta)$, where S is an $s \times q$ matrix, that is to say, $L(S\Theta, \sigma^2, AY)$ is the quadratic loss function of the estimator AY. And then, the risk function can be defined by $R(S\Theta, \sigma^2, AY) = \mathsf{E}[L(S\Theta, \sigma^2, AY)]$. Hence, in the multivariate linear model (1.1), the risk function of the linear estimator of the estimated parameter $S\Theta$ is

$$R(S\Theta, \sigma^2, AY) = p\sigma^2 \mathrm{tr}AVA' + \mathrm{tr}\Theta'(AX - S)'(AX - S)\Theta.$$

In the restriction $H(N, \Theta_1)$, the estimator AY is called as good as BY iff $R(S\Theta, \sigma^2, AY) \le R(S\Theta, \sigma^2, BY)$. In the restriction $H(N, \Theta_1)$, the estimator AY is better than BY iff AY is called as good as BY, and AY has smaller risk than BY at some point in the parameter set. Let L be a class of linear estimators and LH a class of homogeneous linear estimators. Then $d(Y)$ will be said to be admissible in L on $H(N, \Theta_1)$ iff $d(Y) \in L$ and there exists no estimator in L which is better than $d(Y)$ on $H(N, \Theta_1)$. If $d(Y)$ is admissible for $S\Theta$ in L on $H(N, \Theta_1)$, we denote $d(Y) \overset{L}{\sim} S(H(N, \Theta_1))$. For the estimated parameter $S\Theta$ in the model (1.1), we denote by

$$ALH(S, N, \Theta_1) = \{AY : AY \overset{LH}{\sim} S\Theta[H(N, \Theta_1)]\}$$

the class of admissible homogeneous linear estimators with respect to the restriction $H(N, \Theta_1)$.

§2. Main Results

Theorem 2.1 In the multivariate linear model (1.1), suppose that $N - N_1 \geq 0$ for matrices N and N_1. If $AY \overset{LH}{\sim} S\Theta[H(N,\Theta_1)]$, then $AY \overset{LH}{\sim} S\Theta[H(N_1,\Theta_1)]$.

Proof If there exists B such that $R(S\Theta,\sigma^2,BY) \leq R(S\Theta,\sigma^2,AY)$, $\forall\,(\Theta,\sigma^2) \in H(N_1,\Theta_1)$. That is, $\forall\,(\Theta,\sigma^2) \in H(N_1,\Theta_1)$,

$$p\sigma^2 \mathrm{tr} BVB' + \mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta \leq p\sigma^2 \mathrm{tr} AVA' + \mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

Then, $\forall\,(\Theta,\tau^2) \in \mathbf{R}^{q\times p} \times [0,+\infty)$,

$$\begin{aligned} & p\tau^2 \mathrm{tr} BVB' + p\,\mathrm{tr} BVB' \cdot \mathrm{tr}(\Theta-\Theta_1)'N_1(\Theta-\Theta_1) + \mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta \\ \leq\ & p\tau^2 \mathrm{tr} AVA' + p\,\mathrm{tr} AVA' \cdot \mathrm{tr}(\Theta-\Theta_1)'N_1(\Theta-\Theta_1) + \mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta. \end{aligned}$$

Note that Θ and τ^2 vary independently. So, the following two inequalities hold simultaneously

$$\mathrm{tr} AVA' - \mathrm{tr} BVB' \geq 0, \tag{2.1}$$

$$\begin{aligned} & p(\mathrm{tr} AVA' - \mathrm{tr} BVB') \cdot \mathrm{tr}(\Theta-\Theta_1)'N_1(\Theta-\Theta_1) \\ & + \mathrm{tr}\Theta'[(AX-S)'(AX-S) - (BX-S)'(BX-S)]\Theta \geq 0, \qquad \forall\,\Theta \in \mathbf{R}^{q\times p}. \end{aligned} \tag{2.2}$$

Under the assumption $N - N_1 \geq 0$, it then follows from (2.1) and (2.2) that

$$\begin{aligned} & p(\mathrm{tr} AVA' - \mathrm{tr} BVB') \cdot \mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1) \\ & + \mathrm{tr}\Theta'[(AX-S)'(AX-S) - (BX-S)'(BX-S)]\Theta \geq 0, \qquad \forall\,\Theta \in \mathbf{R}^{q\times p}. \end{aligned} \tag{2.3}$$

Furthermore, it follws from (2.1) and (2.3) that $\forall\,(\Theta,\tau^2) \in \mathbf{R}^{q\times p} \times [0,+\infty)$,

$$\begin{aligned} & p\tau^2 \mathrm{tr} BVB' + p \mathrm{tr} BVB' \cdot \mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1) + \mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta \\ \leq\ & p\tau^2 \mathrm{tr} AVA' + p\,\mathrm{tr} AVA' \cdot \mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1) + \mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta. \end{aligned}$$

So, $\forall\,(\Theta,\sigma^2) \in H(N,\Theta_1)$,

$$p\sigma^2 \mathrm{tr} BVB' + \mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta \leq p\sigma^2 \mathrm{tr} AVA' + \mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

Note that $AY \overset{LH}{\sim} S\Theta[H(N,\Theta_1)]$, we then have, $\forall\,(\Theta,\sigma^2) \in H(N,\Theta_1)$,

$$p\sigma^2 \mathrm{tr} BVB' + \mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta \equiv p\sigma^2 \mathrm{tr} AVA' + \mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

Therefore, for all $(\Theta,\sigma^2) \in H(N_1,\Theta_1)$, we have

$$p\sigma^2 \mathrm{tr} BVB' + \mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta \equiv p\sigma^2 \mathrm{tr} AVA' + \mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

That is, in the restriction $H(N_1,\Theta_1)$, there does not exist any homogeneous linear estimator which is better than AY. Therefore, $AY \overset{LH}{\sim} S\Theta[H(N_1,\Theta_1)]$.

The proof of Theorem 2.1 is completed. $\square$

Remark 1 It follows easily from the above theorem that, for the multivariate linear model (1.1), if AY is the admissible linear estimator of $S\Theta$ under an incomplete ellipsoidal restriction, then AY is also admissible in the case with no parameter restriction.

Theorem 2.2 In the multivariate linear model (1.1), assume that $D=\lambda P$, $\lambda\neq 0$, $P'P=I_p$ and $\Theta_1=\Theta_2 D$. If $AY \overset{LH}{\sim} S\Theta[H(N,\Theta_1)]$, then $AY \overset{LH}{\sim} S\Theta[H(N,\Theta_2)]$.

Proof If there exists B such that $R(S\Theta,\sigma^2,BY)\leq R(S\Theta,\sigma^2,AY)$, $\forall\,(\Theta,\sigma^2)\in H(N,\Theta_2)$. Then, $\forall\,(\Theta,\sigma^2)\in H(N,\Theta_2)$,

$$p\sigma^2\mathrm{tr}BVB'+\mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta\leq p\sigma^2\mathrm{tr}AVA'+\mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

Note that the restriction means $\mathrm{tr}(\Theta-\Theta_2)'N(\Theta-\Theta_2)\leq\sigma^2$, $\forall\,(\Theta,\sigma^2)\in H(N,\Theta_2)$. Then, $\forall\,(\Theta,\tau^2)\in\mathbf{R}^{q\times p}\times[0,+\infty)$,

$$\begin{aligned}&p\tau^2\mathrm{tr}BVB'+p\,\mathrm{tr}BVB'\cdot\mathrm{tr}(\Theta-\Theta_2)'N(\Theta-\Theta_2)+\mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta\\ \leq\ &p\tau^2\mathrm{tr}AVA'+p\,\mathrm{tr}AVA'\cdot\mathrm{tr}(\Theta-\Theta_2)'N(\Theta-\Theta_2)+\mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta.\end{aligned}$$

Note that the parameters Θ and τ^2 are not constrained in the above inequality. So, the following two inequalities hold simultaneously

$$\mathrm{tr}AVA'-\mathrm{tr}BVB'\geq 0, \tag{2.4}$$

$$\begin{aligned}&p(\mathrm{tr}AVA'-\mathrm{tr}BVB')\cdot\mathrm{tr}(\Theta-\Theta_2)'N(\Theta-\Theta_2)\\ &+\mathrm{tr}\Theta'[(AX-S)'(AX-S)-(BX-S)'(BX-S)]\Theta\geq 0,\qquad\forall\,\Theta\in\mathbf{R}^{q\times p}.\end{aligned}\tag{2.5}$$

We replace Θ by ΘD^{-1}, (2.5) still hold. So, if $D=\lambda P$, $\lambda\neq 0$, $P'P=I_p$, we have

$$\begin{aligned}&p(\mathrm{tr}AVA'-\mathrm{tr}BVB')\cdot\mathrm{tr}(\Theta-\Theta_2 D)'N(\Theta-\Theta_2 D)\\ &+\mathrm{tr}\Theta'[(AX-S)'(AX-S)-(BX-S)'(BX-S)]\Theta\geq 0,\qquad\forall\,\Theta\in\mathbf{R}^{q\times p}.\end{aligned}\tag{2.6}$$

That is, by (2.4) and (2.5), it follows that (2.4) and (2.6) hold simultaneously.

By (2.4) and (2.6) and the assumption $\Theta_1=\Theta_2 D$, we have $\forall\,(\Theta,\tau^2)\in\mathbf{R}^{q\times p}\times[0,+\infty)$,

$$\begin{aligned}&p\tau^2\mathrm{tr}BVB'+p\,\mathrm{tr}BVB'\cdot\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)+\mathrm{tr}\Theta'(BX-S)'(BX-S)\Theta\\ \leq\ &p\tau^2\mathrm{tr}AVA'+p\,\mathrm{tr}AVA'\cdot\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)+\mathrm{tr}\Theta'(AX-S)'(AX-S)\Theta.\end{aligned}$$

Hence, $\forall\,(\Theta,\sigma^2)\in H(N,\Theta_1)$,

$$p\sigma^2\text{tr}BVB'+\text{tr}\Theta'(BX-S)'(BX-S)\Theta\le p\sigma^2\text{tr}AVA'+\text{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

Note that $AY\overset{LH}{\sim}S\Theta[H(N,\Theta_1)]$, we then have, for all $(\Theta,\sigma^2)\in H(N,\Theta_1)$,

$$p\sigma^2\text{tr}BVB'+\text{tr}\Theta'(BX-S)'(BX-S)\Theta\equiv p\sigma^2\text{tr}AVA'+\text{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

It then follows that, for all $(\Theta,\sigma^2)\in H(N,\Theta_2)$,

$$p\sigma^2\text{tr}BVB'+\text{tr}\Theta'(BX-S)'(BX-S)\Theta\equiv p\sigma^2\text{tr}AVA'+\text{tr}\Theta'(AX-S)'(AX-S)\Theta.$$

That is, in the restriction $H(N,\Theta_2)$, there does not exist any homogeneous linear estimator which is better than AY. Therefore, $AY\overset{LH}{\sim}S\Theta[H(N,\Theta_2)]$.

The proof of Theorem 2.2 is then completed. □

Corollary 2.1 In the multivariate linear model (1.1). If $AY\overset{LH}{\sim}S\Theta[H(N,\Theta_1)]$, then for any N_1 satisfying that $N-N_1\ge 0$ and for any Θ_2 satisfying that $\Theta_2=\Theta_1D$ where $DD'=\lambda I_p$ with $\lambda\ne 0$, we have $AY\overset{LH}{\sim}S\Theta[H(N_1,\Theta_2)]$.

Proof The proof is easy and then omitted. □

Corollary 2.2 In the multivariate linear model (1.1), if $\Theta_2=\Theta_1D+\sum\limits_{i=1}^{d}\lambda_iE_i$, $D=\lambda P$, $\lambda\ne 0$, $PP'=I_p$, E_i satisfying $NE_i=0$ then $ALH(S,N,\Theta_2)=ALH(S,N,\Theta_1)$.

Proof By Theorem 2.2 and the inversity of the matrix D, we can have that the two classes of admissible homogeneous linear estimators is the same in the restriction $H(N,\Theta_1)$ and $H(N,\Theta_1D)$. Moreover, by the assumption $NE_i=0$, we have $H(N,\Theta_1D)=H(N,\Theta_1D+\sum\limits_{i=1}^{d}\lambda_iE_i)$. It then follows that $ALH(S,N,\Theta_2)=ALH(S,N,\Theta_1)$.

The proof of Corollary 2.2 is completed. □

Remark 2 The above results are obtained under the quadratic loss function. We guess that there may be some similar results as the above under the matrix loss function, which deserves further research.

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不完全椭球约束下多元线性模型线性估计的可容许性

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本文研究了多元线性模型当未知参数受不完全椭球约束$\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)\leq\sigma^2$时线性估计的可容许性问题. 具体而言, 我们研究了约束$\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)\leq\sigma^2$中$N$和非中心点$\Theta_1$对线性估计的可容许性的影响. 主要结果表明在两个不同的不完全椭球约束条件$\mathrm{tr}(\Theta-\Theta_1)'N(\Theta-\Theta_1)\leq\sigma^2$与$\mathrm{tr}(\Theta-\Theta_2)'N(\Theta-\Theta_2)\leq\sigma^2$下, 当$\Theta_1$和$\Theta_2$满足一定的关系时, 可容许的齐次线性估计类是相同的.

关 键 词: 可容许性, 不完全椭球约束, 线性估计, 多元线性模型.

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