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# A Note on Weighted Invariance Principle *

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**Abstract**

In this note we generalize Davydov's[1] weak invariance principle for stationary processes to a weighted partial sums of long memory infinite moving average processes. This note also contains some bounds on the second moments of increments of some weighted partial sum processes of a general long memory time series, not necessarily moving average type. These bounds are useful in proving the tightness in uniform metric of these processes. As a consequence of continuous mapping theorem, the probability bounds on certain functions of random variables can be established.

**Keywords:** Fractional Brownian motion, infinite moving average processes, invariance principle, long range dependent data.

**AMS Subject Classification:** Primary 62F12; secondary 60G10, 60G15.

## §1. Introduction

Let $X_1, X_2, \cdots$ be independent identically distributed (i.i.d.) random variables with a distribution $F$ which has mean 0 and variance 1, and $S_n = X_1 + X_2 + \cdots + X_n$. Also let

$$S(t) := \begin{cases} S_k, & \text{if } t = k \in \mathbb{N}; \\ \text{linear on } [k, k+1], & \text{if } t \in (k, k+1),\ k \in \mathbb{N}. \end{cases}$$

Then Donsker Theorem states that $n^{-1/2}S(nt) \longrightarrow_{[0,1]} B(t)$, for all $t \in [0,1]$, i.e., the associated measure on $C[0,1]$ converge weakly to Brownian motion. Since the asymptotic distribution of $S(nt)$ is insensitive to the changes of the distribution $F$ of these variables $X$, we call it an invariance principle. This result is a powerful tool for proving the limit distribution of certain functions of random variables and plays a prominent role also in the vast literature on non-parametric tests involved with approximating stochastic processes.

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Davydov[1] describes an invariance principle for stationary processes which include long memory linear processes. In order to state it formally, let $X_t$ follow an infinite order moving average process:

$$X_t := \sum_{j\in\mathbb{Z}} a_j\varepsilon_{t-j}, \qquad a_j \sim c_\pm |j|^{-(1+\theta)/2} \ (j\to\pm\infty). \tag{1.1}$$

Here $\varepsilon_j$, $j\in\mathbb{Z} := \{0,\pm1,\pm2,\cdots\}$ are i.i.d. standard random variables, and $c_+, c_-$, $0<\theta<1$ are some constants with $c_+^2+c_-^2\neq 0$. (Here and below, $\sim$ indicates the ratio tends to 1.) Note that (1.1) implies, see, e.g., Beran[2],

$$\begin{aligned}
&r(j) = \mathsf{Cov}\,(X_1, X_{1+j}) \sim c_1^2 j^{-\theta}, \qquad j\to\infty,\\
&\mathsf{E}\overline{X}_T^2 \sim c_2^2 T^{-\theta}, \qquad T\to\infty,\\
&c_1^2 := (c_-^2 + c_+^2)\int_0^\infty (u+u^2)^{-(1+\theta)/2}\mathrm{d}u,\\
&c_2^2 := (2/(2-\theta)(1-\theta))c_1^2.
\end{aligned} \tag{1.2}$$

Accordingly, for $0<\theta<1$, the error process $X_t$ of (1.1) has nonsummable serial correlations or long memory.

For $0<\theta<1$, let $B_\theta(s)$ be a fractional Brownian motion in $s$ belonging to the real line $\mathbb{R}$, i.e., a continuous Gaussian process with mean zero and the covariance function $\mathsf{Cov}\,(B_\theta(t), B_\theta(s)) = (1/2)(|t|^{2-\theta}+|s|^{2-\theta}-|t-s|^{2-\theta})$, $t,s\in\mathbb{R}$. Davydov[1] shows that the long memory linear process $X_t$ of (1.1) satisfies the following invariance principle with $c_2$ as in (1.2),

$$\frac{1}{T^{1-\theta/2}}\sum_{t=1}^{[Ts]} X_t \Longrightarrow_{[0,1]} c_2 B_\theta(s). \tag{1.3}$$

Here, and in the sequel, $\Longrightarrow_{[a,b]}$ stands for the weak convergence in the Skorohod space $D[a,b]$, $-\infty<a<b<\infty$, with respect to the uniform metric.

In this note we generalize Davydov's weak invariance principle for stationary processes to a weighted partial sums of long memory infinite moving average processes which is described in Section 2. As a consequence of continuous mapping theorem, the probability bounds for maximal inequalities can be established. These results are useful in change point analysis. The detail proof of the main result is deferred to Section 3.

## §2. Main Results

The following result extends the invariance principle (1.3) to some weighted partial sums of long memory moving average processes. The special case $r=0$ corresponds to

Davydov's weak invariance principle. Invariance principle also is very useful to obtain the bounds on the maximal inequality in change point analysis.

**Theorem 2.1** Assume $X_t$ follow an infinite order moving average process as in (1.1). Then

$$\begin{aligned} \frac{1}{s^r T^{1-\theta/2}} \sum_{t=1}^{[Ts]} X_t &\Longrightarrow_{[0,1]} \frac{c_2}{s^r} B_\theta(s), \qquad 0 \le r < 1 - \frac{\theta}{2}, \\ \frac{1}{s^r T^{1-\theta/2}} \sum_{t=1}^{[Ts]} X_t &\Longrightarrow_{[1,\infty]} \frac{c_2}{s^r} B_\theta(s), \qquad 1 - \frac{\theta}{2} < r < 1, \end{aligned} \tag{2.1}$$

where constant $c_2$ is defined as in (1.2).

**Remark 2.1** Note that $c_1^2 \ge 0$. This follows from

$$\int_0^\infty (u+u^2)^{-(1+\theta)/2} \mathrm{d}u \ge (1/2) \int_0^1 (u-u^2)^{-(1+\theta)/2} \mathrm{d}u.$$

In terms of beta-function,

$$B(\theta, (1-\theta)/2) \ge (1/2) B((1-\theta)/2, (1-\theta)/2).$$

In terms of gamma-function,

$$\frac{\Gamma(\theta)\Gamma((1-\theta)/2)}{\Gamma(\theta+(1-\theta)/2)} \ge \frac{\Gamma^2((1-\theta)/2)}{2\Gamma(1-\theta)}$$

or

$$2\Gamma(\theta)\Gamma(1-\theta) \ge \Gamma((1-\theta)/2)\Gamma((1+\theta)/2).$$

The l.h.s. is $2\pi/\sin(\pi\theta)$, and the r.h.s. is $\pi/\cos(\pi\theta/2)$ (Korn and Korn[3], (21.4-8)). This gives

$$2\cos(\pi\theta/2) \ge \sin(\pi\theta),$$

as $\sin(\pi\theta) = 2\sin(\pi\theta/2)\cos(\pi\theta/2)$.

**Remark 2.2** The process (1.1) is called causal if $a_j = 0$ for all $j < 0$. A particular case of causal linear processes (1.1) are ARFIMA, or autoregressive fractionally integrated moving averages; see e.g. Brockwell and Davis[4] or Beran[2]. For more information on their applications in economics and other sciences, see Robinson[5] and Baillie[6]. For various theoretical results pertaining to the empirical processes of long memory moving averages, see Ho and Hsing[7], Koul and Surgailis[8, 9], Li[10], among others. Although causal long memory processes are most important, our results apply to double-sided moving averages (1.1) as well; Moreover, consideration of the more general class as in (1.1) allows to simplify the proofs below even when one is interested in the causal case only.

The proof of Theorem 2.1 is based on Lemmas 2.1 and 2.2 below, which provide the tightness of the underlying processes. Their proofs are deferred to Section 3.

Let $X_t, t \in \mathbb{Z}$ be a zero mean second order process such that for some $C > 0$ and $0 < \theta < 1$,

$$|r(t,s)| = |\mathsf{E}(X_t X_s)| \le C(1+|t-s|)^{-\theta}, \qquad t,s \in \mathbb{Z}. \tag{2.2}$$

Let

$$Y_T(s) := \frac{1}{s^r T^{1-\theta/2}} \sum_{j=1}^{[Ts]} X_j, \quad 0 < s < \infty,\ r \ge 0; \qquad Y_T(0) := 0 =: Y_T(\infty)$$

be the weighted partial sum process.

**Lemma 2.1** Let $0 \le r < 1-\theta/2$. Then there exist constants $C$, $\delta > 0$ and a finite continuous measure $\mu$ on $[0,1]$ such that for all rational $s < t$, $s,t \in \{p/T : p = 0,1,\cdots,T\}$,

$$\mathsf{E}(Y_T(t) - Y_T(s))^2 \le C(\mu(s,t))^{1+\delta}. \tag{2.3}$$

**Lemma 2.2** Let $1-\theta/2 < r < 1$. Then there exist constants $C$, $\delta > 0$ and a finite continuous measure $\nu$ on $[1,\infty]$ such that for all $s < t$, $s,t \in \{p/T : p = T, T+1, \cdots, \infty\}$,

$$\mathsf{E}(Y_T(t) - Y_T(s))^2 \le C(\nu(s,t))^{1+\delta}. \tag{2.4}$$

**Remark 2.3** The above Lemmas 2.1 and 2.2 might be of an independent interest, as they refer to more general (not necessarily linear or even stationary) processes.

**Proof of Theorem 2.1** We shall prove the second part of (2.1) only, as the proof of the first part is analogous. Fix an $r$ in $1-\theta/2 < r < 1$. Put

$$t^* := \begin{cases} [tT]/T, & \text{if } 1 \le t < \infty; \\ \infty, & \text{if } t = \infty, \end{cases}$$

and let

$$Y_T^*(t) := Y_T(t^*), \qquad Z_T(t) := Y_T(t) - Y_T^*(t).$$

It suffices to show

$$Y_T^*(t) \Longrightarrow_{[1,\infty]} c_2 t^{-r} B_\theta(t), \tag{2.5}$$

$$\sup_{t\ge 1} |Z_T(t)| = o_p(1). \tag{2.6}$$

Since the convergence of the finite-dimensional distributions follows from Davydov[1], in order to show (2.5), we only need to check the tightness of the sequence $\{Y_T^*(t)\}_{T\ge 1}$ in $\mathcal{D}[1,\infty]$ only. According to the well-known criterion (Billingsley[11], Theorem 15.6), it suffices to show that there exist a bounded continuous measure $\nu$ on $[1,\infty]$ and some constants $C$, $\delta>0$ such that for all $1\le t_1\le t\le t_2\le\infty$

$$\mathsf{E}|Y_T^*(t)-Y_T^*(t_1)||Y_T^*(t_2)-Y_T^*(t)|\le C(\nu(t_1,t_2))^{1+\delta}. \tag{2.7}$$

Clearly, it suffices to show (2.7) for $t_2-t_1>1/T$ as $0\le t_2-t_1\le 1/T$ implies either $t^*=t_1^*$, or $t^*=t_2^*$ and $(Y_T^*(t)-Y_T^*(t_1))(Y_T^*(t_2)-Y_T^*(t))=0$ by definition of $Y_T^*(t)$.

According to Lemma 2.2,

$$\begin{aligned}\mathsf{E}|Y_T^*(t)-Y_T^*(t_1)||Y_T^*(t_2)-Y_T^*(t)| &\le (\mathsf{E}|Y_T(t^*)-Y_T(t_1^*)|^2)^{1/2}(\mathsf{E}|Y_T(t_2^*)-Y_T(t_1^*)|^2)^{1/2}\\ &\le C(\nu(t_1^*,t_2^*))^{1+\delta}\end{aligned}$$

with $\nu=\mu_\gamma$ and $\mu_\gamma(s,t)=\displaystyle\int_s^t u^{-\gamma}\mathrm{d}u$, for some $\gamma>1$ (For more details, see (3.7) in the proof of Lemmas in Section 3). Note $\mu_\gamma(t_1^*,t_2^*)\le 2^\gamma\mu_\gamma(t_1,t_2)$ for any $t_2-t_1\ge T^{-1}$, $t_1,t_2,T\ge 1$, which follows easily from definition of the measure $\mu_\gamma$. This proves (2.7) and (2.5) too. The relation (2.6) follows from $Z_T(t)=((t^*/t)^r-1)Y_T(t^*)$ where $|(t^*/t)^r-1|\le C|t^*-t|/t\le C/T\to 0$ uniformly in $t\ge 1$, while $\sup\limits_{t\ge 1}|Y_T(t^*)|=O_p(1)$ according to (2.5). Hence, Theorem 2.1 is proved. □

**Remark 2.4** Above weighted invariance principle is very useful in change point analysis, especially in obtaining the probability bounds for certain functions of random variables or maximal inequality.

Horváth and Kokoszka[12] consider the estimation of the time of change in the mean of Gaussian observations. They (Lemma 4.3) need a probability bound for the term $\sup\limits_{k\ge n}k^{-1}\Big|\sum\limits_{t=1}^k X_t\Big|$, where $X_t$ is a long memory Gaussian process with parameter $H$ $(=1-\theta/2)$ (Long memory Gaussian processes have a wide application in practice, for a reference, see, e.g., Li and Xiao[13, 14]). Apply our Theorem 2.1, we can obtain the same rate for long memory moving average process which is following:

**Proposition 2.1** Assume $X_t$ follow an infinite order moving average process as in (1.1). Then

$$\sup_{k\ge n}\frac{1}{k}\Big|\sum_{t=1}^k X_t\Big|=O_p(n^{-\theta/2}).$$

**Proof** Note, for an $1-\theta/2<r<1$,

$$\sup_{k\ge n}\frac{1}{n^{-\theta/2}k}\Big|\sum_{t=1}^k X_t\Big|=\sup_{s\ge 1}\frac{1}{n^{1-\theta/2}s}\Big|\sum_{t=1}^{[ns]}X_t\Big|\le\sup_{s\ge 1}\frac{1}{s^r n^{1-\theta/2}}\Big|\sum_{t=1}^{[ns]}X_t\Big|.$$

The proof of the above proposition now follows from (2.1) of Theorem 2.1, the continuous mapping theorem and the fact that $\lim\limits_{s\to\infty}s^{-r}B_\theta(s)=0$, a.s.. □

## §3. Proofs of Lemmas

**Proof of Lemma 2.1** It suffices to show the lemma for $r$ arbitrary close to $1-\theta/2$; in particular, for $r+\theta>1$. Write

$$Y_T(t)-Y_T(s)=\sum_{sT<p\le tT}\Delta Y_T\Big(\frac{p}{T}\Big),$$

where

$$\begin{aligned}\Delta Y_T\Big(\frac{p}{T}\Big) &:= Y_T\Big(\frac{p}{T}\Big)-Y_T\Big(\frac{p-1}{T}\Big)\\ &= \frac{1}{T^{1-\theta/2}}\Big\{\Big(\frac{p}{T}\Big)^{-r}\sum_{j=1}^{p}X_j-\Big(\frac{p-1}{T}\Big)^{-r}\sum_{j=1}^{p-1}X_j\Big\}\\ &= \frac{1}{T^{1-\theta/2}}\Big\{\Big(\Big(\frac{p}{T}\Big)^{-r}-\Big(\frac{p-1}{T}\Big)^{-r}\Big)\sum_{j=1}^{p-1}X_j+\Big(\frac{p}{T}\Big)^{-r}X_p\Big\}\\ &= \frac{1}{T^{1-\theta/2}}\Big\{\Delta\tau_{p,T}\sum_{j=1}^{p-1}X_j+\tau_{p,T}X_p\Big\},\end{aligned}$$

with

$$\tau_{p,T}:=(p/T)^{-r},\qquad \Delta\tau_{p,T}:=\tau_{p,T}-\tau_{p-1,T}=(p/T)^{-r}-((p-1)/T)^{-r}.$$

Therefore

$$T^{2-\theta}\mathsf{E}(Y_T(t)-Y_T(s))^2\le 2\Sigma_1+2\Sigma_2,$$

where

$$\Sigma_1:=\mathsf{E}\Big\{\sum_{Ts<p\le Tt}\Delta\tau_{p,T}\sum_{j=1}^{p-1}X_j\Big\}^2,\qquad \Sigma_2:=\mathsf{E}\Big\{\sum_{Ts<p\le Tt}\tau_{p,T}X_p\Big\}^2.$$

Now,

$$\Sigma_1=\Sigma_{11}+2\Sigma_{12},\qquad \Sigma_2=\Sigma_{21}+2\Sigma_{22},$$

where

$$\begin{aligned}\Sigma_{11}&:=\sum_{Ts<p\le Tt}(\Delta\tau_{p,T})^2\mathsf{E}\Big(\sum_{j=1}^{p-1}X_j\Big)^2,\\ \Sigma_{12}&:=\sum_{Ts<q<p\le Tt}\Delta\tau_{p,T}\Delta\tau_{q,T}\mathsf{E}\Big(\sum_{j=1}^{p-1}X_j\sum_{j=1}^{q-1}X_j\Big),\\ \Sigma_{21}&:=\sum_{Ts<p\le Tt}\tau_{p,T}^2\mathsf{E}X_p^2,\\ \Sigma_{22}&:=\sum_{Ts<q<p\le Tt}\tau_{p,T}\tau_{q,T}\mathsf{E}(X_pX_q).\end{aligned}$$

We claim that for all $s$, $t$ as in the formulation of the lemma, and for $k, l = 1, 2$,

$$\Sigma_{kl} \le CT^{2-\theta} \begin{cases} t^{-2r}(t-s)^{2-\theta}, & s > t/2, \\ t^{2-2r-\theta}, & s < t/2. \end{cases} \tag{3.1}$$

Let's consider $\Sigma_{22}$ first. Let

$$J_{22}(s,t) := \int_s^t \int_s^t u^{-r} v^{-r} |u-v|^{-\theta} \mathrm{d}v. \tag{3.2}$$

From (2.2), we have

$$\begin{aligned} |\Sigma_{22}| &\le \sum_{Ts<q<p\le Tt} \tau_{p,T}\tau_{q,T}|r(p,q)| \\ &\le C \sum_{Ts<p\le Tt} (p/T)^{-r} \sum_{Ts<q<p} (q/T)^{-r}(p-q)^{-\theta} \\ &\le CT^{2-\theta} \int_s^t u^{-r} \int_s^u v^{-r}(u-v)^{-\theta} \mathrm{d}v\mathrm{d}u =: CT^{2-\theta} J_{22}(s,t). \end{aligned} \tag{3.3}$$

Suppose $s < t/2$. Using $r + \theta > 1$,

$$\begin{aligned} J_{22}(s,t) &\le C \int_0^t u^{-r} \int_0^1 v^{-r}|u-v|^{-\theta} \mathrm{d}v\mathrm{d}u \\ &\le C \int_0^t u^{1-2r-\theta} \mathrm{d}u = Ct^{2-2r-\theta}. \end{aligned} \tag{3.4}$$

In the case $s > t/2$,

$$\begin{aligned} J_{22}(s,t) &\le s^{-2r} \int_s^t \int_s^t |u-v|^{-\theta} \mathrm{d}u\mathrm{d}v \\ &\le Ct^{-2r} \int_0^{t-s} \int_0^{t-s} |u-v|^{-\theta} \mathrm{d}u\mathrm{d}v = Ct^{-2r}(t-s)^{2-\theta}. \end{aligned}$$

This proves (3.1) for term $\Sigma_{22}$.

Consider $\Sigma_{21}$. Clearly,

$$\Sigma_{21} \le C \sum_{Ts\le p\le Tt} \tau_{p,T}^2 \le CT \int_s^t u^{-2r} \mathrm{d}u.$$

Let $s < t/2$, then $\Sigma_{21} \le CTt^{1-2r} = CT^{2-\theta}t^{2-2r-\theta}(Tt)^{-1+\theta}$, where $(Tt)^{-1+\theta} \le 1$ as $t > 1/T$. Next, let $s > t/2$, then $\Sigma_{21} \le CT(t-s)t^{-2r} \le CT^{2-\theta}(t-s)^{2-\theta}t^{-2r}$ as $T(t-s) \ge 1$. This proves (3.1) for $\Sigma_{21}$.

As to the term $\Sigma_{12}$, we may write $\Sigma_{12} = \Sigma'_{12} + \Sigma''_{12}$, where

$$\Sigma'_{12} := \sum_{Ts<q<p\le Tt} \Delta\tau_{p,T}\Delta\tau_{q,T} \sum_{i=q}^{p-1}\sum_{j=1}^{q-1} r(i,j),$$

$$\Sigma''_{12} := \sum_{Ts<q<p\le Tt} \Delta\tau_{p,T}\Delta\tau_{q,T} \sum_{i=1}^{q-1}\sum_{j=1}^{q-1} r(i,j).$$

We have $|\Delta\tau_{p,T}| \le CT^{-1}(p/T)^{-r-1}$ and therefore

$$\begin{aligned}|\Sigma'_{12}| &\le CT^{-2}\sum_{Ts<q<p\le Tt}(p/T)^{-r-1}(q/T)^{-r-1}\sum_{q\le i<p}\sum_{1\le j<q}(i-j)^{-\theta}\\ &\le C\int_s^t u^{-r-1}\int_s^u v^{-r-1}\int_{Tv}^{Tu}\int_0^{Tv}(x-y)^{-\theta}\mathrm{d}x\mathrm{d}y\mathrm{d}v\mathrm{d}u\\ &= CT^{2-\theta}\int_s^t u^{-r-1}\int_s^u v^{-r-1}(u^{2-\theta}-v^{2-\theta}-(u-v)^{2-\theta})\mathrm{d}v\mathrm{d}u\\ &=: CT^{2-\theta}J'_{12}(s,t). \qquad (3.5)\end{aligned}$$

For $s<t/2$,

$$\begin{aligned}J'_{12}(s,t) &\le \int_0^t u^{-r-1}\int_0^u v^{-r-1}(u^{2-\theta}-(u-v)^{2-\theta})\mathrm{d}v\mathrm{d}u\\ &= \int_0^t u^{1-2r-\theta}\int_0^1 v^{-r-1}(1-(1-v)^{2-\theta})\mathrm{d}v\mathrm{d}u\\ &= C\int_0^t u^{1-2r-\theta}\mathrm{d}u = Ct^{2-2r-\theta},\end{aligned}$$

where we used $|1-(1-v)^{2-\theta}|\le Cv$ for $v\in[0,1]$, as well as $r<1$.

For $s>t/2$,

$$\begin{aligned}J'_{12}(s,t) &\le \int_s^t\int_s^t (uv)^{-r-1}u^{2-\theta}\mathrm{d}u\mathrm{d}v\\ &\le Ct^{-2r}\int_0^{t-s}\int_0^{t-s}u^{-\theta}\mathrm{d}u\mathrm{d}v = Ct^{-2r}(t-s)^{2-\theta}.\end{aligned}$$

This proves (3.1) for $\Sigma'_{21}$.

Turning to $\Sigma''_{12}$, note $\sum_{i,j=1}^{q-1} r(i,j)\le C\sum_{i,j=1}^{q-1}(1+|i-j|)^{-\theta}\le Cq^{2-\theta}$, and we obtain, similarly as above,

$$|\Sigma''_{12}|\le CT^{2-\theta}\int_s^t u^{-r-1}\int_s^u v^{1-r-\theta}\mathrm{d}v\mathrm{d}u =: CT^{2-\theta}J''_{12}(s,t). \qquad (3.6)$$

Let $s<t/2$, then

$$J''_{12}(s,t)\le\int_0^t u^{-r-1}\int_0^u v^{1-r-\theta}\mathrm{d}v\mathrm{d}u\le C\int_0^u u^{1-2r-\theta}\mathrm{d}u = Ct^{2-2r-\theta}.$$

Next, let $s>t/2$, then

$$J''_{12}(s,t)\le s^{-2r}\int_s^t\mathrm{d}u\int_s^t v^{-\theta}\mathrm{d}v\le Ct^{-2r}(t-s)^{2-\theta},$$

thereby proving (3.1) for $\Sigma''_{21}$.

Finally, let's consider term $\Sigma_{11}$. From covariance (2.2),

$$\Sigma_{11} \le C \sum_{Ts<p\le Tt} T^{-2}(p/T)^{-2r-2}p^{2-\theta} \le CT^{2-\theta}J_{11}(s,t),$$

where $J_{11}(s,t) := \sum_{p=Ts+1}^{Tt}(p/T)^{-2r-\theta}T^{-2}$.

Let $s < t/2$. Then

$$J_{11}(s,t) \le \sum_{p=1}^{Tt}(p/T)^{1-2r-\theta}T^{-1} \le C\int_0^t u^{1-2r-\theta}\mathrm{d}u = Ct^{2-2r-\theta},$$

where we used the fact that $1-2r-\theta > -1$. On the other hand, if $s > t/2$, then

$$J_{11}(s,t) \le T^{-1}\sum_{p=Ts+1}^{Tt}(p/T)^{-2r-\theta}T^{-1} \le T^{-1}\int_s^t u^{-2r-\theta}\mathrm{d}u \le \int_s^t u^{1-2r-\theta}\mathrm{d}u,$$

where we used $uT > 1/2$ in the integrand (which follows from $tT \ge 1$ and $s > t/2$). As $\int_s^t u^{1-2r-\theta}\mathrm{d}u \le Ct^{-2r}(t-s)^{2-\theta}$ for $s > t/2$, this ends the proof of the claim (3.1).

Put

$$\mu_\gamma(s,t) := \int_s^t u^{-\gamma}\mathrm{d}u, \qquad 0 < s < t < \infty. \tag{3.7}$$

Then for $\gamma < 1$, $\mu_\gamma$ defines a finite continuous measure on $[0,1]$. Note the following easy property of this measure: for any $\gamma < 1$, there exist constants $0 < C_1 < C_2 < \infty$ such that for all $0 \le s < t \le 1$,

$$C_1t^{-\gamma}(t-s) \le \mu_\gamma(s,t) \le C_2t^{-\gamma}(t-s), \qquad \text{if } s > t/2, \tag{3.8}$$

$$C_1t^{1-\gamma} \le \mu_\gamma(s,t) \le C_2t^{1-\gamma}, \qquad \text{if } s < t/2. \tag{3.9}$$

The statement of the lemma follows from (3.1) and the lower bounds in (3.8) and (3.9). Indeed, let $\gamma := 2r/(2-\theta)$, then $\gamma < 1$ by the condition of the lemma, while $t^{2-2r-\theta} = (t^{1-\gamma})^{2-\theta}$, $t^{-2r}(t-s)^{2-\theta} = (t^{-\gamma}(t-s))^{2-\theta}$. This proves Lemma 2.1 with $\mu = \mu_\gamma$ and $\delta = 1-\theta > 0$. $\square$

**Proof of Lemma 2.2** The proof is similar to that of Lemma 2.1 above and we omit some details. It suffices to consider the case when both $s$ and $t$ are finite. Indeed, (2.4) in the case $1 \le s < t = \infty$ follows by continuity: $\lim_{t=p/T\to\infty}\nu(s,t) = \nu(s,\infty)$,

$$\lim_{t=p/T\to\infty}\mathsf{E}(Y_T(t)-Y_T(s))^2 = \mathsf{E}(Y_T(\infty)-Y_T(s))^2 = \mathsf{E}(Y_T(s))^2;$$

the last convergence is an easy consequence of

$$\lim_{t=p/T\to\infty}\mathsf{E}(Y_T(t))^2 \le C\lim_{t=p/T\to\infty}t^{-2r}T^{-(2-\theta)}[Tt]^{2-\theta} = 0.$$

To prove (2.4), write $T^{2-\theta}\mathsf{E}(Y_T(t)-Y_T(s))^2 \le C\sum\limits_{k,l=1}^{2}|\Sigma_{kl}|$, where $\Sigma_{kl}$ are defined exactly as in the proof of Lemma 2.1. We claim that there exist a constant $C<\infty$ such that for all $s$, $t$ as in the statement of Lemma 2.2, and any $k,l=1,2$

$$\Sigma_{kl} \le CT^{2-\theta}\begin{cases} s^{-2r}(t-s)^{2-\theta}, & s>t/2, \\ s^{2-2r-\theta}, & s<t/2. \end{cases} \tag{3.10}$$

To conclude from (3.10) the statement of the lemma, consider $\mu_\gamma$ defined by (3.7), with $\gamma>1$. Then $\mu_\gamma$ defines a finite continuous measure on $[1,\infty)$ which has the property that there exist constants $0<C_1<C_2<\infty$ such that for all $1\le s<t<\infty$

$$\begin{aligned} &C_1 s^{-\gamma}(t-s) \le \mu_\gamma(s,t) \le C_2 s^{-\gamma}(t-s), \qquad s>t/2,\\ &C_1 s^{1-\gamma} \le \mu_\gamma(s,t) \le C_2 s^{1-\gamma}, \qquad s<t/2. \end{aligned}$$

Let $\gamma := 2r/(2-\theta)$, then $\gamma>1$ by the condition of the lemma, and

$$s^{2-2r-\theta}=(s^{1-\gamma})^{2-\theta}, \qquad s^{-2r}(t-s)^{2-\theta}=(s^{-\gamma}(t-s))^{2-\theta}.$$

Hence Lemma 2.2 follows with $\nu=\mu_\gamma$ and $\delta=1-\theta>0$.

It remains to check the claim (3.10). Again, consider first $|\Sigma_{22}|\le CT^{2-\theta}J_{22}(s,t)$, where $J_{22}$ is as in (3.2). Let $s<t/2$, then

$$\begin{aligned} J_{22}(s,t) &\le \int_s^\infty u^{-r}\mathrm{d}u\int_1^\infty v^{-r}|u-v|^{-\theta}\mathrm{d}v \\ &\le C\int_s^\infty u^{1-2r-\theta}\mathrm{d}u = Cs^{2-2r-\theta}. \end{aligned}$$

Next, let $s>t/2$. Then

$$\begin{aligned} J_{22}(s,t) &\le s^{-2r}\int_s^t\int_s^t |u-v|^{-\theta}\mathrm{d}u\mathrm{d}v \\ &= s^{-2r}\int_0^{t-s}\int_0^{t-s}|u-v|^{-\theta}\mathrm{d}u\mathrm{d}v = Cs^{-2r}(t-s)^{2-\theta}. \end{aligned}$$

This proves (3.10) for $\Sigma_{22}$.

Next, consider $\Sigma_{21}\le CT\int_s^t u^{-2r}\mathrm{d}u$. Let $s<t/2$, then

$$\begin{aligned} \Sigma_{21} &\le CT\int_s^\infty u^{-2r}\mathrm{d}u = CTs^{1-2r} \\ &= CT^{2-\theta}s^{2-2r-\theta}(Ts)^{-(1-\theta)} \le CT^{2-\theta}s^{2-2r-\theta}, \end{aligned}$$

as $T, s \geq 1$. Next, let $s > t/2$, then $\Sigma_{21} \leq CT(t-s)s^{-2r} \leq CT^{2-\theta}(t-s)^{2-\theta}s^{-2r}$ as $T(t-s) \geq 1$. This proves (3.10) for $\Sigma_{21}$.

Now, consider $\Sigma_{12} = \Sigma'_{12} + \Sigma''_{12}$, where $|\Sigma'_{12}| \leq CT^{2-\theta}J'_{12}(s,t)$, $|\Sigma''_{12}| \leq CT^{2-\theta}J''_{12}(s,t)$ and $J'_{12}(s,t)$, $J''_{12}(s,t)$ are the same as in the proof of Lemma 2.1. Let $s < t/2$. Then

$$\begin{aligned} J'_{12}(s,t) &\leq \int_s^\infty v^{-r-1}\mathrm{d}v \int_v^\infty u^{-r-1}(u^{2-\theta} - (u-v)^{2-\theta})\mathrm{d}u \\ &\leq \int_s^\infty v^{1-2r-\theta}\mathrm{d}v \int_1^\infty z^{-r-1}(z^{2-\theta} - (z-1)^{2-\theta})\mathrm{d}z \\ &\leq C\int_s^\infty v^{1-2r-\theta}\mathrm{d}v = Cs^{2-2r-\theta}, \end{aligned}$$

where we used $z^{2-\theta} - (z-1)^{2-\theta} \leq Cz^{1-\theta}$ and the inequality $r + \theta > 1$ implying the boundedness of the integral w.r.t. $z$. Next, let $s > t/2$, then

$$\begin{aligned} J'_{12}(s,t) &\leq s^{-2r-2}\int_s^t\int_s^t u^{2-\theta}\mathrm{d}u\mathrm{d}v \\ &\leq s^{-2r-2}t^2\int_0^{t-s}\int_0^{t-s} u^{-\theta}\mathrm{d}u\mathrm{d}v \leq Cs^{-2r}(t-s)^{2-\theta}. \end{aligned}$$

This proves (3.10) for $\Sigma'_{12}$. The same bound for $\Sigma''_{12}$ follows by

$$J''_{12}(s,t) = \int_s^t v^{1-r-\theta}\mathrm{d}v \int_v^t u^{-1-r}\mathrm{d}u$$

by considering cases $s < t/2$ and $s > t/2$, as above. Indeed, in the first case, we use

$$J''_{12}(s,t) \leq \int_s^\infty v^{1-r-\theta}\mathrm{d}v \int_1^\infty u^{-1-r}\mathrm{d}u \leq Cs^{2-r-\theta},$$

and in the second case,

$$J''_{12}(s,t) \leq Cs^{-2r}\int_s^t v^{-\theta}\mathrm{d}v \int_s^t \mathrm{d}u = Cs^{-2r}(t-s)^{2-\theta}.$$

This proves (3.10) for $\Sigma''_{12}$ and $\Sigma_{12}$, too.

Finally, consider the case $|\Sigma_{11}| \leq CT^{2-\theta}J_{11}(s,t)$, where $J_{11}(s,t) = T^{-1}\int_s^t u^{-2r-\theta}\mathrm{d}u$. Let $s < t/2$, then

$$J_{11}(s,t) \leq T^{-1}\int_s^\infty u^{-2r-\theta}\mathrm{d}u = CT^{-1}s^{1-2r-\theta} \leq Cs^{2-2r-\theta}$$

as $s$, $T \geq 1$. Let $s > t/2$, then

$$\begin{aligned} J_{11}(s,t) &\leq T^{-1}s^{-2r}\int_s^t u^{-\theta}\mathrm{d}u \leq T^{-1}s^{-2r}\int_0^{t-s} u^{-\theta}\mathrm{d}u \\ &= CT^{-1}(t-s)^{1-\theta}s^{-2r} \leq C(t-s)^{2-\theta}s^{-2r} \end{aligned}$$

as $t - s \geq 1/T$. This proves (3.10) and Lemma 2.2, too. $\square$

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# 关于加权不变原理的一个注记

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在这个注记中我们将关于平稳过程的Davydov弱不变原理推广到长记忆无穷滑动平均过程的加权部分和过程, 文中还给出了一些不限于滑动平均过程的一般长记忆时间序列的加权部分和过程增量的二阶矩的边界, 这些边界将有助于证明这些过程关于一致度量的胎紧性. 作为连续映射定理的一个结果, 我们也导出了一些随机变量函数的概率边界.

**关键词:** 分式布朗运动, 无穷滑动平均过程, 不变原理, 长期相依数据.

**学科分类号:** O211.6, O212.7.