文章编号:1671-9352(2007)03-0013-05

NLS方程的守恒数值格式及其收敛性分析

王　震[1],张　[redacted][2]

（1.山东科技大学　信息科学与工程学院,山东　青岛　266510；2.济南大学　信息科学与工程学院,山东　济南　250022）

摘要:研究了一类不稳定非线性Schrödinger方程初边值问题的有限差分方法,证明了差分格式的两个离散守恒律,用能量方法得到了差分解的收敛性和稳定性.给出了数值算例.

关键词:NLS方程;守恒;差分法;收敛性

中图分类号:O241.82　　**文献标识码**:A

A conservative numerical scheme and its convergence analysis for a class of the NLS equation

WANG Zhen[1] and ZHANG Jin[2]

(1. Department of Information, Shandong University of Science and Technology, Qingdao 266510, Shandong, China;
2. Department of Information, Jinan Univ., Jinan 250022, Shandong, China)

Abstract: The finite difference scheme is devised for a class of the unstable nonlinear Schrödinger equation. Two discrete conservation laws of the difference scheme are proved. By using the energy method, the convergence and stability of the difference scheme are proved.

Key words: NLS equation; conservation; difference scheme; convergence

0　引言

非线性Schrödinger在非线性光纤光学、等离子体物理等领域有着广泛的应用.近年来,关于不稳定介质中孤子现象的研究引起了众多科学工作者的兴趣,文献[1]引入了方程 $\mathrm{i}u_x+u_{tt}+2|u|^2u+\varepsilon u_{xt}=0(\varepsilon\ll 1)$,用来描述变化的波列中某些不稳定性,并得到了解的一系列守恒律.文献[2,3]研究了此类方程的一些精确解.

本文在有限区域 $Q=[0,1]\times[0,T]$ 上考虑如下问题:

$$\mathrm{i}u_x+u_{tt}+\varepsilon u_{xt}+f(|u|^2)u=0,\ (0<\varepsilon\ll 1),\tag{1}$$

$$u(x,0)=\phi(x),u_t(x,0)=\Psi(x),\tag{2}$$

$$u(0,t)=u(1,t)=0.\tag{3}$$

其中,$u(x,t)$ 为复值函数,$f(s),\phi(x),\Psi(x)$ 是给定的实值函数,$\mathrm{i}^2=-1$.该问题有着如下的守恒关系[4]:

$$E_1=(\mathrm{i}u_x,u)+\|u_t\|^2+\int_0^1F(|u|^2)\mathrm{d}x=\mathrm{const},\tag{4}$$

$$E_2=\varepsilon(\mathrm{i}u_x,u)+2\mathrm{Im}(u,u_t)=\mathrm{const}.\tag{5}$$

其中,$F(s)=\int_0^sf(r)\mathrm{d}r$,Im,Re分别表示虚部和实部.

收稿日期:2006-05-26

作者简介:王　震(1977-　),男,讲师,硕士,研究方向:偏微分方程数值解.

本文对问题(1) ~ (3) 构造了3层隐式差分格式,用能量方法证明了差分格式满足两个离散守恒律,并证明了差分格式的收敛性和稳定性.

1 差分格式及其守恒律

对平面区域 Q 作网格剖分,$x_j = jh(j = 0,1,\cdots J)$,$t^n = n\tau(n = 0,1,\cdots N)$,其中 $Jh = 1, N\tau = T$; J, N 都是正整数. 记 $u_j^n = u(x_j, t^n)$, $u^n = \{u_j^n\}_{j=0}^J$ 表示第 n 时间层上的网格函数. 本文中的 C 表示不依赖于 h,τ 的正常数,在不同的地方可取不同的值.

引进如下差分记号:

$$(w_j^n)_x = \frac{w_{j+1}^n - w_j^n}{h},\ (w_j^n)_{\hat{x}} = \frac{w_{j+1}^n - w_{j-1}^n}{2h},\ (w_t)_j^n = \frac{w_j^{n+1} - w_j^n}{\tau},\ (w_j^n)_{\hat{t}} = \frac{w_j^{n+1} - w_j^{n-1}}{2\tau},$$

$$(w_j^n)_{\bar{t}} = \frac{w_j^n - w_j^{n-1}}{\tau},\ (v^n, w^n) = \sum_{j=0}^{J-1} v_j^n \bar{w}_j^n h,\ \| w^n \|^2 = (w^n, w^n),\ \| w \|_\infty = \sup_j | w_j^n |.$$

对于方程(1) ~ (3) 提出如下差分格式:

$$\frac{1}{2}\mathrm{i}(u_j^{n+1} + u_j^{n-1})_{\hat{x}} + (u_j^n)_{t\bar{t}} + \varepsilon(u_j^n)_{\hat{x}\hat{t}} + \frac{F(|u_j^{n+1}|^2) - F(|u_j^{n-1}|^2)}{|u_j^{n+1}|^2 - |u_j^{n-1}|^2}\cdot\frac{u_j^{n+1} + u_j^{n-1}}{2} = 0,$$

$$j = 1,\cdots J-1,\ n = 1,\cdots N-1; \tag{6}$$

$$u_j^0 = \phi_j,\ u_j^1 = u_j^{-1} + 2\tau\Psi_j,\ j = 0,\cdots J; \tag{7}$$

$$u_0^n = u_J^n = 0,\ n = 1,\cdots N. \tag{8}$$

对于方程(6),令 $n = 0$,与(7) 联立即可消掉 u_j^{-1}.

引理1 对满足齐次边界条件的网格函数$\{u_j\}$,$\{v_j\}$ 有下列等式成立:

(ⅰ) $\mathrm{Re}(\mathrm{i}u_{\hat{x}}, u) = (\mathrm{i}u_{\hat{x}}, u)$; (ⅱ) $\mathrm{Re}(u_{\hat{x}}, u) = \mathrm{Re}(u, u_{\hat{x}}) = 0$;

(ⅲ) $\mathrm{Re}(u_{\hat{x}}, v) = -\mathrm{Re}(v_{\hat{x}}, u)$; (ⅳ) $\mathrm{Re}(\mathrm{i}u_{\hat{x}}, v) = (\mathrm{i}v_{\hat{x}}, u)$.

证明 令复值网格函数 $u_j = u_{1j} + \mathrm{i}u_{2j}$,其中 u_{1j}, u_{2j} 为满足齐次边界条件的实值网格函数.易于验证上述结论成立.

同理可得下述引理.

引理2 对满足齐次边界条件的网格函数$\{u_j\}$,$\{v_j\}$ 有下列等式成立:

(ⅰ) $\mathrm{Im}(u, v) = -\mathrm{Im}(v, u)$; (ⅱ) $\mathrm{Im}(u_{\hat{x}}, v) = \mathrm{Im}(v_{\hat{x}}, u)$;

(ⅲ) $(u_{\hat{x}}, v) = -(v_{\hat{x}}, u)$; (ⅳ) $\mathrm{Im}(u_{\hat{x}}, u) = -(\mathrm{i}u_{\hat{x}}, u)$.

定理1 差分格式(6) ~ (8) 满足如下守恒关系:

$$E_1^n = E_1^{n-1} = \cdots = E_1^0, \tag{9}$$

$$E_2^n = E_2^{n-1} = \cdots = E_2^0. \tag{10}$$

其中

$$E_1^n = \frac{1}{2}[(\mathrm{i}u_{\hat{x}}^{n+1}, u^{n+1}) + (\mathrm{i}u_{\hat{x}}^n, u^n)] + \| u_t^n \|^2 + \frac{h}{2}\sum_{j=1}^{J} F(|u_j^{n+1}|^2),$$

$$E_2^n = 2\mathrm{Im}(u_t^n, u^{n+1}) - \frac{\varepsilon}{2}[(\mathrm{i}u_{\hat{x}}^{n+1}, u^{n+1}) + (\mathrm{i}u_{\hat{x}}^n, u^n)].$$

证明 (6) 式两边与$(u_j^n + u_j^{n-1})_t$ 做内积,取实部,并应用引理可得:

第1项

$$\frac{1}{2\tau}\mathrm{Re}(\mathrm{i}u_{\hat{x}}^{n+1} + \mathrm{i}u_{\hat{x}}^{n-1}, u^{n+1} - u^{n-1}) = \frac{1}{2\tau}\mathrm{Re}[(\mathrm{i}u_{\hat{x}}^{n+1}, u^{n+1}) - (\mathrm{i}u_{\hat{x}}^{n+1}, u^{n-1}) + (\mathrm{i}u_{\hat{x}}^{n-1}, u^{n+1}) - (\mathrm{i}u_{\hat{x}}^{n-1}, u^{n-1})] = \frac{1}{2\tau}[(\mathrm{i}u_{\hat{x}}^{n+1}, u^{n+1}) - (\mathrm{i}u_{\hat{x}}^{n-1}, u^{n-1})]; \tag{11}$$

第2项

$$\frac{1}{\tau^3}\mathrm{Re}(u^{n+1}-2u^n+u^{n-1},u^{n+1}-u^{n-1})=\frac{1}{\tau^3}(\|u^{n+1}-u^n\|^2-\|u^n-u^{n-1}\|^2)=$$

$$\frac{1}{\tau}(\|u_t^n\|^2-\|u_t^{n-1}\|^2);\tag{12}$$

第 3 项

$$\frac{\varepsilon}{2\tau^2}\mathrm{Re}(u_{\hat{x}}^{n+1}-u_{\hat{x}}^{n-1},u^{n+1}-u^{n-1})=0;\tag{13}$$

第 4 项

$$h\mathrm{Re}\sum_j\frac{F(|u_j^{n+1}|^2)-F(|u_j^{n-1}|^2)}{|u_j^{n+1}|^2-|u_j^{n-1}|^2}\cdot\frac{u_j^{n+1}+u_j^{n-1}}{2}\cdot(\bar{u}_j^{n+1}-\bar{u}_j^{n-1})=$$

$$\frac{h}{2}\sum_j[F(|u_j^{n+1}|^2)-F(|u_j^{n-1}|^2)].\tag{14}$$

综合上述 4 项可得

$$E_1^n=\frac{1}{2}[(\mathrm{i}u_{\hat{x}}^{n+1},u^{n+1})+(\mathrm{i}u_{\hat{x}}^n,u^n)]+\|u_t^n\|^2+\frac{h}{2}\sum_j[F(|u_j^{n+1}|^2)+F(|u_j^n|^2)]=$$

$$\frac{1}{2}[(\mathrm{i}u_{\hat{x}}^n,u^n)+(\mathrm{i}u_{\hat{x}}^{n-1},u^{n-1})]+\|u_t^{n-1}\|^2+\frac{h}{2}\sum_j[F(|u_j^n|^2)+F(|u_j^{n-1}|^2)]=E_1^{n-1}.\tag{15}$$

由上式递推即得

$$E_1^n=E_1^{n-1}=\cdots=E_1^0.$$

(6) 式两边与 $u_j^{n+1}+u_j^{n-1}$ 做内积,取虚部可得:

第 2 项

$$\mathrm{Im}(u_{t\bar{t}}^n,u^{n+1}+u^{n-1})=\mathrm{Im}[\tau^2(u_{t\bar{t}}^n,u_{t\bar{t}}^n)+2(u_{t\bar{t}}^n,u^n)]=2\mathrm{Im}(u_{t\bar{t}}^n,u^n)=$$

$$\frac{2}{\tau}\mathrm{Im}(u_t^n-u_t^{n-1},u^{n+1}-\tau u_t^n)=\frac{2}{\tau}\mathrm{Im}[(u_t^n,u^{n+1})-(u_t^{n-1},u^n)];\tag{16}$$

第 3 项

$$\frac{\varepsilon}{2\tau}\mathrm{Im}(u_{\hat{x}}^{n+1}-u_{\hat{x}}^{n-1},u^{n+1}+u^{n-1})=-\frac{\varepsilon}{2\tau}[(\mathrm{i}u_{\hat{x}}^{n+1},u^{n+1})-(\mathrm{i}u_{\hat{x}}^{n-1},u^{n-1})];\tag{17}$$

第 1 项和第 4 项为 0.

综合上述 4 项可得

$$E_2^n=2\mathrm{Im}(u_t^n,u^{n+1})-\frac{\varepsilon}{2}[(\mathrm{i}u_{\hat{x}}^{n+1},u^{n+1})+(\mathrm{i}u_{\hat{x}}^n,u^n)]=$$

$$2\mathrm{Im}(u_t^{n-1},u^n)-\frac{\varepsilon}{2}[(\mathrm{i}u_{\hat{x}}^n,u^n)+(\mathrm{i}u_{\hat{x}}^{n-1},u^{n-1})]=E_2^{n-1}.\tag{18}$$

由上式递推即得

$$E_2^n=E_2^{n-1}=\cdots=E_2^0.$$

证毕.

易见,E_1^n,E_2^n 是对(4),(5) 式的数值模拟.

2　差分格式的收敛性和稳定性

设 $v_j^n=u(x_j,t^n)$,则有

$$\frac{1}{2}\mathrm{i}(v_j^{n+1}+v_j^{n-1})_{\hat{x}}+(v_j^n)_{t\bar{t}}+\varepsilon(v_j^n)_{\hat{x}\hat{t}}+\frac{F(|v_j^{n+1}|^2)-F(|v_j^{n-1}|^2)}{|v_j^{n+1}|^2-|v_j^{n-1}|^2}\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2}=R_j^n.$$

其中,R_j^n 为截断误差,显然 $R_j^n=O(h^2+\tau^2)$.

定理 2　设 $u(x,t)\in C^{3,4}(Q)$,$f'(s)\in C^1$,则守恒差分格式(6) ~ (8) 的解收敛到定解问题(1) ~ (3) 的解,且收敛阶为 $O(h^2+\tau^2)$.

证明 设

$$e_j^n = v_j^n - u_j^n,\ G(u_j^{n+1}) = F(|u_j^{n+1}|^2) - F(|u_j^{n-1}|^2)/|u_j^{n+1}|^2 - |u_j^{n-1}|^2,$$

则有

$$\frac{1}{2}\mathrm{i}(e_j^{n+1} + e_j^{n-1})_{\hat{x}} + (e_j^n)_{t\bar{t}} + \varepsilon(e_j^n)_{\hat{x}t\bar{t}} = R_j^n - [G(v_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2} - G(u_j^{n+1})\cdot\frac{u_j^{n+1}+u_j^{n-1}}{2}]. \tag{19}$$

上式两边与$(e_j^n + e_j^{n-1})_t$做内积,取实部可得:

等号右端

$$-h\mathrm{Re}\sum_j[G(v_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2} - G(u_j^{n+1})\cdot\frac{u_j^{n+1}+u_j^{n-1}}{2}](\bar{e}_j^n + \bar{e}_j^{n-1})_t =$$

$$-h\mathrm{Re}\sum_j[G(v_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2} - G(u_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2} + G(u_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2}] -$$

$$G(u_j^{n+1})\cdot\frac{u_j^{n+1}+u_j^{n-1}}{2}](\bar{e}_j^n + \bar{e}_j^{n-1})_t \le C(\|e^{n+1}\|^2 + \|e^{n-1}\|^2 + \|e_t^n\|^2 + \|e_t^{n-1}\|^2); \tag{20}$$

$$\mathrm{Re}(R^n, e_t^n + e_t^{n-1}) \le C\tau(\|R^n\|^2 + \|e_t^n\|^2 + \|e_t^{n-1}\|^2). \tag{21}$$

左端各项与(11)~(13)的处理方式相同,可得

$$\frac{1}{2}[(\mathrm{i}e_{\hat{x}}^{n+1}, e^{n+1}) - (\mathrm{i}e_{\hat{x}}^{n-1}, e^{n-1})] + \|e_t^n\|^2 - \|e_t^{n-1}\|^2 \le$$

$$C\tau(\|e^{n+1}\|^2 + \|e^{n-1}\|^2 + \|e_t^n\|^2 + \|e_t^{n-1}\|^2). \tag{22}$$

(19)式两边与$e_j^{n+1} + e_j^{n-1}$做内积,取虚部并注意到$f'(s) \in C^1$,可得:

等号右端

$$-h\mathrm{Im}\sum_j[G(v_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2} - G(u_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2} + G(u_j^{n+1})\cdot\frac{v_j^{n+1}+v_j^{n-1}}{2} -$$

$$G(u_j^{n+1})\cdot\frac{u_j^{n+1}+u_j^{n-1}}{2}](\bar{e}_j^{n+1} + \bar{e}_j^{n-1}) \le C(\|e^{n+1}\|^2 + \|e^{n-1}\|^2); \tag{23}$$

$$\mathrm{Im}(R^n, e^{n+1} + e^{n-1}) \le C\tau(\|R^n\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}\|^2). \tag{24}$$

左端各项与(16),(17)的处理方式相同,可得

$$2\mathrm{Im}[(e_t^n, e^{n+1}) - (e_t^{n-1}, e^n)] - \frac{\varepsilon}{2}[(\mathrm{i}e_{\hat{x}}^{n+1}, e^{n+1}) - (\mathrm{i}e_{\hat{x}}^{n-1}, e^{n-1})] \le C\tau(\|e^{n+1}\|^2 + \|e^{n-1}\|^2). \tag{25}$$

(22)×ε+(25),然后两端对n从1到$N-1$求和,得

$$2\mathrm{Im}(e_t^{N-1}, e^N) + \varepsilon\|e_t^{N-1}\|^2 \le C(\|e^0\|^2 + \|e^1\|^2 + \|e_t^0\|^2 + \|e_t^1\|^2 + h^4 + \tau^4) +$$

$$C\tau\sum_{n=1}^{N-1}(\|e^{n+1}\|^2 + \|e_t^n\|^2). \tag{26}$$

由$e_j^{n+1} - e_j^n = \tau(e_j^n)_t$,两边同乘以$\bar{e}_j^{n+1} + \bar{e}_j^n$,然后对$j$求和取实部,得

$$\|e^N\|^2 \le C\|e^1\|^2 + C\tau\sum_{n=1}^{N-1}(\|e^{n+1}\|^2 + \|e_t^n\|^2). \tag{27}$$

因为

$$2|\mathrm{Im}(e_t^{N-1}, e^N)| \le \frac{\varepsilon}{2}\|e_t^{N-1}\|^2 + \frac{2}{\varepsilon}\|e^N\|^2, \tag{28}$$

(28)式代入(26)+(27)×$(1+\frac{2}{\varepsilon})$,可得

$$\frac{\varepsilon}{2}\|e_t^{N-1}\|^2 + \|e^N\|^2 \le C(\|e^0\|^2 + \|e^1\|^2 + \|e_t^0\|^2 + \|e_t^1\|^2 + h^4 + \tau^4) +$$

$$C\tau\sum_{n=1}^{N-1}(\|e^{n+1}\|^2 + \|e_t^n\|^2). \tag{29}$$

由初值条件易于验证

$$\|e^0\|^2 + \|e^1\|^2 + \|e_t^0\|^2 + \|e_t^1\|^2 = O(h^4 + \tau^4). \tag{30}$$

由(29),(30)并应用离散的 Gronwall 不等式可得

$$\| e_t^{N-1} \| + \| e^N \| \leq C(h^2 + \tau^2). \tag{31}$$

证毕.

类似地,可以证明差分格式的稳定性.

定理 3　在定理 2 的条件下,守恒差分格式(6) ~ (8) 的解依平方模稳定.

3　数值实验

考虑如下算例:

取 $q = 2, \alpha = 2, \beta = 2; x_l = 20, x_m = -20; t \in [0,1]$, 精确解为[2,3]

$$U(x,t) = b\operatorname{sech}\left(\frac{b}{a}(x - vt)\right)\exp(\mathrm{i}(kx - \Omega t)).$$

其中

$$k = 2, \Omega = 3, v = 4, \varepsilon = 0.1 \ll 1;$$
$$a^2 = v^2 + \varepsilon v, b^2 = k + \Omega^2 + \varepsilon k\Omega.$$

初边值条件相应得到. 利用本文所给格式进行计算,结果如图 1,图 2 所示. 由计算结果可见该格式的有效性.

图 1　取不同值时的误差图像

Fig.1　The error image of different values

图 2　精确解与数值解的图像比较

Fig.2　The comparisons between precise solution and numerical solution

(下转第 22 页)

(上接第17页)

参考文献:

[1] Miki Wadati, H Segur, M J Ablowitz. A new Hamiltonian amplitude equation governing modulated wave instabilities[J]. J Phys Soc Japan, 1992, 61(4):1 187 ~ 1 193.

[2] Zhenya Yan. Symbolic computation and new families of solitary wave solutions to a Hamiltonian amplitude equation[J]. Z Angew Math Phys, 2002, 53(5):533 ~ 537.

[3] Yong Chen, Biao Li, Hongqing Zhang. Explicit exact solutions for a new generalized Hamiltonian amplitude equation with nonlinear terms of any order[J]. Z Angew Math Phys, 2004, 55(4):983 ~ 993.

[4] H Beub. On unstable nonlinear schrödinger equations[J]. Nonlinear Analysis, Theory, Methods & Applications, 1996, 26(5):903 ~ 914.

(编辑:冯保初)