文章编号:1671-9352(2007)04-0019-05

多孔介质中可混溶流体驱动的动态网格特征混合有限元方法

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摘要:结合变网格和特征有限元方法来处理多孔介质中可混溶流体驱动模型问题. 在不同的时间层采用不同的有限元空间,在需要时可以进行加密或稀疏网格,进行基函数调整. 并对算法做出了误差估计.

关键词:多孔介质中可混溶流体驱动模型;变网格;特征有限元方法;误差估计
中图分类号:O241.2　　**文献标识码**:A

Characteristic mixed finite-element method of dynamic grid driven by miscible fluid in porous media

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Abstract: A method for numerically solving miscible displacement in porous media is presented and analyzed using a combination of the method of characteristics with the dynamic finite-element method. Different numbers of elements and basis functions are implemented at different levels. It makes the grid refinements or non-refinements and function adjustments possible at any necessary time. Convergence analysis and error estimates are established.

Key words: miscible fluid driven modle in porous media; variable grid; the characteristics finite element method; error estimates

0　引言

考虑多孔介质中可混溶流体驱动模型是耦合的非线性偏微分方程组的初边值问题,该问题可转化为压力方程和浓度方程,压力方程是椭圆的,浓度方程是对流占优的抛物线方程:

$$\begin{cases} \nabla \cdot u = -\nabla \cdot (a(x,c)\nabla p) = q, & (x,t)\in \Omega\times J, \quad (0.1)\\ \phi(x)\dfrac{\partial c}{\partial t} - \nabla\cdot(D\nabla c) + u\cdot\nabla c = g(x,t,c), & (x,t)\in\Omega\times J. \quad (0.2)\end{cases}$$

其中 $\Omega\subset\mathbf{R}^2$, $J=[0,T]$. p,u 分别是混合流体的压力和达西速度. $a(x,c)=(a_1(x,c),a_2(x,c))$是混合流体的渗透率. $q=q(x,t)$是外部流速. ϕ 是介质粘性, c 是注入液体的浓度, g 是已知函数.

$$D=D(x,u)=\phi(x)[d_mI+|u|(d_lE(u)+d_tE^{\perp}(u))],\ E(u)=(u_iu_j/|u|^2)_{2\times 2},\ E^{\perp}=I-E.$$

d_m 是分子扩散系数, d_l, d_t 分别是纵向和横向弥散系数,令 $\sigma=D\nabla c$,初始边界条件:

收稿日期:2006-04-12
基金项目:国家自然科学基金资助项目(10471079;10071044)
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$$\begin{cases} u\cdot v=0, & (x,t)\in \Gamma\times J, & (0.3)\\ \sigma\cdot v=0, & (x,t)\in \Gamma\times J, & (0.4)\\ c(x,0)=c_0(x), & x\in\Omega. & (0.5)\end{cases}$$

假设条件 A:$0<k_1\le a(x,p)\le k_2$, a,D,g 满足一致 lipschitz 连续,

$$u\in L^\infty(J,H^{k+2}(\Omega)),c\in L^\infty(J,H^{k+2}(\Omega)),\frac{\partial^2 c}{\partial l^2}\in L^2(l^2),\ \frac{\partial u}{\partial t}\in L^2(l^2),\ \frac{\partial c}{\partial t}\in L^2(l^2).$$

若 $d_m\neq 0$ 时,D 是一致正定的,并且可逆,$D^{-1}(x,u)$是一致有界的.

为了保证解的存在惟一性 $\int_\Omega p\,\mathrm{d}x=0,\quad \int_\Omega q(x)\,\mathrm{d}x=0.$

对于求解依赖于时间的偏微分方程,变网格有限元法可以根据实际计算的问题的需要,对不同时刻的空间区域采用不同的有限元网格,加密或变疏局部网格,调整基函数,往往比固定网格具有更好的逼近效果,而且在整体上不增加计算量. 变网格有限元方法,国内外已有许多文献: R.Bonnerot 和 P.jamet 首先在文献[1]提出变网格有限元方法,[2]研究了基本的变网格方法;[3,4]研究了线性抛物型方程初边值问题及变动区域的变网格方法;[5,6]用变网格有限元方法处理了其它一些类型问题. 特征有限元方法对于对流占优问题是有效的方法. [7]利用变网格特征有限元方法对对流占优问题进行了讨论. [8]利用动态网格方法讨论了多孔介质中可混溶流体驱动模型问题. 本文结合变网格和特征有限元方法来处理多孔介质中可混溶流体驱动模型问题.

1 变网格特征混合元格式

令 $W=(w\mid w\in H(\mathrm{div};\Omega),\ w\cdot v=0$ 在 Γ 上$)$,$V=L^2(\Omega)\{v\mid v=$ 常数$\}$,$\Psi=L^2(\Omega)$.

令 $\psi(x,u)=[\phi^2(x)+|u(x,t)|^2]^{1/2}$,$\tau=\tau(x,u)$ 表示 $\phi(x)\dfrac{\partial c}{\partial t}+u\cdot\nabla c$ 的特征方向,则

$$\frac{\partial}{\partial\tau}=\psi^{-1}\left(\phi\frac{\partial}{\partial t}+u\cdot\nabla\right),$$

(0.2) 可表示为形式:

$$\psi(x,u)\frac{\partial c}{\partial\tau}-\nabla\cdot(D\nabla C)=g(x,t,c),\quad (x,t)\in\Omega\times J. \tag{1.1}$$

则(0.1) ~ (0.5) 的弱形式为:求解$\{p,u\}:J\to V\times W$, $\{c,\sigma\}:J\to\Psi\times W$ 满足

$$(\mathrm{div}u,v)=(q,v),\qquad \forall v\in V, \tag{1.2}$$

$$(a^{-1}(x,c)u,w)-(p,\mathrm{div}w)=0,\qquad \forall w\in W, \tag{1.3}$$

$$\left(\psi(x,u)\frac{\partial c}{\partial\tau},\psi\right)-(\mathrm{div}\sigma,\psi)=(g,\psi),\ \forall\psi\in\Psi, \tag{1.4}$$

$$(D^{-1}\sigma,w)+(c,\mathrm{div}w)=0,\qquad \forall w\in W. \tag{1.5}$$

将 $J=[0,T]$ 进行剖分 $\Delta t=T/N$,N 是整数 $J_n=[t_n,t_{n+1}]$,我们将在 t_n 处逼近解,$n=0,1,\cdots N$.

$$\psi(x,u^{n+1})\frac{\partial c^{n+1}}{\partial\tau}\approx\psi(x,u^{n+1})\frac{c^{n+1}-\bar c_n}{(|x-\bar x|^2+\Delta t^2)^{1/2}}=\phi(x)\frac{c^{n+1}-\bar c_n}{\Delta t},\ \bar x=x-\frac{u^{n+1}}{\phi(x)}\Delta t.$$

对 Ω 进行拟正则三角剖分或四边形剖分,在时间层 t_n 处关于压力方程建立 Raviart-Thomas $V_n\times W_n\subset V\times W$ 单元直径为 h_p^n,指数为 k_h,建立 Raviart-Thomas 空间 $\Psi_n\times Z_n\subset\Psi\times Z$ 单元为 h_c^n,指数为 l_n,令 $k=\min\limits_n\{k_n\}$, $h_p=\max\limits_n\{h_p^n\}$, $l=\min\limits_n\{l_n\}$, $h_c=\max\limits_n\{h_c^n\}$.

则空间具有下列逼近性质:

$$\inf_{v_h\in V_n}\|v-v_h\|_V\le Kh_p^s\|v\|_s,\qquad 1\le s\le k+1,\ \forall v\in V, \tag{1.6}$$

$$\inf_{w_h\in W_n}\|w-w_h\|_W\le Kh_p^s(\|w\|_s+\|\mathrm{div}\ w\|_s),1\le s\le k+1,\ \forall w\in W, \tag{1.7}$$

$$\inf_{\psi_h\in\Psi_n}\|\psi-\psi_h\|_\Psi\le Kh_c^s\|\psi\|_s,\qquad 1\le s\le l+1,\ \forall\psi\in\Psi, \tag{1.8}$$

$$\inf_{z_h \in Z_n} \| z - z_h \|_W \le K h_c^s (\| z \|_s + \| \mathrm{div} z \|_s), \quad 1 \le s \le l+1, \ \forall z \in W. \tag{1.9}$$

$(P, U; C, \sum)$逼近$\{p, u; c, \sigma\}$,C_0 是 $c_0(x)$在 Ψ_0 空间上的椭圆投影,$\sum_0$ 是 $\sigma_0(x) = D(x, u_0) \nabla c_0$ 在 Z_0 空间上的椭圆投影. 则(0.1)~(0.5),(1.1)的逼近形式为:

(1) 假定 C_n 已知,求解$\{P_n, U_n\} \in V_n \times V_n$ 满足:

$$(\mathrm{div}\ U_n, v_h) = (q_n, v_h), \ \forall v_h \in V_n, \tag{1.10}$$

$$(a^{-1}(x, C_n) U_n, w_h) - (p_n, \mathrm{div} w_h) = 0, \ \forall w_h \in W_n, \qquad n = 0, 1 \cdots N-1. \tag{1.11}$$

(2) 当 U_n 已知,求解$\{C_{n+1}, \sum_{n+1}\} \in \Psi_{n+1} \times Z_{n+1}$满足:

$$(\phi(\hat{C}_n - C_n), \psi_h) = 0, \ \forall \psi_h \in \Psi_{n+1}, \tag{1.12}$$

$$\left(\phi \frac{C_{n+1} - \tilde{\hat{C}}_n}{\Delta t}, \psi_h \right) - (\mathrm{div} \textstyle\sum_{n+1}, \psi_h) = (g(x, t, \hat{C}_n), \psi_h), \ \forall v_h \in \Psi_{n+1}, \tag{1.13}$$

$$(D^{-1}(U_m) \textstyle\sum_{n+1}, z_h) + (C_{n+1}, \mathrm{div} z_h) = 0, \ \forall z_h \in Z_{n+1}, \qquad n = 0, 1 \cdots N-1. \tag{1.14}$$

其中:

$$C = \hat{C}_n(\tilde{x}), \ \tilde{x} = x - \frac{U_n}{\phi(x)} \Delta t, \ \bar{x} = x - \frac{u^{n+1}}{\phi(x)} \Delta t. \tag{1.15}$$

2 误差估计

我们引进椭圆投影

(1) 求解$\{R_n p, R_n u\}: J \to V_n \times W_n$ 满足

$$(\mathrm{div}\ R_n u, v_h) = (q, v_h), \ \forall v_h \in V_h, \tag{2.1}$$

$$(a^{-1}(c) R_n u, w_h) - (R_n p, \ \mathrm{div} w_h) = 0, \ \forall w_h \in W_n. \tag{2.2}$$

(2) 求解$\{R_n c, R_n \sigma\}: J \to \Psi_n \times Z_n$ 满足

$$(\mathrm{div} R_n \sigma, \psi_h) = (\mathrm{div} \sigma, \psi_h), \ \forall \psi_h \in \Psi_n, \tag{2.3}$$

$$(D^{-1}(u) R_n \sigma, z_h) + (R_n c, \mathrm{div} z_h) = 0, \ \forall z_h \in Z_n. \tag{2.4}$$

由[9,10]知(2.1)~(2.4)有惟一解,并且成立:

$$\| u - R_n u \| + \| p - R_n p \| \le K h_p^{k+1} \| p \|_{k+2}, \ t \in J, \qquad n = 0, 1 \cdots N,$$

$$\| \sigma - R_n \sigma \| + \| c - R_n c \| \le K h_c^{l+1} \| c \|_{l+2}, \ t \in J, \qquad n = 0, 1, \cdots N.$$

由(1.10),(1.11),(2.2)可知:

$$(\mathrm{div}(U_n - R_n u_n), v_h) = 0, \ \forall v_h \in V_n,$$

$$(a^{-1}(C_n)(U_n - R_n u_n), w_h) - (p_n - R_n p_n, \ \mathrm{div}\ w_h) = ((a^{-1}(c_n) - a^{-1}(C_n)) R_n u_n, w_h), \ \forall w_h \in W_n.$$

由 Brezzi 引理可知:

$$\| U_n - R_n u_n \| + \| p_n - R_n p_n \| \le K \| R_n u_n \|_\infty \| c_n - C_n \|, \qquad 0 \le n \le N. \tag{2.5}$$

令

$$e_n = C_n - R_n c_n, \ \varepsilon_n = c_n - R_n c_n, \ \alpha_n = \textstyle\sum_n - R_n \sigma_n, \ \beta_n = \sigma_n - R_n \sigma_n, \qquad n = 0, 1 \cdots N,$$

$$\hat{e}_n = \hat{C}_n - R_{n+1} c_n, \ \hat{\varepsilon}_n = c_n - R_{n+1} c_n, \ \hat{\alpha}_n = \hat{\textstyle\sum}_n - R_{n+1} \sigma_n, \ \hat{\beta}_n = \sigma_n - R_{n+1} \sigma_n, \qquad n = 0, 1 \cdots N.$$

利用(1.13),(1.14),(2.3)式可知:

$$\begin{aligned} \left(\phi \frac{e_{n+1} - \tilde{\hat{e}}_n}{\Delta t}, \psi_h \right) - (\mathrm{div} \alpha_{n+1}, \psi_h) = {} & \left(\psi(x, u^{n+1}) \frac{\partial c^{n+1}}{\partial \tau} - \phi \frac{c_{n+1} - \bar{c}_n}{\Delta t}, \psi_h \right) + \left(\phi \frac{\varepsilon_{n+1} - \overline{\hat{\varepsilon}}_n}{\Delta t}, \psi_h \right) + \\ & (g(x, t, \hat{C}_n) - g(x, t, c_{n+1}), \psi_h) + \left(\phi \frac{R_{n+1} \tilde{c}_n - \tilde{c}_n}{\Delta t}, \psi_h \right) - \\ & \left(\phi \frac{R_{n+1} \bar{c}_n - \bar{c}_n}{\Delta t}, \psi_h \right) + \left(\phi \frac{\tilde{c}_n - \bar{c}_n}{\Delta t}, \psi_h \right). \end{aligned} \tag{2.6}$$

由(1.14),(2.4)可得:

$$(D^{-1}(U_n)\alpha_{n+1},z_h)+(e_{n+1},\mathrm{div}z_h)=((D^{-1}(u_{n+1})-D^{-1}(U_n))R_{n+1}\sigma_{n+1},z_h). \tag{2.7}$$

取 $\psi_h=e_{n+1},z_h=\alpha_{n+1}$.将(2.6),(2.7)式合并得:

$$\left(\phi\frac{e_{n+1}-\tilde{e}_n}{\Delta t},e_{n+1}\right)+(D^{-1}(U_n)\partial_{n+1},\partial_{n+1})=$$
$$\left(\psi(x,u^{n+1})\frac{\partial c^{n+1}}{\partial\tau}-\phi\frac{c_{n+1}-\bar{c}_n}{\Delta t},e_{n+1}\right)+\left(\phi\frac{\varepsilon_{n+1}-\bar{\varepsilon}_n}{\Delta t},e_{n+1}\right)+$$
$$(g(x,t,\hat{C}_n)-g(x,t,c_{n+1}),e_{n+1})+((D^{-1}(u_{n+1})-D^{-1}(U_n))R_{n+1}\sigma_{n+1},\alpha_{n+1})+$$
$$\left\{\left(\phi\frac{R_{n+1}\tilde{c}_n-\tilde{c}_n}{\Delta t},e_{n+1}\right)-\left(\phi\frac{R_{n+1}\bar{c}_n-\bar{c}_n}{\Delta t},e_{n+1}\right)\right\}+\left(\phi\frac{\tilde{c}_n-\bar{c}_n}{\Delta t},e_{n+1}\right)=$$
$$T_1+T_2+T_3+T_4+T_5+T_6.$$

$$T_1|\le\|e_{n+1}\|^2+k\Delta t\left\|\frac{\partial^2c}{\partial\tau^2}\right\|^2_{L_2(J_n,L_2)}, \tag{2.8}$$

$$T_2|\le k(\|e_{n+1}\|^2+\Delta t^{-2}\|\varepsilon_{n+1}-\hat{\bar{\varepsilon}}_n\|^2), \tag{2.9}$$

$$T_3\ |\le k\left(\|\hat{e}_n\|^2+\|\hat{\varepsilon}_n\|^2+\Delta t\int_{t_n}^{t_{n+1}}\left|\frac{\partial c}{\partial t}\right|^2\mathrm{d}t+\|e_{n+1}\|^2\right), \tag{2.10}$$

$$T_4\ |\le k\left(h_p^{2(k+1)}\|p\|^2_{L^\infty(H^{k+2})}+h_c^{2(l+1)}\|c\|^2_{L^\infty(H^{l+2})}+\Delta t\int_{t_n}^{t_{n+1}}\left|\frac{\partial u}{\partial t}\right|^2\mathrm{d}t+\|e_n\|^2\right)+\varepsilon\|\alpha_{n+1}\|^2, \tag{2.11}$$

$$T_5\ |\le k(\|e_{n+1}\|^2+\Delta t^{-2}h_c^{2(l+1)}\|c\|^2_{L^\infty(J^{l+2})}), \tag{2.12}$$

$$T_6\ |\le k\left(h_p^{2(k+1)}\|p\|^2_{L^\infty(H^{k+2})}+h_c^{2(l+1)}\|c\|^2_{L^\infty(H^{l+2})}+\Delta t\int_{t_n}^{t_{n+1}}\left|\frac{\partial u}{\partial t}\right|^2\mathrm{d}t+\|e_{n+1}\|^2+\|e_n\|^2\right). \tag{2.13}$$

$$\phi\left(\frac{e_{n+1}-\hat{e}_n}{\Delta t},e_{n+1}\right)+(D^{-1}(U_n)\alpha_{n+1},\alpha_{n+1})\ge$$
$$\frac{1}{2\Delta t}[\|e_{n+1}\|^2_\phi-\|\hat{e}_n\|^2_\phi]+d_0\|\alpha_{n+1}\|^2\ge$$
$$\frac{1}{2\Delta t}[\|e_{n+1}\|^2_\phi-\|\hat{e}_n\|^2_\phi]-k\|\hat{e}_n\|^2_c+d_0\|\alpha_{n+1}\|^2.$$

由以上可得:

$$\|e_{n+1}\|^2_\phi-\|\hat{e}_n\|^2_\phi+d_0\Delta t\|\alpha_{n+1}\|^2\le k[\Delta t(\|e_{n+1}\|^2+\|\hat{e}_n\|+\|e_n\|^2+\|\hat{\varepsilon}_n\|^2)]+$$
$$\Delta t^2\left(\left\|\frac{\partial^2c}{\partial\tau^2}\right\|^2_{L^2(J_n,L^2)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(J_n,L^2)}\right)+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(J_n,L^2)}+$$
$$\Delta t^{-1}(h_c^{2(l+1)}\|c\|^2_{L^\infty(H^{l+2})}+\|\varepsilon_{n+1}-\hat{\bar{\varepsilon}}_n\|^2)+\Delta th_p^{2(k+1)}\|p\|^2_{L^\infty(H^{k+2})}.$$

由等式(1.12)可得:

$$(\phi(\hat{e}_n-e_n),\psi_h)=(\phi(\hat{\varepsilon}_n-\varepsilon_n),\psi_h),\quad\forall\psi_h\in\psi_{n+1},$$

$$\xi\|\hat{e}_n\|^2_\phi-\|e_n\|^2_\phi\le\frac{1}{1-\xi}\|\hat{\varepsilon}_n-\varepsilon_n\|^2_\phi,\quad\xi\in(0,1),$$

$$\xi\|e_{n+1}\|^2_\phi-\|e_n\|^2_\phi+d_0\xi\Delta t\|\alpha_{n+1}\|^2\le$$

$$k\left[\begin{array}{l}\Delta t(\|e_{n+1}\|^2+\|e_n\|^2+\hat{\varepsilon}_n\|^2)+\Delta t^2\left(\left\|\frac{\partial^2c}{\partial\tau^2}\right\|^2_{L^2(J_n,L^2)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(J_n,L^2)}\right)+\left\|\frac{\partial c}{\partial t}\right\|^2_{L^2(J_n,L^2)}+\\ \Delta t^{-1}(h_c^{2(l+1)}\|c\|^2_{L^\infty(H^{l+2})}+\|\varepsilon_{n+1}-\hat{\bar{\varepsilon}}_n\|^2)+\Delta th_p^{2(k+1)}\|p\|^2_{L^\infty(H^{k+2})}\end{array}\right]+$$

$$\frac{1}{1-\xi}\|\hat{\varepsilon}_n-\varepsilon_n\|^2_\phi.$$

$$\eta_{n+1}=\begin{cases}1,&\text{若 }\psi_n=\psi_{n+1},\qquad Z_n=Z_{n+1},\\ \xi,&\text{若 }\psi_n\ne\psi_{n+1},\qquad\text{或 }Z_n\ne Z_{n+1}.\end{cases}$$

$$
\begin{aligned}
&\eta_{n+1}\|e_{n+1}\|_{\phi}^{2}-\|e_{n}\|_{\phi}^{2}+\eta_{n+1}d_{0}\Delta t\|\alpha_{n+1}\|^{2}\leq\\
&\quad k[\Delta t(\|e_{n+1}\|^{2}+\|e_{n}\|^{2}+\|\hat{\varepsilon}_{n}\|^{2})]+\\
&\quad \Delta t^{2}\left(\left\|\frac{\partial^{2}c}{\partial\tau^{2}}\right\|_{L^{2}(J_n,L^{2})}^{2}+\left\|\frac{\partial u}{\partial t}\right\|_{L^{2}(J_n,L^{2})}^{2}+\left\|\frac{\partial c}{\partial t}\right\|_{L^{2}(J_n,L^{2})}^{2}\right)+\\
&\quad \Delta t^{-1}(h_c^{2(l+1)}\|c\|_{L^{\infty}(H^{l+2})}^{2}+\|\varepsilon_{n+1}-\hat{\bar{\varepsilon}}_{n}\|^{2})+\\
&\quad \Delta t h_p^{2(k+1)}\|p\|_{L^{\infty}(H^{k+2})}+\frac{1}{1-\eta_{n+1}}\|\hat{\varepsilon}_{n}-\varepsilon_{n}\|^{2}.
\end{aligned}
$$

利用[7]1 节中类似方法引理可得:

$$
\|e_m\|^2+\sum_{n=0}^{m-1}\Delta t\|\alpha_{n+1}\|^2\leq
$$

$$
k\left[\Delta t^{2}\left(\left\|\frac{\partial^{2}c}{\partial\tau^{2}}\right\|_{L^{2}(L^{2})}^{2}+\left\|\frac{\partial u}{\partial t}\right\|_{L^{2}(L^{2})}^{2}+\left\|\frac{\partial c}{\partial t}\right\|_{L^{2}(L^{2})}^{2}\right)+\Delta t^{-1}h_c^{2(l+1)}\|c\|_{L^{\infty}(H^{l+2})}^{2}+h_p^{2(k+1)}\|p\|_{L^{\infty}(H^{k+2})}^{2}\right].
$$

定理　$\{p,u,c,\sigma\}$ 则(1.2) ~ (1.5) 的解 $\{P,U,C,\sum\}$ 是方法(1.10) ~ (1.15) 的解,则对于 $k\geq 1, l\geq 1, h_c, h_p$ 充分小并且假设 A 成立时有

$$
\begin{aligned}
&\|P_m-p_m\|+\|U_m-u_m\|+\|C_m-c_m\|+\left(\sum_{n=1}^{m}\Delta t\left\|\sum\nolimits_{n}-\sigma_n\right\|^{2}\right)^{1/2}\leq\\
&\qquad k(\Delta t^{-1}h_c^{l+1}+h_p^{k+1}+\Delta t),\qquad 0\leq m\leq N.
\end{aligned}
$$

致谢:衷心感谢芮洪兴教授的热情鼓励和指导!

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(编辑:李晓红)