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二维扩散方程的一类交替分组方法

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摘要:利用第二类 Saul'yev 型非对称格式,给出了二维扩散方程的一类交替分组方法,并证明了其绝对稳定性,最后给出了数值实验的结果.

关键词:交替分组方法;并行计算;显式方法

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A class of alternating group method for solving the two dimensional diffusion equation

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Abstract: A class of alternating group method is given for solving the two-dimensional diffusion equation. The method is unconditionally stable. The result of a numerical example shows the method is suitable for parallel computing with high precision.

Key words: alternating group method; parallel computing; explicit scheme

0　引言

扩散方程是一类重要的偏微分方程,可以描述热传导、气体的扩散运动,液体的渗透等物理现象,研究其数值解法有重要意义. 数值解法中显式方法便于计算,但稳定性差,隐式方法稳定性好,但不适合并行计算. 因此,构造具有稳定性好且具有并行本性的差分方法就有着重要的理论意义和现实意义. D.J.Evans[1,2] 提出了率先提出了具有并行本性的交替分组差分格式,随后,张宝琳[3,4] 等人作了进一步的研究,提出了交替分段、分块的显隐方法. 本文利用第二类 Saul'yev 型非对称格式,给出了二维扩散方程的一类交替分组格式,并分析了该方法的稳定性;数值实验表明,该方法有较高的精度,是绝对稳定的,并且适合于并行计算.

考虑二维扩散方程的初边值问题:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2},\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le t \le T, \\ u(x,y,0) = f(x,y), \\ u(0,y,t) = g_1(y,t), u(1,y,t) = g_2(y,t), \\ u(x,0,t) = f_1(x,t), u(x,1,t) = f_2(x,t). \end{cases} \tag{0.1}$$

将区域 Ω:$(0,1)\times(0,1)\times(0,T)$网格剖分,为简便起见,将 x 轴和 y 轴等分,记空间步长 $h = \Delta x = \Delta y =$

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$1/m$,时间步长为 $\tau=\Delta t$, $x_i=ih(i=0,1,\cdots m)$, $y_j=jh(j=0,1,\cdots m)$, $t_n=n\tau(n=0,1,\cdots T/\tau)$. 上述问题在 $(jh,jh,n\tau)$ 的数值解记为 $u_{i,j}^n$,简记为 (i,j,n),精确解为 $u(x_i,y_j,t_n)$,记 $r=\tau/h^2$.

1 交替分组方法

1.1 基本格式

利用第二类 Saul'yev 型非对称格式,构造本文的16种基本格式(图1-16):

$$(1+r)u_{i,j}^{n+1}-\frac{r}{2}u_{i+1,j}^{n+1}-\frac{r}{2}u_{i,j+1}^{n+1}=ru_{i-1,j}^n+ru_{i,j-1}^n+(1-3r)u_{i,j}^n+\frac{r}{2}u_{i,j+1}^n+\frac{r}{2}u_{i+1,j}^n, \tag{1.1}$$

$$-\frac{r}{2}u_{i,j}^{n+1}+(1+2r)u_{i+1,j}^{n+1}-ru_{i+2,j}^{n+1}-\frac{r}{2}u_{i+1,j+1}^{n+1}=ru_{i+1,j-1}^n+\frac{r}{2}u_{i,j}^n+(1-2r)u_{i+1,j}^n+\frac{r}{2}u_{i+1,j+1}^n, \tag{1.2}$$

$$-ru_{i+1,j}^{n+1}+(1+2r)u_{i+2,j}^{n+1}-\frac{r}{2}u_{i+3,j}^n-\frac{r}{2}u_{i+2,j+1}^{n+1}=ru_{i+2,j-1}^n+(1-2r)u_{i+2,j}^n+\frac{r}{2}u_{i+3,j}^n+\frac{r}{2}u_{i+2,j+1}^n, \tag{1.3}$$

$$-\frac{r}{2}u_{i+2,j}^{n+1}+(1+r)u_{i+3,j}^{n+1}-\frac{r}{2}u_{i+3,j+1}^{n+1}=ru_{i+3,j-1}^n+(1-3r)u_{i+3,j}^n+ru_{i+4,j}^n+\frac{r}{2}u_{i+3,j+1}^n+\frac{r}{2}u_{i+2,j}^n, \tag{1.4}$$

$$-\frac{r}{2}u_{i,j}^{n+1}+(1+2r)u_{i,j+1}^{n+1}-\frac{r}{2}u_{i+1,j+1}^{n+1}-ru_{i,j+2}^{n+1}=ru_{i-1,j+1}^n+(1-2r)u_{i,j+1}^n+\frac{r}{2}u_{i,j}^n+\frac{r}{2}u_{i+1,j+1}^n, \tag{1.5}$$

$$-\frac{r}{2}u_{i+1,j}^{n+1}+(1+3r)u_{i+1,j+1}^{n+1}-\frac{r}{2}u_{i,j+1}^{n+1}-ru_{i+2,j+1}^{n+1}-ru_{i+1,j+2}^{n+1}=\frac{r}{2}u_{i+1,j}^n+\frac{r}{2}u_{i,j+1}^n+(1-r)u_{i+1,j+1}^n, \tag{1.6}$$

$$-\frac{r}{2}u_{i+2,j}^{n+1}-ru_{i+1,j+1}^{n+1}+(1+3r)u_{i+2,j+1}^{n+1}-\frac{r}{2}u_{i+3,j+1}^{n+1}-ru_{i+2,j+2}^{n+1}=\frac{r}{2}u_{i+2,j}^n+\frac{r}{2}u_{i+3,j+1}^n+(1-r)u_{i+2,j+1}^n, \tag{1.7}$$

$$-\frac{r}{2}u_{i+3,j}^{n+1}-\frac{r}{2}u_{i+2,j+1}^{n+1}+(1+2r)u_{i+3,j+1}^{n+1}-ru_{i+3,j+2}^{n+1}=\frac{r}{2}u_{i+3,j}^n+\frac{r}{2}u_{i+2,j+1}^n+(1-2r)u_{i+3,j+1}^n+ru_{i+4,j+1}^n, \tag{1.8}$$

$$-ru_{i,j+1}^{n+1}+(1+2r)u_{i,j+2}^{n+1}-\frac{r}{2}u_{i+1,j+2}^{n+1}-\frac{r}{2}u_{i,j+3}^{n+1}=ru_{i-1,j+2}^n+(1-2r)u_{i,j+2}^n+\frac{r}{2}u_{i+1,j+2}^n+\frac{r}{2}u_{i,j+3}^n, \tag{1.9}$$

$$-ru_{i+1,j+1}^{n+1}-\frac{r}{2}u_{i,j+2}^{n+1}+(1+3r)u_{i+1,j+2}^{n+1}-ru_{i+2,j+2}^{n+1}-\frac{r}{2}u_{i+1,j+3}^{n+1}=\frac{r}{2}u_{i,j+2}^n+(1-r)u_{i+1,j+2}^n+\frac{r}{2}u_{i+1,j+3}^n, \tag{1.10}$$

$$-ru_{i+2,j+1}^{n+1}-ru_{i+1,j+2}^{n+1}+(1+3r)u_{i+2,j+2}^{n+1}-\frac{r}{2}u_{i+3,j+2}^{n+1}-\frac{r}{2}u_{i+2,j+3}^{n+1}=(1-r)u_{i+2,j+2}^n+\frac{r}{2}u_{i+3,j+2}^n+\frac{r}{2}u_{i+2,j+3}^n, \tag{1.11}$$

$$-ru_{i+3,j+1}^{n+1}-\frac{r}{2}u_{i+2,j+2}^{n+1}+(1+2r)u_{i+3,j+2}^{n+1}-\frac{r}{2}u_{i+3,j+3}^{n+1}=\frac{r}{2}u_{i+2,j+2}^n+(1-2r)u_{i+3,j+2}^n+\frac{r}{2}u_{i+3,j+3}^n+ru_{i+4,j+2}^n, \tag{1.12}$$

$$-\frac{r}{2}u_{i,j+2}^{n+1}+(1+r)u_{i,j+3}^{n+1}-\frac{r}{2}u_{i+1,j+3}^{n+1}=ru_{n-1,j+3}^n+\frac{r}{2}u_{i,j+2}^n+(1-3r)u_{i,j+3}^n+\frac{r}{2}u_{i+1,j+2}^n+ru_{i,j+4}^n, \tag{1.13}$$

$$-\frac{r}{2}u_{i+1,j+2}^{n+1}-\frac{r}{2}u_{i,j+3}^{n+1}+(1+2r)u_{i+1,j+3}^{n+1}-ru_{i+2,j+3}^{n+1}=\frac{r}{2}u_{i+1,j+2}^n+\frac{r}{2}u_{i,j+3}^n+(1-2r)u_{i+1,j+3}^n+ru_{i+1,j+4}^n, \tag{1.14}$$

$$-\frac{r}{2}u_{i+2,j+2}^{n+1}-ru_{i+1,j+3}^{n+1}+(1+2r)u_{i+2,j+3}^{n+1}-\frac{r}{2}u_{i+3,j+3}^{n+1}=\frac{r}{2}u_{i+2,j+2}^n+(1-2r)u_{i+2,j+3}^n+\frac{r}{2}u_{i+3,j+3}^n+ru_{i+2,j+4}^n, \tag{1.15}$$

$$-\frac{r}{2}u_{i+3,j+2}^{n+!}-\frac{r}{2}u_{i+2,j+3}^{n+1}+(1+r)u_{i+3,j+3}^{n+1}=\frac{r}{2}u_{i+3,j+2}^n+\frac{r}{2}u_{i+2,j+3}^n+(1-3r)u_{i+3,j+3}^n+ru_{i+4,j+3}^n+ru_{i+3,j+4}^n. \tag{1.16}$$

1.2　基本分组情况

令内点数 $m-1=4p+2$(p 为整数),已知第 n 层上的解,要计算第 $n+1$ 和第 $n+2$ 层上的数值解,构造连续两层独立计算交替分组格式如图 17 和图 18 所示:

图 17　$n+1$ 层分组情况
Fig.17　The grouping at $n+1$ level

图 18　$n+2$ 层分组情况
Fig.18　The grouping at $n+2$ level

1.3　基本点组求解格式的构造

下面介绍用到的基本点组.

(16 点组):基本的 N 点组共有 16 个内点,横向和纵向内点分布数相同,按照先横向后纵向的排列顺序,依次运用格式(1.1)~(1.16). 这样组合的目的有两个:其一,分组计算以达到并行目的;其二,单个非对称格式(1.1)~(1.16)产生的误差经组合后可部分抵消,从而可显著提高求解精度.

若设 $\bar{\boldsymbol{u}}_{i,j}^{n}=(\boldsymbol{u}_{j}^{n},\boldsymbol{u}_{j+1}^{n},\boldsymbol{u}_{j+2}^{n},\boldsymbol{u}_{j+3}^{n})^{\mathrm{T}}$, $\boldsymbol{u}_{j+k}^{n}=(u_{i,j+k}^{n},u_{i+1,j+k}^{n},u_{i+2,j+k}^{n},u_{i+3,j+k}^{n})^{\mathrm{T}}$, $k=0,1,2,3$,

$$\bar{\boldsymbol{F}}_{i,j}^{n}=(\boldsymbol{F}_{j}^{n},\boldsymbol{F}_{j+1}^{n},\boldsymbol{F}_{j+2}^{n},\boldsymbol{F}_{j+3}^{n})^{\mathrm{T}},\ \boldsymbol{F}_{j}^{n}=(ru_{i-1,j}^{n}+ru_{i,j-1}^{n},ru_{i+1,j-1}^{n},ru_{i+2,j-1}^{n},ru_{i+4,j}^{n}+ru_{i+3,j-1}^{n})^{\mathrm{T}},$$

$$\boldsymbol{F}_{j+1}^{n}=(ru_{i-1,j+1}^{n},0,0,ru_{i+4,j+1}^{n})^{\mathrm{T}},\ \boldsymbol{F}_{j+2}^{n}=(ru_{i-1,j+2}^{n},0,0,ru_{i+4,j+2}^{n})^{\mathrm{T}},$$

$$\boldsymbol{F}_{j+3}^{n}=(ru_{i-1,j+3}^{n}+ru_{i,j+4}^{n},ru_{i+1,j+4}^{n},ru_{i+2,j+4}^{n},ru_{i+3,j+4}^{n}+ru_{i+4,j+3}^{n})^{\mathrm{T}}.$$

则 16 点格式为:

$$(\boldsymbol{I}+r\boldsymbol{A}_1)\bar{\boldsymbol{u}}_{i,j}^{n+1}=(\boldsymbol{I}-r\boldsymbol{B}_1)\bar{\boldsymbol{u}}_{i,j}^{n}+\bar{\boldsymbol{F}}_{i,j}^{n}. \tag{1.17}$$

令 $\boldsymbol{C}_{11}=\begin{bmatrix}1 & -1/2\\ -1/2 & 2\end{bmatrix}$, $\boldsymbol{C}_{12}=\begin{bmatrix}2 & -1/2\\ -1/2 & 1\end{bmatrix}$, $\boldsymbol{C}_{13}=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}$, $\boldsymbol{C}_{14}=\begin{bmatrix}-1/2 & 0\\ 0 & -1/2\end{bmatrix}$, $\boldsymbol{C}_{15}=\begin{bmatrix}-1\\ 0\end{bmatrix}$,有:

$$\boldsymbol{A}_1=\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12} & & \\ \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \boldsymbol{A}_{14} & \\ & \boldsymbol{A}_{14} & \boldsymbol{A}_{13} & \boldsymbol{A}_{12}\\ & & \boldsymbol{A}_{12} & \boldsymbol{A}_{11}\end{bmatrix}_{16\times 16},\ \boldsymbol{B}_1=\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12} & & \\ \boldsymbol{B}_{12} & \boldsymbol{B}_{13} & & \\ & & \boldsymbol{B}_{13} & \boldsymbol{B}_{12}\\ & & \boldsymbol{B}_{12} & \boldsymbol{B}_{11}\end{bmatrix}_{16\times 16},\ \boldsymbol{A}_{11}=\begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{13}^{\mathrm{T}}\\ \boldsymbol{C}_{13} & \boldsymbol{C}_{12}\end{bmatrix},$$

$$\boldsymbol{A}_{12}=\begin{bmatrix}\boldsymbol{C}_{14} & 0\\ 0 & \boldsymbol{C}_{14}\end{bmatrix},\ \begin{cases}\boldsymbol{A}_{13}=\boldsymbol{I}+\boldsymbol{A}_{11}\\ \boldsymbol{A}_{14}=2\boldsymbol{A}_{12}\end{cases};\ \boldsymbol{B}_{11}=\boldsymbol{I}+\begin{bmatrix}\boldsymbol{C}_{12} & 0\\ 0 & \boldsymbol{C}_{11}\end{bmatrix},\ \boldsymbol{B}_{12}=\begin{bmatrix}\boldsymbol{C}_{14} & 0\\ 0 & \boldsymbol{C}_{14}\end{bmatrix},\ \boldsymbol{B}_{13}=\boldsymbol{B}_{11}-\boldsymbol{I}.$$

(Lx 组):按照先横向后纵向顺序运用格式(1.3),(1.4),(1.7),(1.8),(1.11),(1.12),(1.15),(1.16).

若设 $\bar{\boldsymbol{v}}_{i,j}^{n+1}=(\boldsymbol{v}_j^{n+1},\boldsymbol{v}_{j+1}^{n+1},\boldsymbol{v}_{j+2}^{n+1},\boldsymbol{v}_{j+3}^{n+1})^{\mathrm{T}}$,其中 $\boldsymbol{v}_{j+k}^{n+1}=(v_{1,j+k}^{n+1},v_{2,j+k}^{n+1})^{\mathrm{T}}$, $k=0,1,2,3$,

$$\bar{\boldsymbol{w}}_{i,j}^{n}=(\boldsymbol{w}_j^n,\boldsymbol{w}_{j+1}^n,\boldsymbol{w}_{j+2}^n,\boldsymbol{w}_{j+3}^n)^{\mathrm{T}},\ \boldsymbol{w}_j^n=(ru_{1,j-1}^n+ru_{0,j}^{n+1},ru_{2,j-1}^n+ru_{3,j}^n)^{\mathrm{T}},\ \boldsymbol{w}_{j+1}^n=(ru_{0,j+1}^{n+1},ru_{3,j+1}^n)^{\mathrm{T}},$$

$$\boldsymbol{w}_{j+2}^n=(ru_{0,j+2}^{n+1},ru_{3,j+2}^n)^{\mathrm{T}},\ \boldsymbol{w}_{j+3}^n=(ru_{0,j+3}^{n+1}+ru_{1,j+4}^n,ru_{2,j+4}^n+ru_{3,j+3}^n)^{\mathrm{T}},$$

则有

$$(\boldsymbol{I}+r\boldsymbol{A}_2)\bar{\boldsymbol{v}}_{i,j}^{n+1}=(\boldsymbol{I}-r\boldsymbol{B}_2)\bar{\boldsymbol{v}}_{i,j}^{n}+\bar{\boldsymbol{w}}_{i,j}^{n}. \tag{1.18}$$

$$\boldsymbol{A}_2=\begin{bmatrix}\boldsymbol{A}_{21} & \boldsymbol{A}_{22}\\ \boldsymbol{A}_{23} & \boldsymbol{A}_{24}\end{bmatrix}_{8\times 8},\ \boldsymbol{B}_2=\begin{bmatrix}\boldsymbol{B}_{21} & O\\ O & \boldsymbol{B}_{22}\end{bmatrix}_{8\times 8},\ \boldsymbol{B}_{21}=\begin{bmatrix}\boldsymbol{I}+\boldsymbol{C}_{11} & \boldsymbol{C}_{14}\\ \boldsymbol{C}_{14} & \boldsymbol{C}_{11}\end{bmatrix},\ \boldsymbol{B}_{22}=\begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{14}\\ \boldsymbol{C}_{14} & \boldsymbol{I}+\boldsymbol{C}_{11}\end{bmatrix},$$

$$\boldsymbol{A}_{21}=\begin{bmatrix}\boldsymbol{C}_{12} & \boldsymbol{C}_{14}\\ \boldsymbol{C}_{14} & \boldsymbol{I}+\boldsymbol{C}_{12}\end{bmatrix},\ \boldsymbol{A}_{22}=\begin{bmatrix}0 & 0\\ 2\boldsymbol{C}_{14} & 0\end{bmatrix},\ \boldsymbol{A}_{23}=\boldsymbol{A}_{22}^{\mathrm{T}},\ \boldsymbol{A}_{24}=\begin{bmatrix}\boldsymbol{I}+\boldsymbol{C}_{12} & \boldsymbol{C}_{14}\\ \boldsymbol{C}_{14} & \boldsymbol{C}_{12}\end{bmatrix}.$$

(其它点组):对 Ly 组、Rx 组、Ry 组格式的运用,分别取 16 点组的上半部、左半部、下半部;而对 G 组和 H 组,分别取 16 点组的右上角和左下角.

1.4 交替分组总求解格式的构造

按照前面的分组模式,设 $\bar{\boldsymbol{u}}^n=(\bar{\boldsymbol{u}}_1^n,\bar{\boldsymbol{u}}_5^n,\bar{\boldsymbol{u}}_9^n,\cdots\bar{\boldsymbol{u}}_{m-6}^n,\bar{\boldsymbol{u}}_{m-2}^n)^{\mathrm{T}}$, $\bar{\boldsymbol{u}}_j^n=(\bar{\boldsymbol{u}}_{1,j}^n,\bar{\boldsymbol{u}}_{5,j}^n,\bar{\boldsymbol{u}}_{9,j}^n,\cdots\bar{\boldsymbol{u}}_{m-2}^n)^{\mathrm{T}}$, $\bar{\boldsymbol{u}}_j^n=(\bar{\boldsymbol{u}}_{1,j}^n,\bar{\boldsymbol{u}}_{5,j}^n,\bar{\boldsymbol{u}}_{9,j}^n,\cdots\bar{\boldsymbol{u}}_{m-6,j}^n,\bar{\boldsymbol{u}}_{m-2,j}^n)^{\mathrm{T}}$, $j=1,5,9,\cdots m-6$; 其中 $\bar{\boldsymbol{u}}_{i,j}^n(i=1,5,\cdots\cdots m-6)$定义如前,

$$\bar{\boldsymbol{u}}_{m-2,j}^n=(\boldsymbol{u}_{m-2,j}^n,\boldsymbol{u}_{m-1,j}^n,\boldsymbol{u}_{m-2,j+1}^n,\boldsymbol{u}_{m-1,j+1}^n,\boldsymbol{u}_{m-2,j+2}^n,\boldsymbol{u}_{m-1,j+2}^n,\boldsymbol{u}_{m-2,j+3}^n,\boldsymbol{u}_{m-1,j+3}^n)^{\mathrm{T}};$$

$$\bar{\boldsymbol{u}}_{m-2}^n=(\bar{\boldsymbol{u}}_{1,m-2}^n,\bar{\boldsymbol{u}}_{5,m-2}^n,\bar{\boldsymbol{u}}_{9,m-2}^n,\cdots\bar{\boldsymbol{u}}_{m-6,m-2}^n,\bar{\boldsymbol{u}}_{m-2,m-2}^n)^{\mathrm{T}},$$

$$\bar{\boldsymbol{u}}_{i,m-2}^n=(u_{i,m-2}^n,u_{i,m-1}^n,u_{i+1,m-2}^n,u_{i+1,m-1}^n,u_{i+2,m-2}^n,u_{i+2,m-1}^n,u_{i+3,m-2}^n,u_{i+3,m-1}^n)^{\mathrm{T}},i=1,5,9,\cdots m-6$$

$$\bar{\boldsymbol{u}}_{m-2,m-2}^n=(u_{m-2,m-2}^n,u_{m-2,m-1}^n,,u_{m-1,m-2}^n,u_{m-1,m-1}^n)^{\mathrm{T}},$$

则有连续两层上的交替分组计算格式:

$$\begin{cases}(\boldsymbol{I}+r\boldsymbol{G}_1)\bar{\boldsymbol{u}}_{n+1}=(\boldsymbol{I}-r\boldsymbol{G}_2)\bar{\boldsymbol{u}}_n+\boldsymbol{b}_1^n,\\ (\boldsymbol{I}+r\boldsymbol{G}_2)\bar{\boldsymbol{u}}_{n+2}=(\boldsymbol{I}-r\boldsymbol{G}_1)\bar{\boldsymbol{u}}_{n+1}+\boldsymbol{b}_2^n.\end{cases} \tag{1.19}$$

其中 $\boldsymbol{b}_1^n$ 和 $\boldsymbol{b}_2^n$ 为已知的向量. 设$(m-1)^2=a$, $2(m-3)+4=b$, $4(m-3)+8=c$, 则有:

$$\boldsymbol{G}_1=\begin{bmatrix}\boldsymbol{G}_{11} & & & & \\ & \boldsymbol{G}_{11} & & & \\ & & \cdots & & \\ & & & \boldsymbol{G}_{11} & \\ & & & & \boldsymbol{G}_{12}\end{bmatrix}_{a\times a},\ \boldsymbol{G}_{11}=\begin{bmatrix}\boldsymbol{A}_1 & & & & \\ & \boldsymbol{A}_1 & & & \\ & & \cdots & & \\ & & & \boldsymbol{A}_1 & \\ & & & & \boldsymbol{A}_3\end{bmatrix}_{c\times c},$$

$$\boldsymbol{G}_{12}=\begin{bmatrix}\boldsymbol{A}_3 & & & & \\ & \boldsymbol{A}_3 & & & \\ & & \cdots & & \\ & & & \boldsymbol{A}_3 & \\ & & & & \boldsymbol{A}_4\end{bmatrix}_{b\times b},\ \boldsymbol{A}_3=\begin{bmatrix}\boldsymbol{A}_{31} & \boldsymbol{A}_{32}\\ \boldsymbol{A}_{33} & \boldsymbol{A}_{34}\end{bmatrix}_{8\times 8},\ \begin{cases}\boldsymbol{A}_{31}=\boldsymbol{B}_{22}\\ \boldsymbol{A}_{33}=\boldsymbol{A}_{22}^{\mathrm{T}}\end{cases},\ \begin{cases}\boldsymbol{A}_{32}=\boldsymbol{A}_{22}\\ \boldsymbol{A}_{34}=\boldsymbol{B}_{21}\end{cases},$$

$$\begin{cases} \boldsymbol{A}_4 = \boldsymbol{B}_{22} \\ \boldsymbol{B}_4 = \boldsymbol{A}_{24} \end{cases},$$

$$\boldsymbol{G}_2 = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{M}^{\mathrm{T}} & & & & \\ \boldsymbol{M} & \boldsymbol{P} & \boldsymbol{M}^{\mathrm{T}} & & & \\ & \cdots & \cdots & \cdots & & \\ & & \boldsymbol{M} & \boldsymbol{P} & \boldsymbol{M}^{\mathrm{T}} & \\ & & & \boldsymbol{M} & \boldsymbol{P} & \boldsymbol{N}^{\mathrm{T}} \\ & & & & \boldsymbol{N} & \boldsymbol{Q} \end{bmatrix}_{a \times a}, \quad \boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{E}_1^{\mathrm{T}} & & & & \\ \boldsymbol{E}_1 & \boldsymbol{P}_1 & \boldsymbol{E}_1^{\mathrm{T}} & & & \\ & \cdots & \cdots & \cdots & & \\ & & \boldsymbol{E}_1 & \boldsymbol{P}_1 & \boldsymbol{E}_1^{\mathrm{T}} & \\ & & & \boldsymbol{E}_1 & \boldsymbol{P}_1 & \boldsymbol{F}_1^{\mathrm{T}} \\ & & & & \boldsymbol{F}_1 & \boldsymbol{P}_2 \end{bmatrix}_{c \times c},$$

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_1 & \boldsymbol{E}_2^{\mathrm{T}} & & & & \\ \boldsymbol{E}_2 & \boldsymbol{Q}_1 & \boldsymbol{E}_2^{\mathrm{T}} & & & \\ & \cdots & \cdots & \cdots & & \\ & & \boldsymbol{E}_2 & \boldsymbol{Q}_1 & \boldsymbol{E}_2^{\mathrm{T}} & \\ & & & \boldsymbol{E}_2 & \boldsymbol{Q}_1 & \boldsymbol{F}_2^{\mathrm{T}} \\ & & & & \boldsymbol{F}_2 & \boldsymbol{Q}_2 \end{bmatrix}_{b \times b}, \boldsymbol{P}_1 = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} & O & O \\ \boldsymbol{P}_{12} & \boldsymbol{P}_{13} & O & O \\ O & O & \boldsymbol{P}_{13} & \boldsymbol{P}_{12} \\ O & O & \boldsymbol{P}_{12} & \boldsymbol{P}_{11} \end{bmatrix}_{16 \times 16}, \quad \boldsymbol{P}_2 = \begin{bmatrix} \boldsymbol{P}_{21} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{P}_{22} \end{bmatrix},$$

$$\boldsymbol{P}_{11} = \begin{bmatrix} \boldsymbol{I} + \boldsymbol{C}_{12} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} + \boldsymbol{C}_{11} \end{bmatrix}, \ \boldsymbol{P}_{12} = \begin{bmatrix} \boldsymbol{C}_{14} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{14} \end{bmatrix}, \ \boldsymbol{P}_{13} = \begin{bmatrix} \boldsymbol{C}_{1,2} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{1,1} \end{bmatrix}, \ \begin{cases} \boldsymbol{P}_{2,1} = \boldsymbol{A}_{2,4}, \\ \boldsymbol{P}_{2,2} = \boldsymbol{A}_{2,1} \end{cases},$$

$$\boldsymbol{E}_1 = \begin{bmatrix} \boldsymbol{E}_{1,1} & & & \\ & \boldsymbol{E}_{1,1} & & \\ & & \boldsymbol{E}_{1,1} & \\ & & & \boldsymbol{E}_{1,1} \end{bmatrix}_{16 \times 16}, \quad \boldsymbol{F}_1 = \begin{bmatrix} \boldsymbol{F}_{1,1} & & & \\ & \boldsymbol{F}_{1,1} & & \\ & & \boldsymbol{F}_{1,1} & \\ & & & \boldsymbol{F}_{1,1} \end{bmatrix}_{8 \times 16},$$

$$\boldsymbol{E}_{1,1} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{C}_{1,3} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}_{4 \times 4}, \ \boldsymbol{F}_{1,1} = [\boldsymbol{O} \quad \boldsymbol{C}_{13}]_{2 \times 4}, \ \boldsymbol{E}_2 = \begin{bmatrix} \boldsymbol{O} & 2\boldsymbol{C}_{1,4} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}_{4 \times 8}, \ \boldsymbol{Q}_1 = \boldsymbol{P}_2, \ \boldsymbol{Q}_2 = \boldsymbol{A}_{2,4},$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_1 & & & \\ & \cdots & & \\ & & \boldsymbol{M}_1 & \\ & & & \boldsymbol{M}_2 \end{bmatrix}_{c \times c}, \begin{cases} \boldsymbol{M}_1 = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{M}_{1,1} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}_{16 \times 16} \\ \boldsymbol{M}_{1,1} = \begin{bmatrix} 2\boldsymbol{C}_{1,4} & \boldsymbol{O} \\ \boldsymbol{O} & 2\boldsymbol{C}_{1,4} \end{bmatrix}_{4 \times 4} \end{cases}, \ \boldsymbol{M}_2 = \begin{bmatrix} \boldsymbol{O} & 2\boldsymbol{C}_{1,4} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}_{8 \times 8},$$

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{N}_1 & & & \\ & \cdots & & \\ & & \boldsymbol{N}_1 & \\ & & & \boldsymbol{N}_2 \end{bmatrix}_{c \times b}, \begin{cases} \boldsymbol{N}_1 = [O \quad \boldsymbol{N}_{11}]_{8 \times 16} \\ \boldsymbol{N}_2 = [O \quad \boldsymbol{N}_{21}]_{4 \times 8}, \end{cases} \boldsymbol{N}_{2,1} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{N}_{1,1} = \begin{bmatrix} \boldsymbol{C}_{15} & & & \\ & \boldsymbol{C}_{15} & & \\ & & \boldsymbol{C}_{15} & \\ & & & \boldsymbol{C}_{15} \end{bmatrix}.$$

2　稳定性分析

Kellog 引理[5]　设 $r>0$,矩阵 $\boldsymbol{G}$ 是非负定实阵,则成立

$$\begin{cases} \|(\boldsymbol{I}+r\boldsymbol{G})^{-1}\|_2 \le 1, \\ \|(\boldsymbol{I}-r\boldsymbol{G})(\boldsymbol{I}+r\boldsymbol{G})^{-1}\|_2 \le 1. \end{cases} \tag{2.1}$$

从矩阵的构造过程易知矩阵 $\boldsymbol{G}_1$ 和 $\boldsymbol{G}_2$ 都是对角占优的,即矩阵 $\boldsymbol{G}_1$ 和 $\boldsymbol{G}_2$ 都是非负定实矩阵,所以 $\|(\boldsymbol{I}+r\boldsymbol{G}_1)^{-1}\|_2\le 1$, $\|(\boldsymbol{I}-r\boldsymbol{G}_1)(\boldsymbol{I}+r\boldsymbol{G}_1)^{-1}\|_2\le 1$; $\|(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}\|_2\le 1$, $\|(\boldsymbol{I}-r\boldsymbol{G}_2)(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}\|_2\le 1$. 设 n 为偶数,有 $\boldsymbol{u}^n=\boldsymbol{G}\boldsymbol{u}^{n-2}$,其中 $\boldsymbol{G}=(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}(\boldsymbol{I}-r\boldsymbol{G}_1)(\boldsymbol{I}+r\boldsymbol{G}_1)^{-1}(\boldsymbol{I}-r\boldsymbol{G}_2)$为增长矩阵,记 $\overline{\boldsymbol{G}}=(\boldsymbol{I}+r\boldsymbol{G}_2)\boldsymbol{G}(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}=(\boldsymbol{I}-r\boldsymbol{G}_1)(\boldsymbol{I}+r\boldsymbol{G}_1)^{-1}(\boldsymbol{I}-r\boldsymbol{G}_2)(\boldsymbol{I}+r\boldsymbol{G}_2)^{-1}$,显然有 $\rho(\boldsymbol{G})=\rho(\overline{\boldsymbol{G}})\le\|\overline{\boldsymbol{G}}\|_2\le 1$,所以由(1.19)式定义的交替分组方法绝对稳定. 从而得到下面的定理:

定理　由(1.19)式定义的交替分组方法是绝对稳定的.

3 数值算例

考虑如下初边值问题:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2},\ 0\leq x\leq 1,\ 0\leq y\leq 1,\ 0\leq t\leq T, \\ u(x,y,0)=\sin(\pi x)\sin(\pi y), \\ u(0,y,t)=0,u(1,y,t)=0, \\ u(x,0,t)=0,u(x,1,t)=0, \end{cases} \tag{3.1}$$

其精确解[3]为 $u(x,y,t)=e^{-2\pi^2 t}\sin(\pi x)\sin(\pi y)$,下面将本文提出的交替分组求解方法(1.19)算得的数值结果与文献[3]的结果及精确解作了比较. 其中 A.E 和 P.E 表示本文方法的绝对误差和相对误差;A.E[3] 和 P.E[3] 表示文献[3]方法的绝对误差和相对误差;Exact 表示精确解.

(1) 表 1 给出了 $r=0.4$, $m=19$, $h=1/19$, $\tau=r/361$, $t=390\tau$, $y=16h$ 时本文格式与文献[3]提出的格式数值结果比较,可以发现,本文所采用的格式具有较高的数值求解精度.

表 1 $r=0.4$, $m=19$, $h=1/19$, $\tau=r/361$, $t=390\tau$, $y=16h$ 时与文献[3]的数值比较结果

Table 1 Comparison of results with thesis[3] at $r=0.4$, $m=19$, $h=1/19$, $\tau=r/361$, $t=390\tau$, $y=16h$

Error	x						
	$6h$	$7h$	$8h$	$9h$	$10h$	$11h$	$12h$
A.E	1.185×10^{-7}	8.08×10^{-8}	1.05×10^{-7}	1.088×10^{-7}	9.48×10^{-8}	1.022×10^{-7}	8.82×10^{-8}
P.E	1.506×10^{-1}	9.385×10^{-2}	1.152×10^{-2}	1.162×10^{-1}	1.102×10^{-1}	1.121×10^{-1}	1.025×10^{-1}
A.E[3]	6.677×10^{-7}	6.488×10^{-7}	7.800×10^{-7}	7.881×10^{-7}	7.651×10^{-7}	7.099×10^{-7}	6.520×10^{-7}
P.E[3]	8.487×10^{-1}	7.539×10^{-1}	8.562×10^{-1}	8.415×10^{-1}	8.168×10^{-1}	7.793×10^{-1}	7.576×10^{-1}
Exact	7.867×10^{-5}	8.607×10^{-5}	9.111×10^{-5}	9.366×10^{-5}	9.366×10^{-5}	9.111×10^{-5}	8.607×10^{-5}

(2) 图 19 给出了 $r=0.4$, $m=23$, $h=1/23$, $\tau=r/361$, $t=390\tau$, $y=16h$ 时数值解与精确解的比较,可以发现用本文给出的分组格式所求的数值解与精确解十分吻合.

图 19 $r=0.4, m=23, h=1/23, \tau=r/361, t=390\tau, y=16h$ 数值解与精确解比较图

Fig.19 Comparison between numerical results and exact results with $r=0.4, m=23, h=1/23, \tau=r/361, t=390\tau, y=16h$

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