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Finite-time stabilization for a class of first-order nonlinear systems with unknown control direction

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Abstract: In this paper, the finite time stabilization via state-feedback and adaptive technique was investigated for a class of first-order nonlinear systems with unknown control direction. Using the Nussbaum gain method, an adaptive state-feedback controller is successfully constructed, which guarantees the global stability of the closed-loop system, and the global finite time stability of the original system state (This was rigorously proven with the help of the celebrated L'Hôspital's Rule). A simulation example was provided to illustrate the effectiveness of the proposed approach.

Key Words: nonlinear systems; unknown control direction; adaptive control; Nussbaum gain; finite time stability; global stability

一类一阶控制系数未知非线性系统有限时间镇定

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摘要:应用状态反馈和自适应技术,研究了一类一阶控制系数未知非线性系统有限时间镇定. 基于 Nussbaum 增益方法, 成功地构造了自适应状态反馈控制器,确保了闭环系统的全局稳定性, 并且原系统的状态是有限时间稳定的(其严格证明是借助于著名的罗彼塔法则完成的). 仿真算例验证了该方法的有效性.

关键词:非线性系统;未知控制系数;自适应控制;Nussbaum 增益;有限时间稳定;全局稳定
中图分类号:O231;TP273 **文献标志码**:A

0 Introduction

The control problems of the systems with unknown control directions have received much attention in the past three decades[1-9]. When the signs of control coefficients are unknown, the control problem becomes much more difficult, because in this case, we cannot decide the direction along which the control operates. This control problem had remained open till the early 1 980 s, the breakthrough solution was originally given for a class of first-order linear systems by introducing the Nussbaum function[2]. Using Nussbaum gain methods, adaptive control was given for first-order nonlinear systems in [1]. As the first step toward higher order systems, backstepping with Nussbaum function was then developed for second-order systems in [5]. Later, backstepping with Nussbaum function was successfully developed for arbitrary any finite order of nonlinear systems in the triangular structure, with constant unknown control coefficients in [6] and with

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time varying unknown control coefficients[7-8], respectively.

Fair recently, the finite-time stabilization has been roundly studied for nonlinear systems. Different from the asymptotical stabilization with infinite settling time, finite-time stabilization gives the convergence with finite settling time. Moreover, finite-time stabilization is of much interest partially because of its faster convergence, higher tracking precision and robustness to uncertainties[10-11]. The non-smooth (continuous) analysis has been the prevailing methodology to finite-time stabilization for a lot of classes of nonlinear systems[11-18]. More specifically, [11] and [12] provided a rigorous foundation for the theory of finite-time stability. Works [14] and [15] proposed the explicit design scheme for finite-time stable controller for two classes of nonlinear systems. Work [13] first considered the case of output-feedback, and the further results see the recent paper[16]. As the last development on this topic, [17] presented the adaptive finite-time control methodology for a class of uncertain nonlinear systems.

In this paper, the global adaptive finite-time stabilization is investigated for a class of first-order nonlinear systems with unknown control direction. Despite the aforementioned progress, this control problem is quite complicated and has remained open and unsolved up to now. The major difficulty of the problem is the absence of effective methods to analyze the finite-time stability when the control direction is unknown although it is not hard to design a continuous state-feedback controller. Inspired by the recent works[6, 9, 17], a Nussbaum function is incorporated into the estimation for both value and direction of the unknown control coefficient, and then an adaptive state-feedback controller is successfully constructed. It is rigorously proven, with the help of the celebrated L'Hôspital's Rule, that the state of the original nonlinear system under the designed controller is globally finite-time stable while the other closed-loop signal is bounded on $[0,\infty)$. The effectiveness of the designed controller is illustrated by a simulation example, regardless of the actual sign and value of the unknown control coefficient.

2 Preliminary knowledge

Throughout this paper, IN denotes the set of all natural numbers; $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ the set of all nonnegative real numbers, and $\mathbb{R}^n$ the real n-dimensional space, $n\in$ IN; a constant is said to be *unknown*, if both its value and sign are unknown.

Definition 1 *A function*, $N:[0,\infty)\to\mathbb{R}$, *is called a* Nussbaum function *if it satisfies*

$$\begin{cases}\limsup\limits_{s\to\infty}\left(\dfrac{1}{s}\displaystyle\int_0^s N(v)\,\mathrm{d}v\right)=+\infty,\\ \liminf\limits_{s\to\infty}\left(\dfrac{1}{s}\displaystyle\int_0^s N(v)\,\mathrm{d}v\right)=-\infty.\end{cases}\tag{1}$$

For example, $v\mapsto e^{v^2}\cos(\pi v/2)$, $v\in\mathbb{R}^+$; $v\mapsto\ln(v+1)\cos(\sqrt{\ln(v+1)})$, $v\in\mathbb{R}^+$, and $v\mapsto v^2\cos(\pi v/2)$, $v\in\mathbb{R}^+$ are all Nussbaum functions according to the above definition [1, 4].

The following two lemmas demonstrate some basic properties of Nussbaum functions.

Lemma 1 *If* $N:[0,\infty)\to\mathbb{R}$ *is a Nussbaum function and* $g\in\mathbb{R}$ *a nonzero constant, then* $gN(\cdot)$ *is also a Nussbaum function.*

Proof If $g>0$, then from Definition 1, it is easy to see that $gN(\cdot)$ is also a Nussbaum function.

On the other hand, if $g<0$, then we have

$$\limsup_{s\to\infty}\left(\frac{1}{s}\int_0^s gN(v)\,\mathrm{d}v\right)=g\liminf_{s\to\infty}\left(\frac{1}{s}\int_0^s N(v)\,\mathrm{d}v\right)=+\infty$$

and

$$\liminf_{s\to\infty}\left(\frac{1}{s}\int_0^s gN(v)\,\mathrm{d}v\right)=g\limsup_{s\to\infty}\left(\frac{1}{s}\int_0^s N(v)\,\mathrm{d}v\right)=-\infty,$$

which show that $gN(\cdot)$ is a Nussbaum function as well.

Lemma 1 is restricted to the case of $g\in\mathbb{R}\setminus\{0\}$ being a constant. More generally, the following lemma is devoted to the case of g being a function.

Lemma 2 *Let $g:[0,\infty)\to\mathbb{R}$ be nonzero and always positive (or negative). If $N:[0,\infty)\to\mathbb{R}$ is a Nussbaum function, then $gN:[0,\infty)\to\mathbb{R}$ is also a Nussbaum function if for any $\varepsilon>0$, $g(\upsilon)>0$, $\forall \upsilon\geqslant 0$ satisfies*

$$\limsup_{s\to\infty}\left(\frac{1}{s}\int_0^s \upsilon^{\varepsilon} g(\upsilon)\mathrm{d}\upsilon\right)=+\infty, \tag{2}$$

or $g(\upsilon)<0$, $\forall \upsilon\geqslant 0$,

$$\liminf_{s\to\infty}\left(\frac{1}{s}\int_0^s \upsilon^{\varepsilon} g(\upsilon)\mathrm{d}\upsilon\right)=-\infty. \tag{3}$$

Proof This can be shown by a contradiction argument. Suppose that when $g(\upsilon)>0$, $\forall \upsilon\geqslant 0$, (2) does not hold, that is,

$$\limsup_{s\to\infty}\left(\frac{1}{s}\int_0^s \upsilon^{\varepsilon} g(\upsilon)\mathrm{d}\upsilon\right)<+\infty, \tag{4}$$

for some $\varepsilon>0$. Then, one can construct the following piecewise right-continuous Nussbaum function ($0<\varepsilon<1$):

$$N(\upsilon)=\begin{cases}0, & \upsilon\in[0,3),\\ \upsilon^{\varepsilon}, & \upsilon\in[3^n,3^{n+1}), \quad \text{when } n \text{ is positive odd number},\\ -\upsilon^{\varepsilon}, & \upsilon\in[3^n,3^{n+1}), \quad \text{when } n \text{ is positive even number},\end{cases}$$

for which, we have

$$-\frac{1}{s}\int_0^s \upsilon^{\varepsilon} g(\upsilon)\mathrm{d}\upsilon\leqslant\frac{1}{s}\int_0^s g(\upsilon)N(\upsilon)\mathrm{d}\upsilon\leqslant\frac{1}{s}\int_0^s \upsilon^{\varepsilon} g(\upsilon)\mathrm{d}\upsilon.$$

From this and (4), it follows that

$$-\infty<\limsup_{s\to\infty}\int_0^s g(\upsilon)N(\upsilon)\mathrm{d}\upsilon<+\infty,$$

which obviously contradicts the known fact that $gN:[0,\infty)\to\mathbb{R}$ is a Nussbaum function and especially $\limsup_{s\to\infty}(\frac{1}{s}\int_0^s g(\upsilon)N(\upsilon)\mathrm{d}\upsilon)=+\infty$.

Analogously, the assertion for the case of $g(\upsilon)<0$, $\forall \upsilon\geqslant 0$ can be proven.

From Definition 1 and Lemmas 1 and 2, it can be seen that a Nussbaum function is not restricted to be even (or odd) and smooth (or continuous). However, in the latter development of the paper, we will choose $\upsilon\mapsto N(\upsilon)=\mathrm{e}^{\upsilon^2}\cos(\pi\upsilon/2)$ as an even smooth Nussbaum function mainly for the convenient when feedback design.

The following lemma is much fundamental and plays an important role the stability analysis of nonlinear control systems with unknown control direction, which has been explicitly proven in [6].

Lemma 3[6] *Let t_f be some finite time in $\mathbb{R}^+$ or $+\infty$, $V:[0,t_f)\to\mathbb{R}^+$ and $\zeta:[0,t_f)\to\mathbb{R}$ continuously differentiable functions, and $N(\cdot)$ an even smooth Nussbaum function defined as $\zeta\mapsto\mathrm{e}^{\zeta^2}\cos(\pi\zeta/2)$. If the following inequality holds:*

$$V(t)\leqslant c+\int_0^t (gN(\zeta(\upsilon))+1)\mathrm{d}\zeta(\upsilon),\ \forall t\in[0,t_f), \tag{5}$$

where $g\in\mathbb{R}$ is a nonzero constant and $c\in\mathbb{R}$ represents some suitable constant, then $t\mapsto\zeta(t)$, $t\mapsto V(t)$ and $t\mapsto\int_0^t N(\zeta(\upsilon))\mathrm{d}\zeta(\upsilon)$ are all bounded on $[0,t_f)$.

Let's next turn to presenting some preliminary results on finite-time stability. The following definition on the concept of *finite-time stability* was rigorously introduced in the seminal paper[11] and today has been well recognized and widely adopted in the relevant literature.

Definition 2[11] *Consider the continuous n-dimensional system $\dot{x}(t)=f(x(t))$ on domain $D\subseteq\mathbb{R}^n$ with $f(0)=0$ and $x(0)=x_0$. Its zero solution is finite-time stable if there exist an open neighborhood $\mathscr{B}\subseteq D$ of the origin and a function $T:\mathscr{B}\setminus\{0\}\to(0,\infty)$, called the settling-time function, such that the following two statements hold:*

(Ⅰ) (*Finite-time convergence*) *For every* $x_0 \in \mathscr{B} \setminus \{0\}$, $x(t)$ *is defined on* $[0, T(x_0))$, $x(t) \in \mathscr{B} \setminus \{0\}$, *for any* $t \in [0, T(x_0))$, *and* $\lim_{t \to T(x_0)} x(t) = 0$.

(Ⅱ) (*Lyapunov stability*) *For any open set* $\mathscr{U}_\varepsilon$ *of the origin in* $\mathscr{B}$, *there exists an open set* $\mathscr{U}_\delta$ *such that* $0 \in \mathscr{U}_\delta \subseteq \mathscr{B}$ *and for every* $x_0 \in \mathscr{U}_\varepsilon \setminus \{0\}$, $x(t) \in \mathscr{U}_\delta$ *for any* $t \in [0, T(x_0))$.

The zero solution of the considered system is said to be globally finite-time stable if it is finite-time stable and $D = \mathscr{B} = \mathbb{R}^n$.

The following lemma already available in the literature gives the sufficient conditions of global finite-time stability in the Lyapunov manner.

Lemma 4[12] *For a continuous n-dimensional system* $\dot{x}(t) = f(x(t))$ *with* $f(0) = 0$ *and* $x(0) = x_0$, *suppose there exists a* C^1 *positive definite and proper function* $V: \mathbb{R}^n \to \mathbb{R}$, *and real numbers* $k > 0$ *and* $\alpha \in (0, 1)$ *such that* $\dot{V} + kV^\alpha$ *is negative semidefinite. Then, the zero solution of the system is globally finite-time stable, and moreover the settling time* $T \leqslant \dfrac{V^{1-\alpha}(x_0)}{k(1-\alpha)}$.

2 Finite-time stabilization of first-order nonlinear systems

Consider the problem of the global finite-time stabilization for control-affine first-order nonlinear systems in the following form:

$$\dot{x} = gu + x\phi(x), \tag{6}$$

where x is the scalar system state with the initial data $x(0) = x_0$, g is a nonzero constant with the unknown sign and the unknown value, and $\phi: \mathbb{R} \to \mathbb{R}$ is continuous and restricted to satisfy the following assumption:

A1. *There exists an open neighborhood* $\mathscr{N}$ *of the origin, such that* $\phi(x) < 0$ *for every* $x \in \mathscr{N}$.

Obviously, the system (6) is with unknown control direction due to the unknown sign of g. For such simple nonlinear control systems, there has been available methods (see e.g. [6-7]) to asymptotically stabilize the system states, but which cannot realize the finite-time convergence for any system state.

The objective of this paper is to search for the following adaptive continuous state-feedback control law:

$$\dot{\zeta} = \mu(\zeta, x),$$
$$u = \rho(\zeta, x),$$

for the system (6) such that the closed-loop system state x is globally finite-time stable, that is, there exists a finite *settling time* $\tau \geqslant 0$ such that

$$\lim_{t \to \tau} x(t) = 0, \quad \text{and} \quad x(t) = 0, \quad t > \tau,$$

while the other closed-loop state $\zeta \in \mathbb{R}$ is bounded on $[0, \infty)$.

Remark 1 *It is apparent that under Assumption A1, the origin solution of system (6) is locally asymptotically stable when no control effect since for the positive definite* $W(x) = x^2$, *its time-derivative* $\dot{W} = x^2\phi(x)$ *is negative definite in* $\mathscr{N}$. *From the later development, it can be seen that the existence of such neighborhood* $\mathscr{N}$ *will play a crucial role in obtaining the previous objective of the paper.*

2.1 Finite-time control design

Because of the presence of unknown control coefficient, g, i.e., both its value and sign are unknown, finite-time stable control design is very hard, if not impossible. It has been shown that in this case, the method based on Nussbaum function is an effectual tool without worrying too much about their practical applications. The newest result available thus far in the finite-time-control literature is probably that in [17], which, however, is applicable to the class of uncertain systems with determinate control coefficient.

In this subsection, a continuous adaptive controller is designed for system (6) based on the Lyapunov function together with the universal adaptive technique based on a Nussbaum function. Also, an important theorem and a key lemma are presented to establish the stability of the closed-loop system and reveal the most intrinsic properties of such type of control problems, respectively.

Let $V = \dfrac{x^{2-v}}{2-v}$ be the Lyapunov candidate function for the control design, where $v \in (0, 1)$ is a ratio as $\dfrac{2p_v}{q_v}$ with p_v being a positive integer and q_v a positive odd integer. Then, along the trajectories of system (6), it is easy to get

$$\dot{V} = x^{1-v}(gu + x\phi(x)). \tag{7}$$

As mentioned earlier, choose the Nussbaum function $N(\cdot)$ as $k \mapsto N(k) = e^{k^2}\cos(\pi k/2)$ and design an adaptive continuous state-feedback controller as follows:

$$\begin{cases} \dot{k} = x^{1-v}(x^{1-v_1} + x\,|\,\phi(x)\,|\,), \quad k(0) = k_0 > 0, \\ u = N(k)(x^{1-v_1} + x\,|\,\phi(x)\,|\,), \end{cases} \tag{8}$$

where $v_1 \in (0, 1)$ is another ratio as $\dfrac{2p_{v_1}}{q_{v_1}}$ with p_{v_1} being a positive integer and q_{v_1} a positive odd integer, and satisfies $v + 2v_1 > 2$ (v has been defined above), and $t \mapsto k(t) \in \mathbb{R}^+_0$, $t \in \mathbb{R}^+$ is called the adaptive updating signal. Substituting (8) into (7) concludes that

$$\begin{aligned} \dot{V} &= gN(k)(x^{2-v-v_1} + x^{2-v}\,|\,\phi(x)\,|\,) + x^{2-v}\phi(x) \leqslant \\ &\quad - x^{2-v-v_1} + (gN(k) + 1)(x^{2-v-v_1} + x^{2-v}\,|\,\phi(x)\,|\,) = \\ &\quad - x^{2-v-v_1} + (gN(k) + 1)\dot{k}, \end{aligned} \tag{9}$$

in some small interval $[0, t_f)$, where $t_f > 0$. From this, it follows that for any t in the interval $[0, t_f)$,

$$V(t) \leqslant c + \int_0^t (gN(k(\mu)) + 1)\dot{k}(\mu)\mathrm{d}\mu, \tag{10}$$

where c represents some appropriate constant depending on the initial data $x(0)$ of system(6).

The following theorem is an important intermediate result to ultimately establish the global finite-time stability of the resulting closed-loop system.

Theorem 1 *Under the assumption A1, the solution $(x(t), k(t))$ of system (6) with (8) in the loop is well defined and bounded on $[0, \infty)$ for any initial data $x(0)$ and $k(0)$. More specifically, the system state $x(t)$ converges to 0 as time t goes to $+\infty$, while $\lim_{t\to\infty} k(t) = k_c$ for some $k_c > 0$.*

Proof As already mentioned, there exists some interval $[0, t_f)$, $0 < t_f < \infty$, in which the closed-loop system has solutions due to the continuous dynamics. Moreover, by Lemma 2 in Page 107 of [20] and Theorem 4.3 in Page 59 of [21], and the continuity of the systems dynamics, it can be shown that these solutions are unique in forward time and are continuously dependent on the closed-loop system initial data[11, 18]. Therefore, the solution of the closed-loop system is well-defined on the interval $[0, t_f)$. Without loss of generality, suppose that the interval $[0, t_f)$ can be maximized to the maximal interval $[0, T_f)$ for some T_f, where $t_f \leqslant T_f \leqslant \infty$. Then by the aforementioned Lemma 3, we easily achieve the boundedness of $k(t)$ and $V(t)$, as well as $x(t)$ on the maximal interval $[0, T_f)$.

We next prove $T_f = +\infty$ by a contradiction argument. Suppose that $T_f < +\infty$. Then, T_f would be the finite escape time of the closed-loop system. This evidently contradicts to the fact that any solution of the closed-loop system (6) and (8) is bounded on the maximal interval $[0, T_f)$, and hence also bounded at $t = T_f$ due to the continuity of any solution of the system.

As an immediate result, $\dot{k}$, u, $\dot{x}$ and $\ddot{x}$ are all bounded on $[0, \infty)$, and in turn, using the well-known Barbalat's Lemma, it can be concluded that $\lim_{t\to\infty} x(t) = 0$. Additionally, according to (8), i.e., $\dot{k}(t) \geqslant 0$, $\forall t \geqslant 0$, $k(0) > 0$ and the proven fact $k(t) < \infty$ on $[0, \infty)$, it is clear that $k(t)$ monotonically increasingly converges to its finite limit as time goes to infinity, that is, $\lim_{t\to\infty} k(t) = k_c$ for a finite positive constant k_c. This completes the proof.

The following lemma reveals the most intrinsic properties of such type of control problems, that is, by the Nussbaum function, the unknown sign of the control coefficient g can be successfully estimated, as will be seen that the sign of the Nussbaum function $t \mapsto N(k(t))$ eventually becomes the opposite sign of g when time goes sufficiently large, while $N(k)$ is a measurable signal and hence applicable for feedback design.

Lemma 5 *For the adaptive updating signal $k(t)$ given by (8) with the nonzero closed-loop system initial data (i.e., $x(0) \neq 0$ and $k(0) \neq 0$), its limit exists as $t \to \infty$ (denoted by k_c), and satisfies the following relation:*

$$gN(k_c) = g\mathrm{e}^{(k_c^2)}\cos(\pi k_c/2) \leqslant 0. \tag{11}$$

Proof It has been shown in Theorem 1 that the existence of the limit $k_c > 0$ of $k(t)$ as $t \to \infty$. We next only prove the correctness of relation (11). Suppose that (11) does not hold. This implies that

$$g\mathrm{e}^{(k_c^2)}\cos(\pi k_c/2) > 0. \tag{12}$$

We will show this is not true by a contradiction argument. By some evident observation, we can easily see that whether $x(t) \neq 0$ for all $t \geqslant 0$ or $x(t)$ reaches zero at a finite time and remains zero forever after. Therefore, the rest proof is broken up into two separate parts: (Ⅰ) for any $t \geqslant 0$, $x(t) \neq 0$, though $\lim_{t\to\infty} x(t) = 0$ (This has been proven in above Theorem 1); and (Ⅱ) there exists a finite time t_e at which $x(t_e) = 0$, and $x(t) = 0$ for any $t \geqslant t_e$.

(Ⅰ) For the case of $\forall t \geqslant 0$, $x(t) \neq 0$, from (12) and the continuity of the variable $k(t)$ in t, it follows that for some constant $c_g > 0$, there is a finite time t_{c_g} (may be very large) such that $g\exp(k^2(t))\cos(\pi k(t)/2) \geqslant c_g > 0$ in $[t_{c_g}, \infty)$. On the other hand, from the proven fact $\lim_{t\to\infty} x(t) = 0$ and the continuity of $x(t)$ in t, it concludes that for a sufficiently small constant $c_1 > 0$, there exists another finite time $t_d \in [t_{c_g}, \infty)$ at which $|x(t_d)| = c_1$ and such that

$$c_1^{2-v-v_1}\Big(c_g + c_g c_1^{v_1} \min_{x\in[-c_1,\, c_1]} |\phi(x)| - c_1^{v_1} \max_{x\in[-c_1,\, c_1]} |\phi(x)|\Big) > 0,$$

which implies that $\dot V(t_d) > 0$ and in turn concludes that $|x(t)| \geqslant c_1$, $\forall t \in [t_d, \infty)$. This obviously contradicts the proven fact $\lim_{t\to\infty} x(t) = 0$.

(Ⅱ) In the case, suppose that $t_e \geqslant 0$ is the first finite time at which $x(t_e) = 0$. Then $\dot V(t_e) = 0$ and $\dot k(t_e) = 0$, and hence the closed-loop system will settle down at this finite time, or equivalently, $x(t) = 0$, $\forall t \in [t_e, \infty)$ and $k(t) = k_c$, $\forall t \in [t_e, \infty)$. In the following, we restrict our attention to the case of $t_e > 0$ since $t_e = 0$ is the trivial case and hence it does not satisfy the conditions of this lemma. From (12), $k(t_e) = k_c$ and the continuity of $k(\cdot)$, it concludes that $g\exp(k^2(t))\cos(\pi k(t)/2) \geqslant c_2$ on some interval $[t_e - \varepsilon, t_e)$ for some constants $\varepsilon > 0$ and $c_2 > 0$. Additionally, at the time $t_e - \varepsilon$, there holds $|x(t_e - \varepsilon)| \geqslant c_3$ for some constant $c_3 > 0$. Similar to the argument of (Ⅰ), if ε is chosen sufficiently small, then the constant c_3 will be sufficiently small so that

$$c_3^{2-v-v_1}\Big(c_2 + c_2 c_3^{v_1} \min_{x\in[-c_3,\, c_3]} |\phi(x)| - c_3^{v_1} \max_{x\in[-c_3,\, c_3]} |\phi(x)|\Big) > 0.$$

This will result in $\dot V(t_e - \varepsilon) > 0$, and in turn $|x(t)| \geqslant c_3 > 0$, $\forall t \in [t_e - \varepsilon, \infty)$. This contradicts the assumed fact $x(t) = 0$, $\forall t \in [t_e, \infty)$.

The both contradictions for parts (Ⅰ) and (Ⅱ) under (12) show that the relation (11) is correct. This completes the proof.

2.2 Finite-time stability analysis

We have the following theorem which summarizes the main results of the paper.

Theorem 2 *With the designed adaptive state-feedback control law (8) in the loop, the system (6) under Assumption A1 is globally stable, and furthermore, the system state $x(t)$ is globally finite-time stable, that is, for any closed-loop system initial data, there exists a finite time τ, such that $\lim_{t\to\tau} x(t) = 0$, and $x(t) = 0$ for all $t \geqslant \tau$.*

Proof The global stability of the closed-loop system is evident from the previous discussion (e.g. Theorem 1).

It suffices to show the finite-time stability of the closed-loop system state $x(t)$. Obviously, when the initial data $x(0) = 0$, the conclusions automatically hold. Therefore, we next restrict our attention to the case $x(0) \neq 0$. Without loss of generality, suppose that $\mathrm{sgn}(g) = +1$ and $x(0) > 0$. Then, from Lemma 5, it follows that $N(k_c) = \exp(k_c^2)\cos(\pi k_c/2) \leqslant 0$. The rest proceeds in the following two different cases: (Ⅰ) $N(k_c) < 0$; and (Ⅱ) $N(k_c) = 0$.

(Ⅰ) When $N(k_c) < 0$, then apparently for a sufficiently large time $t_g \geqslant 0$, there is a constant $c_g > 0$, such that $N(k(t)) \leqslant -c_g$, $\forall t \in [t_g, \infty)$. On the other hand, suppose that in the interval $[0, t_g]$, $x(t) \neq 0$ otherwise the conclusions would automatically hold. Then there is another finite time $t_d \in (t_g, \infty)$, at which $|x(t_d)| \neq 0$, and for sufficiently small constant $c_1 > 0$ such that $|x(t)| \leqslant c_1$, $\forall t \geqslant t_d$ and

$$\frac{gc_g}{2} - c_1^{v_1} \max_{x \in [-c_1, c_1]} |\phi(x)| \geqslant 0,$$

and hence we have

$$\dot{V} \leqslant -x^{2-v-v_1} + (gN(k) + 1)(x^{2-v-v_1} + x^{2-v}|\phi(x)|) \leqslant -\frac{gc_g}{2}x^{2-v-v_1} = -\bar{k}V^{\alpha}, \tag{13}$$

for any $x \in [-c_1, c_1]$, where $0 < \alpha = \dfrac{2-v-v_1}{2-v} < 1$ and $\bar{k} = \dfrac{gc_g}{2}(2-v)^{\alpha} > 0$. Therefore, by Lemma 4, the finite time stability of the closed-loop system state $x(t)$ can be established, and the settling time $T \leqslant t_d + \dfrac{V^{1-\alpha}(t_d)}{(1-\alpha)}$.

(Ⅱ) For the case of $N(k_c) = 0$, if there is a finite time $\tau \geqslant 0$ such that

$$\lim_{t\to\tau} N(k(t)) = 0, \quad N(k(t)) = 0, \quad \forall t > \tau, \tag{14}$$

which means that in $[\tau, \infty)$, $k(t) = k_c$, i.e., $\dot{k}(t) = 0$ and in turn $x(t) = 0$ in $[\tau, \infty)$, and therefore, the system state $x(t)$ is globally finite-time stable with the settling time $T \leqslant \tau$.

The next turns to the other case of no finite time $\tau \geqslant 0$ satisfying (13), meanwhile $\lim_{t\to\infty} N(k(t)) = N(k_c) = 0$. This means that for any large enough but finite time t, there would always hold $N(k(t)) < 0$ and $x(t) \neq 0$. We will show this is impossible by a contradiction argument, that is, there must be a finite time $\tau > 0$, such that $k(t) = k_c$ and $x(t) = 0$ for any $t \in [\tau, \infty)$. For the aim, we need first to examine the limit of $\dfrac{N(k(t))}{x^{v_1}(t)}$ as the time t goes to $+\infty$. In fact, we have

$$\lim_{t\to\infty}\frac{\mathrm{d}N(k(t))/\mathrm{d}t}{\mathrm{d}x^{v_1}(t)/\mathrm{d}t} = \lim_{t\to\infty}\frac{\mathrm{d}N(k(t))}{\mathrm{d}x^{v_1}(t)} = \lim_{t\to\infty}\left(\frac{\mathrm{d}N(k(t))}{\mathrm{d}k(t)}\cdot\frac{\mathrm{d}k(t)}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x(t)}\cdot\frac{\mathrm{d}x(t)}{\mathrm{d}x^{v_1}(t)}\right).$$

From this and noting that

$$\frac{\mathrm{d}N(k)}{\mathrm{d}k} = 2k\exp(k^2)\cos(\pi k/2) - \pi/2\exp(k^2)\sin(\pi k/2) \to$$
$$\pi/2\exp(k_c^2) \text{ as } t \to +\infty \text{ since } \cos(\pi k/2) \to 0^-,$$

$$\frac{\mathrm{d}x}{\mathrm{d}x^{v_1}} = 1/(\mathrm{d}x^{v_1}/\mathrm{d}x) = 1/(v_1 x^{v_1-1}),$$

$$\frac{\mathrm{d}k(t)}{\mathrm{d}t} = \dot{k}(t) = x^{1-v}(x^{1-v_1} + x|\phi(x)|),$$

$$\frac{\mathrm{d}t}{\mathrm{d}x(t)} = 1/\dot{x}(t) = \frac{1}{gN(k(t))(x^{1-v_1} + x|\phi(x)|) + x\phi(x)},$$

we easily have

$$\lim_{t\to\infty}\frac{-\mathrm{d}N(k(t))}{\mathrm{d}x^{v_1}(t)} = -\pi/2\exp(k_c^2)\lim_{t\to\infty}\frac{x^{1-v}(x^{1-v_1} + x|\phi(x)|)}{gv_1N(k(t)) + gv_1N(k(t))x^{v_1}|\phi(x)| + v_1x^{v_1}\phi(x)} =$$
$$\frac{\pi\exp(k_c^2)}{2v_1}\lim_{t\to\infty}\frac{x^{2-v-v_1}}{-gN(k(t)) - x^{v_1}\phi(x)}. \tag{15}$$

Clearly, it can be seen that the time t is large enough to ensure that $gN(k(t)) < 0$ and $x(t)$ will enter the set $\mathcal{N}$ defined in assumption A1 and remain in it forever after, and thus we have $\phi(x(t)) < 0$. Therefore, if $N(k(t)) = o(x^{v_1})$ when x is sufficiently small in magnitude, then by (14) and noting $2 - v - 2v_1 < 0$, we have

$$\lim_{t\to\infty}\frac{-\mathrm{d}N(k(t))}{\mathrm{d}x^{v_1}(t)} = \lim_{t\to\infty}\frac{-\mathrm{d}N(k(t))}{\mathrm{d}x^{v_1}(t)} = \frac{\pi\exp(k_c^2)}{2v_1}\lim_{x\to0^+}\frac{x^{2-v-v_1}}{-x^{v_1}\phi(x)} = \frac{\pi\exp(k_c^2)}{2v_1}\lim_{x\to0^+}\frac{x^{2-v-2v_1}}{-\phi(0)} = +\infty,$$

and in turn by L'Hospital's Rule [19] (since it can be easily verified that $\mathrm{d}x^{v_1}/\mathrm{d}t \neq 0$ in every right-hand neighborhood of 0 in x),

$$\lim_{t\to\infty}\frac{-N(k(t))}{x^{v_1}(t)} = \lim_{t\to\infty}\frac{-\mathrm{d}N(k(t))}{\mathrm{d}x^{v_1}(t)} = +\infty,$$

which contradicts the following

$$\lim_{t\to\infty}\frac{-N(k(t))}{x^{v_1}(t)} = \lim_{x\to0^+}\frac{o(x^{v_1})}{x^{v_1}} = 0.$$

If $N(k(t)) = O(x^{v_1})$ when x is sufficiently small, a contradiction would establish by the similar analysis to the above development.

It can be turned out that the only feasibility is that when $x \neq 0$ is sufficiently small,

$$k_1|x(t)|^{v_2} < -N(k(t)) < k_2|x(t)|^{v_3}, \tag{16}$$

for some constants $k_1 > 0$, $k_2 > 0$, $v_1 > v_2 > 1 - \frac{v}{2} > 0$ satisfying $v_2 = \frac{2q_{v_2}}{p_{v_2}}$ with q_{v_2} being positive integer and p_{v_2} positive odd integer, and $0 < v_3 < 2 - v - v_1$. Otherwise, when $-N(k(t)) \geqslant k_2|x(t)|^{v_3}$ for sufficiently small $x \neq 0$, by (14) and L'Hospital's Rule, we have

$$\lim_{t\to\infty}\frac{-N(k(t))}{x^{v_1}(t)} \geqslant \lim_{x\to0^+}\frac{k_2|x|^{v_3}}{x^{v_1}} = +\infty,$$

$$\lim_{t\to\infty}\frac{-N(k(t))}{x^{v_1}(t)} = \frac{\pi\exp(k_c^2)}{2v_1}\cdot\frac{1}{gk_2}\lim_{x\to0^+}|x|^{2-v-v_1-v_3} = 0,$$

which is clearly a contradiction. On the other hand, when $-N(k(t)) \leqslant k_1|x(t)|^{v_2}$ for small enough $x \neq 0$, we similarly yeild

$$\lim_{t\to\infty}\frac{-\mathrm{d}N(k(t))}{\mathrm{d}x^{v_2}(t)} = -\frac{\pi\exp(k_c^2)}{2v_2}\lim_{t\to\infty}\frac{x^{1-v}(x^{1-v_1}+x|\phi(x)|)}{gN(k(t))x^{v_2-v_1}+gN(k(t))x^{v_2}|\phi(x)|+x^{v_2}\phi(x)} =$$
$$\frac{\pi\exp(k_c^2)}{2v_2}\lim_{t\to\infty}\frac{x^{2-v-v_1}}{-gN(k(t))x^{v_2-v_1}-x^{v_2}\phi(x)}.$$

Then, by L'Hospital's Rule, and noting that

$$2 - v - v_1 < 2v_2 - v_1,$$
$$2 - v - v_1 < v_2,$$

we have

$$\lim_{t\to\infty}\frac{-N(k(t))}{x^{v_2}(t)} \leqslant \lim_{x\to0^+}\frac{k_1|x|^{v_2}}{x^{v_2}} = k_1,$$

$$\lim_{t\to\infty}\frac{-N(k(t))}{x^{v_2}(t)} = \lim_{t\to\infty}\frac{-\mathrm{d}N(k(t))}{\mathrm{d}x^{v_2}(t)} \geqslant \frac{\pi\exp(k_c^2)}{2v_2}\lim_{t\to\infty}\frac{x^{2-v-v_1}}{-gk_1x^{2v_2-v_1}-x^{v_2}\phi(x)} = +\infty,$$

which is impossible and simultaneously results in a contradiction, and in turn shows that (15) holds.

Thus far, we proceed to finite-time stability analysis based on the above proven relationship (15) after large enough time t_d. Notably, we have

$$\dot V = gN(k)(x^{2-v-v_1} + x^{2-v}|\phi(x)|) + x^{2-v}\phi(x) \leqslant -c|x|^{2-v-v_1+v_2} = -\bar kV^{\alpha}(t),$$

when $t \geqslant t_d$ for some constant $c > 0$, and $0 < \alpha = \frac{2-v-v_1+v_2}{2-v} < 1$ and $\bar k = c(2-v)^{\alpha}$. Then by Lemma 4, the closed-loop system state $x(t)$ is globally finite-time stable and the settling time $T \leqslant t_d + \frac{V^{1-\alpha}(t_d)}{(1-\alpha)}$. This means that $x(t) = 0$, $\forall t \geqslant T$, which contradicts the assumed $x(t) \neq 0$, $\forall t \geqslant 0$ and there must be a finite time $\tau > 0$ such that (13) holds. This completes the proof.

Remark 2 *From the above proof, one can easily see the sufficient conditions to guarantee* (8) *be a global finite-time stable controller for the system state x of* (6) *under Assumption A1 is the following*:

$$v = \frac{2p_v}{q_v} \in (0,\ 1) \ \text{and} \quad v_1 = \frac{2p_{v_1}}{q_{v_1}} \in (0,\ 1) \ \text{such that}\ v + 2v_1 > 2$$

with p_v, p_{v_1} *being positive integers and* q_v, q_{v_1} *positive odd integers*.

3 Simulation

Consider the first-order system:

$$\dot{x} = gu - x(0.1 - x),$$

where $g \neq 0$ is an unknown control coefficient, namely, its sign and value are unknown.

By virtue of the control design scheme given in the previous section, we can easily obtain the universal controller as follows.

$$\begin{cases} \dot{k} = x^{1/5}(x^{1/5} + x \mid 0.1 - x \mid), \quad k(0) = k_0 > 0, \\ u = e^{k^2}\cos(\pi k/2)(x^{1/5} + x \mid 0.1 - x \mid). \end{cases}$$

With the initial conditions: $x(0) = 0.2$ and $k(0) = 0.1$, the simulation results are shown in Figures 1 ~ 4, which are classified into two groups corresponding to the cases of $g = 1$ and $g = -1$, respectively.

Figures 1 and 3 show the finite-time convergence of the state x, regardless of the direction and the value of g, while Figures 2 and 4, for two cases with oppositive directions, illustrate the boundedness of the adaptive signal k which is introduced to identify the direction of the unknown control coefficient implicitly.

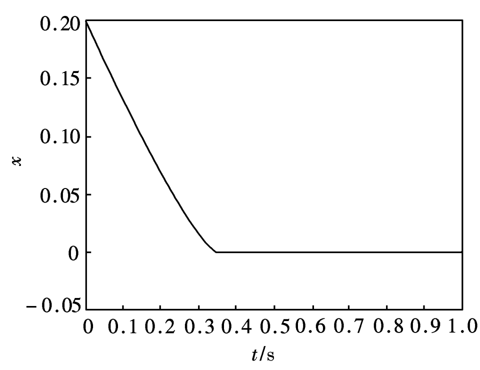

Fig.1　State x for the case of $g = -1$

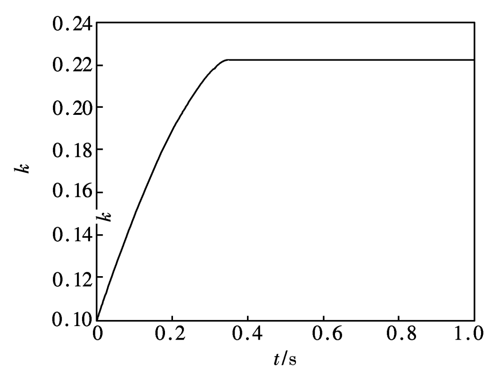

Fig.2　Signal k for the case of $g = -1$

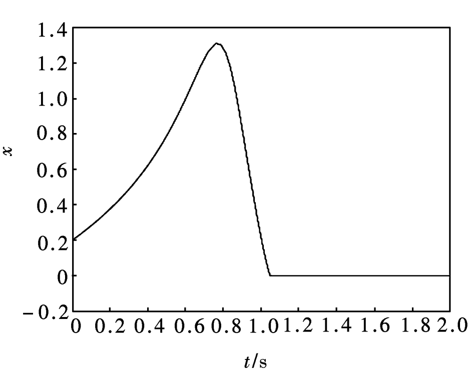

Fig.3　State x for the case of $g = 1$

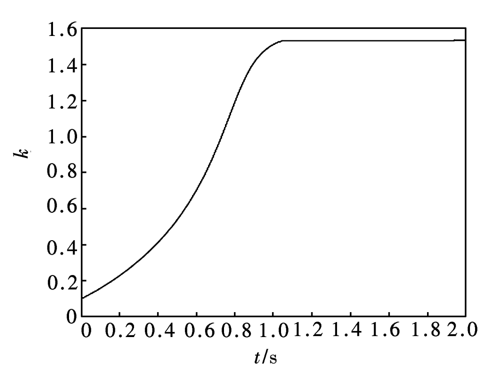

Fig.4　Signal k for the case of $g = 1$

4 Conclusion

In this paper, the global finite-time stabilization has been investigated for a class of first-order uncertain nonlinear systems. By using the adaptive technique integrated with a suitable Nussbaum function, an adaptive finite-time stabilizing controller is constructed so that the state of the original system is globally finite-time stable while other closed-loop signal is bounded on $[0,\infty)$. Another interesting problem is that for higher order nonlinear systems with unknown control directions, how does design a controller to realize the global finite-time stability. This problem greatly differs from that of first-order nonlinear systems and hence is of much interest.

References:

[1] NUSSBAUM R D. Some remarks on a conjecture in parameter adaptive control[J]. Systems & Control Letters, 1983, 3(5):243-246.
[2] WILLEMS J C, BYRNES C I. Global adaptive stabilization in the absence of information on the sign of the high frequency gain[R]. Lecture Notes in Control and Information Sciences No. 62, Berlin: Springer-Verlag, 1984: 49-57.
[3] MÅRTESSON B. Remarks on adaptive stabilization of first order nonlinear systems[J]. Systems & Control Letters, 1990, 14(1):1-7.
[4] ILCHMANN A. Non-identifier-based high-gain adaptive control[R]. Lecture Notes in Control and Information Sciences No. 189, London: Springer-Verlag, 1993.
[5] KALOUST J, QU Z. Continuous robust control design for nonlinear uncertain systems without a priori knowledge of control direction[J]. IEEE Transaction on Automatic Control, 1995, 40(2):276-281.
[6] YE X, JIANG J. Adaptive nonlinear design without a priori knowledge of control directions[J]. IEEE Transaction on Automatic Control, 1998, 43(11):1617-1621.
[7] YE X. Asymptotic regulation of time-varying uncertain nonlinear systems with unknown control directions[J]. Automatica, 1999, 35(5):929-935.
[8] GE S S, WANG J. Robust adaptive tracking for time-varying uncertain nonlinear systems with unknown control coefficients[J]. IEEE Transaction on Automatic Control, 2003, 48(8):1463-1469.
[9] LIU Y G, GE S S. Output-feedback adaptive stabilization for nonlinear systems with unknown direction control coefficients[C]// Proceedings of the 2005 American Control Conference. Portland OR, USA:[s.n.], 2005: 4696-4700.
[10] HAIMO V T. Finite-time controllers[J]. SIAM Journal on Control and Optimization, 1986: 24(4):760-770.
[11] BHAT S, BERNSTEIN D. Continuous finite-time stabilization of the translational and rotational double integrators[J]. IEEE Transaction on Automatic Control, 1998, 43(5):678-682.
[12] BHAT S, BERNSTEIN D. Finite-time stability of continuous autonomous systems[J]. SIAM Journal on Control and Optimization, 2000, 38(3):751-766.
[13] HONG Y, HUANG J, XU Y. On an output feedback finite-time stabilization problem[J]. IEEE Transaction on Automatic Control, 2001, 46(2):305-309.
[14] HONG Y. Finite-time stabilization and stabilizability of a class of controllable systems[J]. Systems & Control Letters, 2002, 46(2):231-236.
[15] HUANG X, LIN W, YANG B. Global finite-time stabilization of a class of uncertain nonlinear systems[J]. Automatica, 2005, 41(5):881-888.
[16] QIAN C, LI J. Global finite-time stabilization by output feedback for planar systems without observable linearization[J]. IEEE Transaction on Automatic Control, 2005, 50(6):885-890.
[17] HONG Y, JIANG Z -P. Finite-time stabilization of nonlinear systems with parametric and dynamic uncertainties[J]. IEEE Transaction on Automatic Control, 2006, 51(12):1950-1956.
[18] NERSESOV S G, HADDAD W M, HUI Q. Finite-time stabilization of nonlinear dynamcial systems via control vector Lyapunov functions [C]// Proceedings of the 2007 American Control Conference. New York City, USA:[s.n.],2007: 4810-4816.
[19] BOAS R P. Counterexamples to L'Hôspital's rule[J]. American Mathematical Monthly, 1986, 93(8):644-645.
[20] FILIPPOV A F. Differential equations with discontinuous right-hand sides[M]. Dordercht: Kluwer Academic Publishers, 1988.
[21] CODDINGTON E A, LEVINSON N. Theory of ordinary differential eqautions[M]. New York: McGraw-Hill, 1955.

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