云南大学学报（自然科学版），2008，30（4）：325～330　　CN 53－1045/N　ISSN 0258－7971
Journal of Yunnan University

一个－3齐次核的Hilbert型积分不等式*

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摘要：通过估算权函数，建立一个核含参数且－3齐次的Hilbert型积分不等式及其等价式，并用复分析方法算出其最佳常数因子，还考虑了逆向的情形及一些特殊结果.

关键词：Hilbert型积分不等式；权系数；核；等价式
中图分类号：O 178　　**文献标识码**：A　　**文章编号**：0258－7971(2008)04－0325－06

设 $f(t)$，$g(t)$ 为非负可测函数，使得 $0<\int_0^\infty f^2(t)\mathrm{d}t<\infty$，$0<\int_0^\infty g^2(t)\mathrm{d}t<\infty$，则有如下具有最佳常数因子 π 的 Hilbert 积分不等式[1]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\pi\left(\int_0^\infty f^2(t)\mathrm{d}t\int_0^\infty g^2(t)\mathrm{d}t\right)^{\frac{1}{2}}. \tag{1}$$

1925年，Hardy－Reisz 引入一对共轭指数 (p,q) $\left(p>1,\frac{1}{p}+\frac{1}{q}=1\right)$，推广式(1)如下[2]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{\frac{1}{q}}, \tag{2}$$

这里，常数因子 $\frac{\pi}{\sin(\pi/p)}$ 为最佳值. 称式(2)为 Hardy－Hilbert 积分不等式. 式(1)，(2)在分析学有重要的应用[3]. 1998年，文献[4,5]由估算权函数入手，给式(1)以如下推广

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y<B\left(\frac{\lambda}{2}\frac{\lambda}{2}\right)\left(\int_0^\infty t^{1-\lambda}f^2(t)\mathrm{d}t\int_0^\infty t^{1-\lambda}g^2(t)\mathrm{d}t\right)^{\frac{1}{2}}, \tag{3}$$

这里 $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)(\lambda>0)$ 为最佳值. 2005年，文献[6]（式(21)，当 $n=2$）引入参数 $\lambda>0$ 及一对共轭指数 $(r,s)\left(r>1,\frac{1}{r}+\frac{1}{s}=1\right)$ 推广式(2)，(3)为

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y<B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)\left(\int_0^\infty t^{p\left(1-\frac{\lambda}{r}\right)-1}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty t^{q\left(1-\frac{\lambda}{s}\right)-1}g^q(t)\mathrm{d}t\right)^{\frac{1}{q}}, \tag{4}$$

这里，$B\left(\frac{\lambda}{r},\frac{\lambda}{s}\right)$ 为最佳值（$B(u,v)$ 为 β 函数）. 2006年，文献[7]考虑了一个混合核的 $-\lambda$ 齐次 Hilbert 型积分不等式；文献[8]建立了一个－2齐次核的 Hilbert 型不等式；2007年，文献[9]考虑了式(4)的逆向及等价形式. 当 $\lambda=3$，$r=s=2$ 时，式(4)变为

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^3}\mathrm{d}x\mathrm{d}y<\frac{\pi}{8}\left(\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t\right)^{\frac{1}{q}}. \tag{5}$$

* 收稿日期：2007－05－23
基金项目：广东高校自然科学基金重点研究资助项目(05Z026)；广东教育学院教授博士专项经费资助项目.
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本文引入多参数，由估算权函数入手，推广式(5)这一特殊结果. 考虑一个核为 $\frac{1}{(x+ay)(x+by)(x+cy)}(a,b,c>0)$ 的 -3 齐次的 Hilbert 型积分不等式及其等价式,并用复分析方法算出其最佳常数因子,还考虑了逆向及其等价形式.

1 引 理

引理 1 设 $a,b,c>0$,定义权函数 $\omega(x,a,b,c),\bar{\omega}(y,a,b,c)$ 为:

$$\omega(x,a,b,c):=\int_0^\infty \frac{y^{1/2}x^{3/2}}{(x+ay)(x+by)(x+cy)}\mathrm{d}y\ , \tag{7}$$

$$\bar{\omega}(y,a,b,c):=\int_0^\infty \frac{x^{1/2}y^{3/2}}{(x+ay)(x+by)(x+cy)}\mathrm{d}x,\quad x,y\in(0,\infty). \tag{8}$$

则有如下等式

$$\omega(x,a,b,c)=\bar{\omega}(y,a,b,c)=k:=\frac{\pi}{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{c})}\ . \tag{9}$$

证明 在式(7),(8)中,作变换 $u^2=x/y$,可算得

$$\omega(x,a,b,c)=\bar{\omega}(y,a,b,c)=\int_{-\infty}^\infty \frac{u^2}{(u^2+a)(u^2+b)(u^2+c)}\mathrm{d}u. \tag{10}$$

不妨设 $a\geqslant b\geqslant c$. 设 $\varepsilon>0,\tilde{c}=c^{1/2},\tilde{b}=b^{1/2}+\varepsilon,\tilde{a}=a^{1/2}+2\varepsilon$,则有 $\tilde{a}>\tilde{b}>\tilde{c}>0$. 设复有理分式函数 $f_\varepsilon(z)=\frac{z^2}{(z^2+\tilde{a}^2)(z^2+\tilde{b}^2)(z^2+\tilde{c}^2)}$. 因为 $z=\tilde{a}\mathrm{i},\tilde{b}\mathrm{i},\tilde{c}\mathrm{i}$ 都为 $f_\varepsilon(z)$ 的一阶极点,易得

$$\operatorname*{Res}_{z=\tilde{a}\mathrm{i}}f_\varepsilon(z)=\left.\frac{z^2}{(z+\tilde{a}\mathrm{i})(z^2+\tilde{b}^2)(z^2+\tilde{c}^2)}\right|_{z=\tilde{a}\mathrm{i}}=\frac{-\tilde{a}}{2\mathrm{i}(-\tilde{a}^2+\tilde{b}^2)(-\tilde{a}^2+\tilde{c}^2)}\ ,$$

$$\operatorname*{Res}_{z=\tilde{b}\mathrm{i}}f_\varepsilon(z)=\frac{-\tilde{b}}{2\mathrm{i}(-\tilde{b}^2+\tilde{a}^2)(-\tilde{b}^2+\tilde{c}^2)}\ ,\operatorname*{Res}_{z=\tilde{c}\mathrm{i}}f_\varepsilon(z)=\frac{-\tilde{c}}{2\mathrm{i}(-\tilde{c}^2+\tilde{a}^2)(-\tilde{c}^2+\tilde{b}^2)}\ .$$

由应用残数计算实积分的理论(见文献[10],定理 6.7),有

$$\int_{-\infty}^\infty f_\varepsilon(x)\mathrm{d}x=2\pi\mathrm{i}\left(\operatorname*{Res}_{z=\tilde{a}\mathrm{i}}f_\varepsilon(z)+\operatorname*{Res}_{z=\tilde{b}\mathrm{i}}f_\varepsilon(z)+\operatorname*{Res}_{z=\tilde{c}\mathrm{i}}f_\varepsilon(z)\right)=$$
$$\pi\left[\frac{-\tilde{a}}{(\tilde{a}^2-\tilde{b}^2)(\tilde{a}^2-\tilde{c}^2)}+\frac{\tilde{b}}{(\tilde{a}^2-\tilde{b}^2)(\tilde{b}^2-\tilde{c}^2)}+\frac{-\tilde{c}}{(\tilde{a}^2-\tilde{c}^2)(\tilde{b}^2-\tilde{c}^2)}\right]=$$
$$\frac{\pi}{(\tilde{a}+\tilde{b})(\tilde{b}+\tilde{c})(\tilde{a}+\tilde{c})}\ . \tag{11}$$

由于 $f_\varepsilon(x)$ 当 $\varepsilon\to0^+$ 时是递增的,由 Levi 定理[11] 及式(11),有

$$\int_{-\infty}^\infty \frac{u^2}{(u^2+a)(u^2+b)(u^2+c)}\mathrm{d}u=\int_{-\infty}^\infty \lim_{\varepsilon\to0^+}f_\varepsilon(x)\mathrm{d}x=\lim_{\varepsilon\to0^+}\int_{-\infty}^\infty f_\varepsilon(x)\mathrm{d}x=k.$$

再由式(10),可得式(9). 证毕.

引理 2 设 $p>0(p\neq1),\frac{1}{p}+\frac{1}{q}=1,a,b,c>0,0<\varepsilon<p,k$ 由式(9)所定义,$\tilde{I}:=\int_1^\infty\int_1^\infty \frac{x^{\frac{1}{2}-\frac{\varepsilon}{p}}y^{\frac{1}{2}-\frac{\varepsilon}{q}}}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\mathrm{d}y$. 则有

$$k+\tilde{o}(1)<\varepsilon\tilde{I}<k+o(1)\ (\varepsilon\to0^+). \tag{12}$$

证明 在式(7)中作变换 $u=\frac{x}{y}$,由式(9),有 $k=\int_0^\infty \frac{u^{1/2}}{(u+a)(u+b)(u+c)}\mathrm{d}u$. 易证

$$\int_0^\infty \frac{u^{\frac{1}{2}-\frac{\varepsilon}{p}}}{(u+a)(u+b)(u+c)}\mathrm{d}u=k+o(1)\ (\varepsilon\to0^+). \tag{13}$$

事实上,我们有

$$\left|\int_0^\infty \frac{u^{\frac{1}{2}-\frac{\varepsilon}{p}}}{(u+a)(u+b)(u+c)}\mathrm{d}u - k\right| = \left|\int_0^\infty \frac{u^{\frac{1}{2}}(u^{-\frac{\varepsilon}{p}-1})}{(u+a)(u+b)(u+c)}\mathrm{d}u\right| =$$

$$\int_0^1 \frac{u^{\frac{1}{2}}(u^{-\frac{\varepsilon}{p}}-1)}{(u+a)(u+b)(u+c)}\mathrm{d}u + \int_1^\infty \frac{u^{\frac{1}{2}}(1-u^{-\frac{\varepsilon}{p}})}{(u+a)(u+b)(u+c)}\mathrm{d}u \leqslant$$

$$\frac{1}{abc}\int_0^1 (u^{-\frac{\varepsilon}{p}}-1)\mathrm{d}u + \int_1^\infty \frac{u^{\frac{1}{2}}(1-u^{-\frac{\varepsilon}{p}})}{u^3}\mathrm{d}u = \frac{1}{abc}\left(\frac{p}{p-\varepsilon}-1\right) + \left(\frac{2}{3}-\frac{2p}{3p+2\varepsilon}\right) \to 0(\varepsilon\to 0^+).$$

作变换 $u = \frac{x}{y}$,因 $p > 0(p \neq 1)$,$\frac{1}{p}+\frac{1}{q}=1$,联系式(13),有

$$\varepsilon\tilde{I} = \varepsilon\int_1^\infty y^{-1-\varepsilon}\int_{\frac{1}{y}}^\infty \frac{1}{(u+a)(u+b)(u+c)}u^{\frac{1}{2}-\frac{\varepsilon}{p}}\mathrm{d}u\mathrm{d}y =$$

$$\varepsilon\int_1^\infty y^{-1-\varepsilon}\left[\int_0^\infty \frac{1}{(u+a)(u+b)(u+c)}u^{\frac{1}{2}-\frac{\varepsilon}{p}}\mathrm{d}u - \varepsilon\int_0^{\frac{1}{y}} \frac{1}{(u+a)(u+b)(u+c)}u^{\frac{1}{2}-\frac{\varepsilon}{p}}\mathrm{d}u\right]\mathrm{d}y >$$

$$\int_0^\infty \frac{1}{(u+a)(u+b)(u+c)}u^{\frac{1}{2}-\frac{\varepsilon}{p}}\mathrm{d}u - \frac{\varepsilon}{abc}\int_1^\infty y^{-1}\int_0^{\frac{1}{y}} u^{\frac{1}{2}-\frac{\varepsilon}{p}}\mathrm{d}u\mathrm{d}y \geqslant$$

$$(k+o(1)) - \frac{\varepsilon}{abc\left(\frac{3}{2}-\frac{\varepsilon}{p}\right)^2} = k + \tilde{o}(1)(\varepsilon\to 0^+);$$

$$\varepsilon\tilde{I} < \varepsilon\int_1^\infty\int_0^\infty \frac{x^{\frac{1}{2}-\frac{\varepsilon}{p}}y^{\frac{1}{2}-\frac{\varepsilon}{q}}}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\mathrm{d}y =$$

$$\int_0^\infty \frac{u^{\frac{1}{2}-\frac{\varepsilon}{p}}\mathrm{d}u}{(u+A)(u+b)(u+c)} = k + o(1)(\varepsilon\to 0^+).$$

故式(12)成立.证毕.

2　定　理

定理 1　设 $p > 1$,$\frac{1}{p}+\frac{1}{q}=1$,$a,b,c>0$,$f(t)$,$g(t)$为非负可测函数,使得 $0 < \int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t < \infty$,$0 < \int_0^\infty \frac{1}{t^{1+q/2}}g^q(f)\mathrm{d}t < \infty$,则有如下等价不等式:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\mathrm{d}y < k\left(\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t\right)^{\frac{1}{q}}, \tag{14}$$

$$\int_0^\infty y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^p\mathrm{d}y < k^p\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t, \tag{15}$$

这里,常数因子 $k = \frac{\pi}{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{c})}$ 与 k^p 为最佳值.特别地,当 $c=b$ 时,有

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+ay)(x+by)^2}\mathrm{d}x\mathrm{d}y < \frac{\pi}{2\sqrt{b}(\sqrt{a}+\sqrt{b})^2}\left(\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t\right)^{\frac{1}{q}}, \tag{16}$$

$$\int_0^\infty y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+ay)(x+by)^2}\mathrm{d}x\right]^p\mathrm{d}y < \left[\frac{\pi}{2\sqrt{b}(\sqrt{a}+\sqrt{b})^2}\right]^p\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t. \tag{17}$$

证明　由带权的Hölder不等式[12]与式(7),(8),有

$$I := \int_0^\infty\int_0^\infty \frac{1}{(x+ay)(x+by)(x+cy)}\left(\frac{y^{1/(2p)}}{x^{1/(2q)}}f(x)\right)\left(\frac{x^{1/(2q)}}{x^{1/(2p)}}g(y)\right)\mathrm{d}x\mathrm{d}y \leqslant$$

$$\left\{\int_0^\infty\int_0^\infty \frac{1}{(x+ay)(x+by)(x+cy)}\frac{y^{1/2}}{x^{(p-1)/2}}f^p(x)\mathrm{d}y\mathrm{d}x\right\}^{\frac{1}{p}}\times$$

$$\left\{\int_0^\infty\int_0^\infty \frac{1}{(x+ay)(x+by)(x+cy)}\frac{x^{1/2}}{y^{(q-1)/2}}g^q(y)\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{q}}=$$

$$\left\{\int_0^\infty \omega(x,a,b,c)\frac{1}{x^{1+p/2}}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty \bar{\omega}(y,a,b,c)\frac{1}{y^{1+q/2}}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{18}$$

若式(18)中间取等号,则有不全为零的常数 A,B,使 $A\frac{y^{1/2}}{x^{(p-1)/2}}f^p(x)=B\frac{x^{1/2}}{y^{(q-1)/2}}g^q(y)$ 几乎处处于$(0,\infty)\times(0,\infty)$[10].即有常数 C,使 $A\frac{1}{x^{-1+p/2}}f^p(x)=B\frac{1}{y^{-1+q/2}}g^q(y)=C$ 几乎处处于$(0,\infty)\times(0,\infty)$.不妨设 $A\neq 0$,则有$\frac{1}{x^{1+p/2}}f^p(x)=\frac{C}{Ax^2}$ 几乎处处于$(0,\infty)$.这矛盾于$0<\int_0^\infty\frac{1}{x^{1+p/2}}f^p(x)\mathrm{d}x<\infty$.再由式(10),有式(14).

任取 $0<\varepsilon<p$,设 $\tilde{f}(x)=\tilde{g}(x)=0,x\in(0,1)$;$\tilde{f}(x)=x^{\frac{1}{2}-\frac{\varepsilon}{p}}$,$\tilde{g}(x)=x^{\frac{1}{2}-\frac{\varepsilon}{q}}$,$x\in[1,\infty)$.若 $\tilde{k}\leqslant k$ 是式(14)的最佳值,则由式(12)有

$$k+\tilde{o}(1)<\varepsilon\tilde{I}=\varepsilon\int_0^\infty\int_0^\infty\frac{\tilde{f}(x)\tilde{g}(y)}{(x+ay)(x+by)}\mathrm{d}x\mathrm{d}y<\varepsilon\tilde{k}\left(\int_0^\infty\frac{1}{t}\tilde{f}^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty\frac{1}{t}\tilde{g}^q(t)\mathrm{d}t\right)^{\frac{1}{q}}=\tilde{k},$$

及由极限保号性,有 $k\leqslant\tilde{k}(\varepsilon\to 0^+)$.故 $\tilde{k}=k$ 为式(14)的最佳值.

设 T 足够大,使 $g(y,T)=\left[y^{1+\frac{q}{2}}\int_0^T\frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^{p-1}>0(y\in(0,\infty)$.则由式(14)有

$$0<\int_0^T\frac{1}{y^{1+q/2}}g^q(y,T)\mathrm{d}y=\int_0^T y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^T\frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^p\mathrm{d}y=$$

$$\int_0^T\int_0^T\frac{f(x)g(y,T)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\mathrm{d}y<k\left(\int_0^T\frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^T\frac{1}{y^{1+q/2}}g^q(y,T)\mathrm{d}y\right)^{\frac{1}{q}}, \tag{19}$$

$$\int_0^T\frac{1}{y^{1+q/2}}g^q(y,T)\mathrm{d}y=\int_0^T y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^T\frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^p\mathrm{d}y<$$
$$k^p\int_0^T\frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t. \tag{20}$$

令 $T\to\infty$,有 $0<\int_0^T\frac{1}{y^{1+p/2}}g^q(y,\infty)\mathrm{d}y<\infty$,再由式(14),知式(19),(20)取严格不等号,故有式(15).

反之,设式(15)成立.由Hölder不等式[11],有

$$I=\int_0^\infty\left[y^{\left(1+\frac{q}{2}\right)\frac{1}{q}}\int_0^\infty\frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]\left[\frac{1}{y^{\left(1+\frac{q}{2}\right)/q}}g(y)\right]\mathrm{d}y\leqslant$$

$$\left\{\int_0^\infty y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^\infty\frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^p\mathrm{d}y\right\}^{\frac{1}{p}}\left\{\int_0^\infty\frac{1}{y^{1+q/2}}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{21}$$

再由式(15),有式(14).故式(14)与式(15)等价.若式(15)的常数因子不是最佳值,则由式(21),可得出式(14)的常数因子也不是最佳值的矛盾.证毕.

定理 2 设 $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$,$a,b,c>0$,$f(t),g(t)$ 为非负可测函数,使得 $0<\int_0^\infty\frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t<\infty$,$0<\int_0^\infty\frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t<\infty$,,则有如下等价不等式:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\mathrm{d}y > k\left(\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t\right)^{\frac{1}{q}}, \tag{22}$$

$$\int_0^\infty y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^p\mathrm{d}y > k^p\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t, \tag{23}$$

$$\int_0^\infty x^{\left(1+\frac{p}{2}\right)(q-1)}\left[\int_0^\infty \frac{g(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}y\right]^q\mathrm{d}x < k^q\int_0^\infty \frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t, \tag{24}$$

这里,常数因子 $k = \dfrac{\pi}{(\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{c})}$,$k^p$ 与k^q 为最佳值.特别当 $c = b$ 时,有如下等价式:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+ay)(x+by)^2}\mathrm{d}x\mathrm{d}y > \frac{\pi}{2\sqrt{b}(\sqrt{a}+\sqrt{b})^2}\left(\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t\right)^{\frac{1}{q}}, \tag{25}$$

$$\int_0^\infty y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+ay)(x+by)^2}\mathrm{d}x\right]^p\mathrm{d}y > \left[\frac{\pi}{2\sqrt{b}(\sqrt{a}+\sqrt{b})^2}\right]^p\int_0^\infty \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t, \tag{26}$$

$$\int_0^\infty x^{\left(1+\frac{p}{2}\right)(q-1)}\left[\int_0^\infty \frac{g(y)}{(x+ay)(x+by)^2}\mathrm{d}y\right]^q\mathrm{d}x < \left[\frac{\pi}{2\sqrt{b}(\sqrt{a}+\sqrt{b})^2}\right]^q\int_0^\infty \frac{1}{t^{1+q/2}}g^q(t)\mathrm{d}t. \tag{27}$$

证明　由带权的逆向Hölder 不等式[12] 与式(7),(8),类似于建立式(18) 的配方法,有

$$I \geqslant \left\{\int_0^\infty \omega(x,a,b,c)f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_0^\infty \bar{\omega}(y,a,b,c)g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{28}$$

类似于证明式(18) 取严格不等式的方法及式(9),有式(22).

任取 $0<\varepsilon<p$,设 $\tilde{f}(x) = \tilde{g}(x) = 0, x\in(0,1)$;$\tilde{f}(x) = x^{\frac{1}{2}-\frac{\varepsilon}{p}}$,$\tilde{g}(x) = x^{\frac{1}{2}-\frac{\varepsilon}{q}}$,$x\in[1,\infty)$.若有 $K\geqslant k$,使式(22) 成立.则由(12) 有

$$k+o(1) > \varepsilon\tilde{I} = \varepsilon\int_0^\infty\int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\mathrm{d}y >$$
$$\varepsilon K\left(\int_0^\infty \frac{1}{t^{1+p/2}}\tilde{f}^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{t^{1+q/2}}\tilde{g}^q(t)\mathrm{d}t\right)^{\frac{1}{q}} = K,$$

及由极限保号性,有 $k\geqslant K(\varepsilon\to 0^+)$.故 $K = k$ 为式(22) 的最佳值.

设 T 足够大,使 $g(y,T) = \left[y^{1+\frac{q}{2}}\int_0^T \frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^{p-1} > 0(y\in(0,\infty))$.则由式(22),有

$$\int_0^T \frac{1}{y^{1+q/2}}g^q(y,T)\mathrm{d}y = \int_0^T y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^T \frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^p\mathrm{d}y =$$

$$\int_0^T\int_0^T \frac{f(x)g(y,T)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\mathrm{d}y > k\left(\int_0^T \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^T \frac{1}{y^{1+q/2}}g^q(y,T)\mathrm{d}y\right)^{\frac{1}{q}}, \tag{29}$$

$$\int_0^T \frac{1}{y^{1+q/2}}g^q(y,T)\mathrm{d}y = \int_0^T y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^T \frac{f(x)\mathrm{d}x}{(x+ay)(x+by)(x+cy)}\right]^p\mathrm{d}y > k^p\int_0^T \frac{1}{t^{1+p/2}}f^p(t)\mathrm{d}t. \tag{30}$$

令 $T\to\infty$,若$\int_0^T \frac{1}{y^{1+q/2}}g^q(y,\infty)\mathrm{d}y = \infty$,则式(30) 自然取严格不等号;若 $0<\int_0^T \frac{1}{y^{1+q/2}}g^q(y,\infty)\mathrm{d}y<\infty$,则在应用式(22) 时,式(29),(30) 仍取严格不等号.故式(23) 成立.

反之,设式(23) 成立.由类似于式(18) 的配方法及反向不等式,有

$$I \geqslant \left\{\int_0^\infty y^{\left(1+\frac{q}{2}\right)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+ay)(x+by)(x+cy)}\mathrm{d}x\right]^p\mathrm{d}y\right\}^{\frac{1}{p}}\left\{\int_0^\infty \frac{1}{y^{1+q/2}}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{31}$$

再由式(23),有式(22).故式(22)与式(23)等价.若式(23)的常数因子不是最佳值,则由式(31),可得出式(22)的常数因子也不是最佳值的矛盾.

若设 $f(x,T)=\left[x^{1+\frac{p}{2}}\int_0^T \frac{g(y)}{(x+ay)(x+by)}\mathrm{d}y\right]^{q-1}>0(x\in(0,\infty))$,类似上面的方法及 $q<0$ 可得式(24),及证得式(24)与式(22)等价,且式(24)的常数因子亦为最佳值.故式(22),(23)与(24)等价.证毕.

评注 当 $a=b=c=1$ 时,式(14)变为式(5).故式(14)(及式(16))是式(5)的最佳推广.

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A Hilbert - type integral inequality with the kernel of - 3 - order homogeneous

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Abstract:By obtaining the weight function,a Hilbert-type inequality with the kernel of order homogeneous and parameters and the equivalent form are given. The best constant factor are calculated by the way of Complex Analysis. The cases of reverse inequalities and particular results are considered.

Key words:Hilbert-type integral inequality;weight function;kernel;equivalent form