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张量点态性质的一个应用

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摘要:将流形上的有关公式用分量表示,取局部正交基,借助张量分量的运算规律、技巧进行推算,最后再利用张量的点态性质,证明了一种特殊的Sasakian流形的φ-截曲率为$c-3$。由此给出了一个计算流形上的几何量的方法。

关 键 词:Sasakian流形;φ-截曲率;张量点态性质

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设CD^n为具有常全纯截曲率c的复单连通有界域,(J,G)是CD^n上的一个Sasakian结构。Kaehlerian结构的基本2-形式是闭的,故$\Omega=\mathrm{d}\omega$,ω为实的解析1-形式。设t表示R上的坐标,对积空间$R\times CD^n$取$\eta=\omega+\mathrm{d}t$(文献[1]中误为$\eta=2\omega+\mathrm{d}t$),若将R看成一个加群,则η是平凡线丛(R,CD^n)上的无穷小联络形式。令$\xi=\frac{\partial}{\partial t}$,$g=\pi^*G+\eta\odot\eta$,其中$\pi:(R,CD^n)\to CD^n$是投射,$\eta$又可写成$\eta=\pi^*\omega+\mathrm{d}t$,这些张量定义了$(R,CD^n)$上具有常$\varphi$-截曲率$k=c-3$的Sasakian结构。

许多关于Sasakian结构的文献引用(R,CD^n)作为Sasakian空间形式的例子,但都未给予证明。类似的问题又多采用整体符号来证明或计算[1],然而整体符号具有局限性。本文利用局部法正交基、张量的分量形式及其点态性质,证明了(R,CD^n)的φ-截曲率为$c-3$。由此给出了一个计算或证明流形上的几何量的补充方法。

1 基本公式

1.1 (R,CD^n)上的度量矩阵及联络系数

取(R,CD^n)上的衬准正交基$\{e_1,e_2,\cdots,e_{2n}\}$,由$\Omega$为基本2-形式得

$$G(e_i,Je_j)=\mathrm{d}\omega(e_i,e_j)。\tag{1}$$

在一点$P\in CD^n$处,必要时作坐标变换使$\frac{\partial}{\partial u^i}=e_i$,利用文献[1]及$\eta=\omega+\mathrm{d}t$得

$$J_{ij}=\frac{1}{2}(\partial_i\omega_j-\partial_j\omega_i)=\frac{1}{2}(\partial_i\eta_j-\partial_j\eta_i),$$

$$\text{其中 }\partial_i=\frac{\partial}{\partial u^i}。\tag{2}$$

而(R,CD^n)上的度量矩阵

$$g=\begin{pmatrix} G_{11}+\eta_1\eta_1 & G_{12}+\eta_1\eta_2 & \cdots & G_{1,2n}+\eta_1\eta_{2n} & \eta_1 \\ G_{21}+\eta_2\eta_1 & G_{22}+\eta_2\eta_2 & \cdots & G_{2,2n}+\eta_2\eta_{2n} & \eta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ G_{2n,1}+\eta_{2n,2}\eta_1 & G_{2n,2}+\eta_{2n}\eta_2 & \cdots & G_{2n,2n}+\eta_{2n}\eta_{2n} & \eta_{2n} \\ \eta_1 & \eta & \eta_2 & \eta_{2n} & \eta_1 \end{pmatrix},\tag{3}$$

$$g^{-1}=\begin{pmatrix} G^{11} & G^{12} & \cdots & G^{1,2n} & -G^{1s}\eta_s \\ G^{21} & G^{22} & \cdots & G^{2,2n} & -G^{2s}\eta_s \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ G^{2n,1} & G^{2n,2} & \cdots & G^{2n,2n} & -G^{2n,s}\eta_s \\ -\eta_sG^{s1} & -\eta_sG^{s2} & \cdots & -\eta_sG^{s,2n} & 1+\eta_jG^{ij}\eta_i \end{pmatrix}。$$

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我们规定：指标 $A,B,\cdots$ 取 $1,2,\cdots,2n+1$；指标 $i,j,k\cdots$ 取 $1,2,\cdots,2n$；$\dot{\nabla},\dot{\Gamma}^A_{BC},\dot{R}^A_{BCD}$ 与 $\nabla,\Gamma_{ij}{}^k,R_{ijk}{}^h$ 依次表示 (R,CD^n) 与 CD^n 上的联络、联络系数、曲率分量。

由式(3)

$$\dot{\Gamma}_{ij}{}^k=\frac{1}{2}g^{kA}\left(\frac{\partial g_{iA}}{\partial u^j}+\frac{\partial g_{jA}}{\partial u^i}-\frac{\partial g_{ij}}{\partial u^A}\right)=\frac{1}{2}(G^{K_1},G^{K_2},\cdots,G^{K_{2n}},-G^{K_t}\eta_t)\cdot$$

$$\left[\begin{pmatrix}\frac{\partial G_{i1}}{\partial u^j}+\frac{\partial G_{j1}}{\partial u^i}-\frac{\partial G_{ij}}{\partial u^1}\\ \frac{\partial G_{i2}}{\partial u^j}+\frac{\partial G_{j2}}{\partial u^i}-\frac{\partial G_{ij}}{\partial u^2}\\ \cdots\cdots\\ \frac{\partial G_{i,2n}}{\partial u^j}+\frac{\partial G_{j,2n}}{\partial u^i}-\frac{\partial G_{2j}}{\partial u^{2n}}\end{pmatrix}+\begin{pmatrix}\frac{\partial\eta_i}{\partial u^j}\eta_1+\frac{\partial\eta_j}{\partial u^i}\eta_1\\ \frac{\partial\eta_i}{\partial u^j}\eta_2+\frac{\partial\eta_j}{\partial u^i}\eta_2\\ \cdots\cdots\\ \frac{\partial\eta_i}{\partial u^j}\eta_{2n}+\frac{\partial\eta_j}{\partial u^i}\eta_{2n}\\ \frac{\partial\eta_i}{\partial u^j}+\frac{\partial\eta_j}{\partial u^i}\end{pmatrix}+\begin{pmatrix}\eta_i(\frac{\partial\eta_1}{\partial u^j}-\frac{\partial\eta_j}{\partial u^1})+\eta_j(\frac{\partial\eta_1}{\partial u^i}-\frac{\partial\eta_i}{\partial u^1})\\ \eta_i(\frac{\partial\eta_2}{\partial u^j}-\frac{\partial\eta_j}{\partial u^2})+\eta_j(\frac{\partial\eta_2}{\partial u^i}-\frac{\partial\eta_i}{\partial u^2})\\ \cdots\cdots\\ \eta_i(\frac{\partial\eta_{2n}}{\partial u^j}-\frac{\partial\eta_j}{\partial u^{2n}})+\eta_j(\frac{\partial\eta_{2n}}{\partial u^i}-\frac{\partial\eta_i}{\partial u^{2n}})\\ 0\end{pmatrix}\right]=\Gamma_{ij}{}^k-\eta^tJ_j-\eta^k_jJ_i{}^k。\quad(4)$$

同理

$$\dot{\Gamma}_{ij}{}^{2n+1}=-\Gamma_{ij}{}^k\eta_k-\eta_jJ_i^k\eta_k+\eta_iJ_j{}^k\eta_k+\frac{1}{2}(\partial_j\eta_i+\partial_i\eta_j)。\quad(5)$$

其中 $\frac{\partial}{\partial u^{2n+1}}=\frac{\partial}{\partial t}$。而 (G,J) 是 (CD^n) 的 Kachlerian 结构，设 (φ,η,ζ,g) 是 (R,CD^n) 的 Sasakian 结构[2]。

$$\eta=\omega+\mathrm{d}t\Rightarrow\mathrm{d}\eta=\mathrm{d}\omega\Rightarrow\bigcirc=\Omega\Rightarrow g(\partial_i,\varphi\partial_j)=G(\partial_i,j\partial_j)\Rightarrow g_{iA}\varphi_j^A=G_{ik}J_j^k\Rightarrow\varphi_{ji}=J_{ji}=\frac{1}{2}(\partial_i\eta_j-\partial_j\eta_i),\quad(6)$$

$$g(\zeta,\varphi(\partial_i))=0\Rightarrow\varphi_i{}^Ag_{2n+1,A}=0\Rightarrow\varphi_{i,2n+1}=0,\quad(7)$$

$$\varphi_i^j=\varphi_{iA}g^{jA}=\varphi_{ik}g^{jk}+\varphi_{i,2n+1}g^{j,2n+1}\Rightarrow\varphi_i^j=J_i{}^j。\quad(8)$$

由

$$\varphi(\zeta)=0\Rightarrow\varphi_{2n+1}{}^A=0,\quad(9)$$

$$\eta(\varphi(\partial_i))=0\Rightarrow\varphi_i^{2n+1}=-\varphi_i^k\eta_k=-J_i^k\eta_k,\quad(10)$$

而

$$\dot{\nabla}\partial_i\zeta=-\varphi(\partial_i)\Rightarrow\dot{\Gamma}^A_{i,2n+1}=-\varphi_i{}^A=\begin{cases}-\varphi_i{}^k,\\-\varphi_i^{2n+1}=J_i{}^k\eta_k。\end{cases}\quad(11)$$

$$\dot{\nabla}_\zeta\zeta=-\varphi(\zeta)=0\Rightarrow\dot{\Gamma}^A_{2n+1,2n+1}=0。\quad(12)$$

由 CD^n 为 Kachlerian 流形，故 $\nabla J=0$，则 $\nabla_iJ_j^k=0$（或 $\nabla_iJ_{jk}=0$）。

由此得

$$\partial_iJ_j{}^k=\Gamma_{ij}{}^lJ_l{}^k,$$

$$\partial_iJ_{jk}=\Gamma_{ij}{}^lJ_{lk}+\Gamma_{ik}{}^lJ_{jl}。\quad(13)$$

在 $P\in CD^n$ 选取局部标架，使 $J\partial_i=\partial_{n+i},J\partial_{n+i}=-\partial_i$ 且 $\{\partial_i\}_1^{2n}$ 是法正交基，易证 $(\partial_i-\eta_i\partial t,\partial_{n+1}-\eta_{n+i}\partial t,\partial t)i=1,2,\cdots n$，是 (R,CD^n) 的 φ 基，记 $n+i$ 为 i^*，则在 P 处有

$$J_{ii^*}=1,J_{i^*i}=-1,$$

$$J_{i^*}{}^i=1,J_i{}^{i^*}=-1，其他\ J_k{}^l=0。\quad(14)$$

此时有

$$\partial_iJ_j{}^k=-\Gamma_{ij}{}^k-\Gamma_{ij^*}{}^{k^*},$$

$$\partial_iJ_{jk}=-\Gamma_{ij}{}^k+\Gamma_{ik}{}^j。\quad(15)$$

1.2 曲率计算

由 1.1 中诸公式，(R,CD^n) 中曲率分量可如下求得

$$\dot{R}^{2n+1}_{ii^*i}=\frac{\partial\dot{\Gamma}^{2n+1}_{i^*i}}{\partial u^i}-\frac{\partial\dot{\Gamma}^{2n+1}_{ii}}{\partial u^{i^*}}+\dot{\Gamma}^{2n+1}_{Ai}\dot{\Gamma}^A_{i^*i}-\dot{\Gamma}^{2n+1}_{Ai^*}\dot{\Gamma}^A_{ii}=$$

$$\partial_i(-\Gamma^k_{i^*i}\eta_k+2J^k_{i^*}\eta_i\eta_k+\partial_{i^*}\eta_i)-\partial_{i^*}[-\Gamma^k_{ii}\eta_k+J^k_i\eta_k\eta_{i^*}+J^k_i\eta_k\eta_i+J_{ki^*}\eta_k\eta_i+\frac{1}{2}(\partial_{i^*}\eta_i+\partial_i\eta_{i^*})]+[-\Gamma^k_{ji}\eta_k+J^k_i\eta_k\eta_j+\Gamma^k_j\eta_k\eta_i+\frac{1}{2}(\partial_j\eta_i+\partial_i\eta_j)](\Gamma^j_{i^*i}-2J^j_{i^*}\eta_{i^*})+J^l_i\eta_l(-\Gamma^k_{i^*i}\eta_k+2J^k_{i^*}\eta_{i^*}\eta_k+\partial_{i^*}\eta_i)-J^l_{i^*}\eta_l[-\Gamma^k_{ii}\eta_k+J^k_i\eta_k\eta_{i^*}+J^k_{i^*}\eta_k\eta_i+\frac{1}{2}(\partial_i\eta_i+\partial_i\eta_{i^*})]-[-\Gamma^k_{ji^*}+J^k_i\eta_k\eta_{i^*}+J^l_{i^*}\eta_l\eta_j+\frac{1}{2}(\partial_{i^*}\eta_j+\partial_j\eta_{i^*})](\Gamma^j_{ii}-J^j_i\eta_{i^*}-J^j_{i^*}\eta_i)=$$

$$-R^k_{ii^*i}\eta_k-\Gamma^k_{i^*i}\partial_i\eta_k+2(\partial_iJ^k_{i^*})\eta_i\eta_k-2(\partial_i\eta_{i^*})\eta_i-2(\partial_i\eta_i)\eta_{i^*}-(\partial_{i^*}J^k_i)\eta_k\eta_{i^*}-2(\partial_{i^*}\eta_{i^*})\eta_{i^*}-(\partial_{i^*}-J^k_{i^*})\eta_k\eta_i+2(\partial_{i^*}\eta_i)\eta_i+\partial_iJ_{i^*i}-2\Gamma^k_{ii}\eta_k\eta_{i^*}+\Gamma^j_{i^*i}\eta_j\eta_{i^*}+2\eta^2_{i^*}\eta_i+\Gamma^j_{i^*i}\eta_{j^*}\eta_i+2\eta^2_{i^*}\eta_i+$$

$\frac{1}{2}\Gamma^{j}_{i^*i^*}(\partial_i\eta_j + \partial_j\eta_i) + 2(\partial_i\eta_j)\eta_{i^*} -$
$\Gamma^{k}_{i^*i^*} - 2\eta^2_{i^*}\eta_i + \eta_{i^*}\partial_i\cdot\eta_{i^*} - \Gamma^{k}_{ii^*}\eta_k\eta_i +$
$\eta^2_{i^*}\eta_i - \eta^3_i + \frac{1}{2}\eta_i(\partial_{i^*}\eta_i + \partial_i\eta_{i^*}) -$
$\Gamma^{k}_{i^*i}\eta_k\eta_{i^*} + \Gamma^{k}_{ii}\eta_k\eta_i - \Gamma^{j}_{ii^*}\eta_{j^*}\eta_{i^*} - \eta^2_{i^*}\eta_i -$
$\eta^2_{i^*}\eta_i + \Gamma^{j}_{ii^*}\eta_j\eta_i - \eta^2_{i^*}\eta_i + \eta^3_i -$
$\frac{1}{2}\Gamma^{j}_{ii^*}(\partial_{i^*}\eta_j + \partial_j\eta_{i^*}) + (\partial_i\cdot\eta_{i^*})\eta_{i^*} -$
$\frac{1}{2}\eta_i(\partial_{i^*}\eta_i + \partial_i\eta_{i^*}) =$
$-R^{k}_{ii^*i^*}\eta_k + \Gamma^{j}_{i^*i^*}J_{ij} - \Gamma^{j}_{ii^*}J_{ji^*} - 2\Gamma^{k}_{ij}\eta_k\eta_{i^*} +$
$\Gamma^{j}_{i^*i^*}(\eta_{j^*}\eta_i - \eta_j\eta_{i^*}) + \Gamma^{j}_{ii^*}(\eta_j\eta_i - \eta_{j^*}\eta_{i^*}) +$
$4\eta_i J_{ii^*} + 2(\partial_i J^{k}_{i^*})\eta^{k^*}\eta_i - (\partial_{i^*}J^{k})\eta_k\eta_{i^*} -$
$(\partial_{i^*}J^{k}_{i^*})\eta_k\eta_i + \partial_i\cdot J_{i^*i} =$
$-R^{k}_{ii^*i^*}\eta_k + \Gamma^{i}_{i^*i^*} + \Gamma^{i}_{ii^*} - 2\Gamma^{k}_{ii}\eta_k\eta_{i^*} +$
$\Gamma^{j}_{i^*i^*}(\eta^{j^*}\eta_i - \eta_j\eta_{i^*}) + \Gamma^{j}_{ii^*}(\eta_j\eta_i - \eta_{j^*}\eta_{i^*}) +$
$4\eta_i + 2\eta_{i^*}\eta_k(-\Gamma^{k^*}_{ii^*} + \Gamma^{k}_{ii}) - \eta_k\eta_{i^*}(-\Gamma^{k^*}_{i^*i} - \Gamma^{k}_{i^*i^*}) -$
$\eta_k\eta_i(-\Gamma^{k^*}_{i^*i^*} + \Gamma^{k}_{i^*i^*}) - \Gamma^{i^*}_{i^*i^*} - \Gamma^{i^*}_{ii} =$
$-R^{k}_{ii^*i^*} + 4\eta_i$。　　(16)

其中用到　$(j^*)^* = -j$,

$$\left.\begin{aligned}
&\dot{R}^{i}_{ii^*i^*} = R^{i}_{ii^*i^*} - 3\delta^i_i - \delta^i_{i^*}\eta_i\eta_{i^*} + \delta^i_i\eta^2_{i^*}\\
&\dot{R}^{2n+1}_{2n+1,ii} = 1 + \eta^2_i,\\
&\dot{R}^{i}_{2n+1,ii} = -\delta^i_i\eta_i,\\
&\dot{R}^{2n+1}_{2n+1,i^*i^*} = 1 + \eta^2_{i^*}\\
&\dot{R}^{i}_{2n+1,i^*i^*} = -\delta^i_{i^*},\\
&\dot{R}^{2n+1}_{2n+1,ii^*} = \eta_i\eta_{i^*},\\
&\dot{R}^{i}_{2n+1,ii^*} = -\delta^i_i\eta_{i^*}
\end{aligned}\right\}。\tag{17}$$

而

$\dot{R}_{ii^*i^*i} = \dot{R}^{2n+1}_{ii^*i^*}g_{2n+1,i} + \dot{R}^{j}_{ii^*i^*}g_{ji} =$
$(-R^{k}_{ii^*i^*}\eta_k + 4\eta_i)\eta_i + (R^{j}_{ii^*i^*} - \delta^j_{i^*}\eta_i\eta_{i^*} + \delta^j_i\eta^2_{i^*} - 3\delta^j_i)(\delta_{ij} + \eta_i\eta_j) =$
$c - 3 + \eta^2_i + \eta^2_{i^*}$,

其中

$g_{2n+1i} = \eta_i, g_{ji} = \delta_{ij} + \eta_i\eta_j$

法标架下显然成立。同理

$$\left.\begin{aligned}
&\dot{R}_{ii^*i^*2n+1} = \eta_i\\
&\dot{R}_{2n+1ii2n+1} = 1\\
&\dot{R}_{2n+1ii^*i} = -\eta_{i^*}\\
&\dot{R}_{2n+1,i,i^*,2n+1} = 0\\
&\dot{R}_{2n+1,i^*i^*,2n+1} = 1
\end{aligned}\right\}。\tag{18}$$

2　结　论

定　理　(R,CD^n) 是 φ-截曲率为 $c-3$ 的 Sasakian 空间形式。

证　明　由 φ-截曲率定义[2]

$g(\dot{R}_{(\partial_i-\eta_i\partial_{2n+1},\partial_{i^*}-\eta_{i^*}\partial_{2n+1})}\partial_i, \partial_i - \eta_i\partial_t) =$
$\dot{R}_{ii^*i^*i} - 2\eta_i\dot{R}_{ii^*i^*,2n+1} +$
$\eta^2_i\dot{R}_{2n+1,i^*i^*,2n+1} - 2\eta_{i^*}\dot{R}_{ii^*,2n+1,i} +$
$2\eta_{i^*}\eta_i\dot{R}_{2n+1,i^*,2n+1,i} =$
$c - 3 + \eta^2_i + \eta^2_{i^*} - 2\eta^2_i + \eta^2_i - 2\eta_{i^*} + \eta^2_{i^*} =$
$c - 3$。

由张量点态性知 (R,CD^n) 有常 φ-截曲率 $c-3$。
证毕。

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(编　辑　曹大刚)

The application of pointwise property of tensor

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Abstract: Using the local orthogonal basis, the pointwise property and component representation of tensor is demonstrated as a special Sasakian manifold. It is Sasakian space form with φ-sectional curveture $k=c-3$. Meanwhile it provides a complemental method for computing some geometric quatity of manifold.

Key words: Sasakian manifold; φ-sectional curveture; pointwise peroprity of tensor