空间离散点的三角 Bézier 整体 VC^1，局部 C^2 光滑插值曲面*

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TRIANGULAR BÉZIER INTERPOLATION SURFACE WITH WHOLE VC^1, LOCAL C^2 CONTINUITY OVER DISCRETE POINTS IN R^3

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Abstract

In this paper, a method for constructing a smooth interpolation surface that is piecewise parametric triangular Bézier surface with whole tangent plane continuity (VC^1) and local C^2 continuity over a given spatial triangular grid satisfied some conditions is given. The Bézier points needed in the Method are reduced. It is possible for us to generate a more complex surfaces on microcomputers.

一、构造插值曲面的基本思想

已知空间 R^3 中一组离散点 $\{p_i=(x_i, y_i, z_i)\}_{i=1}^m$ 和每点 p_i 处的法向量 n_{p_i}，$i=1,2,\cdots,m$. 假设对点集 $\{p_i\}$ 已建立了空间三角形网格 Δ，其中要求以非边界网格点为顶点的三角形的个数为奇数.

本文在已知的Δ上，构造出每片三角曲面片是 C^2 光滑的，而各三角曲面片之间为 VC^1 拼接(即，在公共边界曲线上具有切平面连续)的分片四次三角 Bézier 光滑曲面，并且整张曲面插值给出的空间各离散点 p_i，$i=1,2,\cdots,m$.

为了达到所需的光滑性要求，首先关于Δ中的每个三角形构造出三次三角 Bézier C^2 光滑曲面片，而不考虑各三角曲面片之间的 VC^1 拼接. 其次利用升阶公式将所得的三角曲面片变成四次三角 Bézier 曲面片. 最后修改每个三角曲面片的某些 Bézier 点，使其满足 VC^1 拼接的充分性条件[1]，并且使三角曲面片内部仍是 C^2 光滑的.

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二、插值曲面的构造

显然只需求出所有的 Bézier 点即可.

1. 三次三角 C^2 光滑曲面的构造

首先将任意的三角形 $T=\langle T_1,T_2,T_3\rangle\in\Delta$ 作分割，各网点标号为 T_i，$i=1,2,\cdots,64$，如图 1 所示. 记 T_i 点对应的分片三次三角 Bézier 曲面片的 Bézier 点为 b_i，$i=1,2,\cdots,64$. 根据 C^2 光滑的充分条件[2]点集 $\{b_i\}_{i=1}^{64}$ 可由下列各式求出：

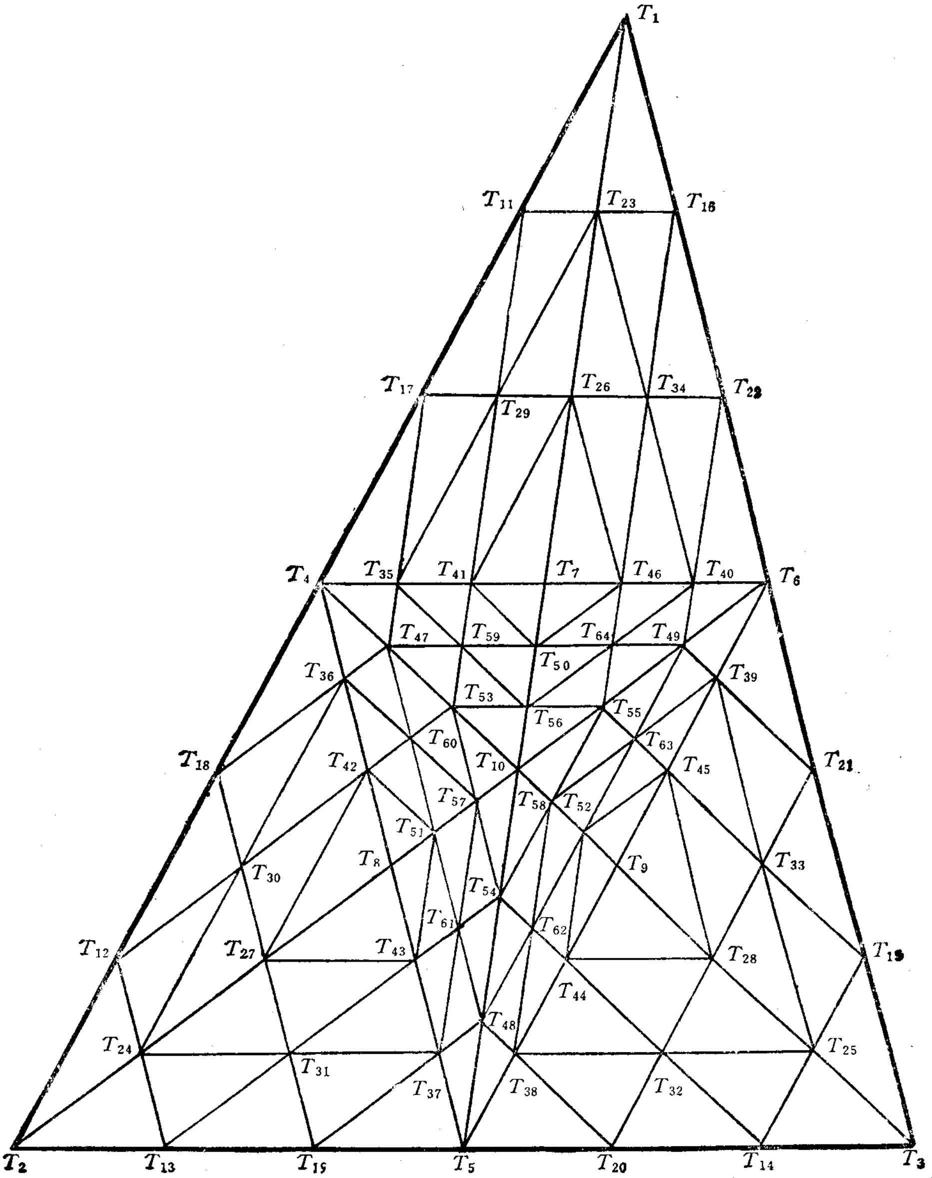

图 1

$$b_i = T_i,\quad b_{i+3} = \frac{1}{2}(b'_{2i+9} + b'_{2i+10}),\ i = 1, 2, 3, \tag{2.1.1}$$

$$b_i = \frac{1}{3} b_j + \frac{2}{3} b'_i,\ j = \text{int}\left(\frac{i-10}{2}\right) + 1(\text{mod}4),\ i = 11,12,\cdots,16, \tag{2.1.2}$$

其中 b' 点的具体算法参见 [3].

$$b_i = b_j + \frac{(-1)^i}{4}(b_k - b_{k+1}),\ j = \text{int}\left(\frac{i-11}{2}\right) + 1,\ k = 2 \cdot \text{int}\left(\frac{i-1}{2}\right) + 5,\ i = 17,18,\cdots,22, \tag{2.1.3}$$

$$b_i = b_j + A_T \cdot (\boldsymbol{a}_i \times \boldsymbol{n}_j)/h_i,\ j = i - 43,\ i = 47,48,49, \tag{2.1.4}$$

其中 $A_T = \text{area}(T_1, T_2, T_3)$ 为三角形 T 的有向面积; $\boldsymbol{a}_i = \overrightarrow{b_j b_{2i-77}}$; $h_i = 27 \cdot \|\boldsymbol{a}_i\| \cdot \|\boldsymbol{a}_i \times \boldsymbol{n}_j\|$; $\boldsymbol{n}_j$ 由下式给出 ($0 < q_j < 1$):

$$\begin{cases} \boldsymbol{n}_j = q_j \cdot \boldsymbol{n}_T + (1 - q_j) \cdot (\boldsymbol{n}_T \times \overrightarrow{b_j b_{2j+9}}), & T_j \in \partial\Delta \text{ 时}, \\ \boldsymbol{n}_j = q_j \cdot \boldsymbol{n}_T + (1 - q_j) \cdot \boldsymbol{n}_{\hat{T}^{(j)}}, & T_j \notin \partial\Delta \text{ 时}, \end{cases} \tag{2.1.5}$$

这里 $\partial\Delta$ 表示Δ的边界. 当 $T_j \notin \partial\Delta$ 时,有 $T_j \in \hat{T}^{(j)} \cap T$, $\hat{T}^{(j)} \in \Delta\backslash T$; $\boldsymbol{n}_T$, $\boldsymbol{n}_{\hat{T}}$ 分别为 T, $\hat{T}$ 所在平面的外法向量.

$$b_i = \frac{3}{4} b_j + \frac{1}{4} b_{i-18},\ j = \text{int}\left(\frac{i-33}{2}\right) + 46,\ i = 35,36,\cdots,40, \tag{2.1.6}$$

$$\begin{cases} b_i = \frac{1}{2} D_1 + (-1)^i [D_2 - D_3 - (b_{17} - b_{18} + b_{19} + b_{20} - b_{21} - b_{22})/4],\ i = 29, 30, \\ b_j = \frac{1}{2} D_2 + (-1)^j [D_3 - D_1 - (-b_{17} - b_{18} + b_{19} - b_{20} + b_{21} + b_{22})/4],\ j = 31, 32, \\ b_k = \frac{1}{2} D_3 + (-1)^k [D_1 - D_2 - (b_{17} + b_{18} - b_{19} - b_{20} + b_{21} - b_{22})/4],\ k = 33, 34, \end{cases} \tag{2.1.7}$$

其中

$$D_l = \frac{3}{2} b_{l+46} + \frac{1}{4}(b_{2l+9} + b_{2l+10}),\ l = 1,2,3,$$

$$b_i = \frac{1}{2}(b_k + b_{k+1}),\ k = 2i - 36 + 6\left[\text{int}\left(\frac{23}{i}\right) + \left(2 + \text{int}\left(\frac{26}{i}\right)\right) \cdot \text{int}\left(\frac{i}{26}\right)\right],\ i = 23,24,\cdots,28, \tag{2.1.8}$$

$$b_7 = \frac{3}{4}(b_{35} + b_{40}) - \frac{1}{4}(b_4 + b_6),\quad b_{41} = b_7 - \frac{1}{4}(b_{40} - b_{35}), \tag{2.1.9}$$

$$\begin{cases} b_i = b_{i-1} - \frac{1}{2}(b_{i-13} - b_{i-12}) + \frac{1}{16}(b_{i-25} - b_{i-24}),\ i = 42, 44, 46, \\ b_j = b_{j-1} - \frac{1}{2}(b_{j-7} - b_{j-6}),\ j = 43, 45, \end{cases} \tag{2.1.10}$$

$$b_i = \frac{1}{2}(b_{2i+26} + b_{2i+27}),\ i = 8, 9, \tag{2.1.11}$$

$$b_i = \frac{4}{3} b_{i-18} - \frac{1}{3} b_{i-30},\ i = 59, 60, \cdots, 64, \tag{2.1.12}$$

$$b_{50} = \frac{1}{2}(b_{64} + b_{59}),\ b_i = \frac{1}{2}(b_{2i-42} + b_{2i-41}),\ i = 51, 52, \tag{2.1.13}$$

$$b_i = \frac{1}{9}(12 \cdot b_{2i-47} + b_{2i-95} - 4 \cdot b_{2i-77}),\ i = 53, 54, 55 \tag{2.1.14}$$

$$b_{56} = \frac{1}{2}(b_{55} + b_{53}),\ b_i = \frac{1}{2}(b_{i-4} + b_{i-3}),\ i = 57, 58, \tag{2.1.15}$$

$$b_{10} = \frac{1}{3}(b_{56} + b_{57} + b_{58}). \tag{2.1.16}$$

2. 三角曲面片之间 VC^1 拼接的确定

利用升阶公式[2]，将得到的三角曲面片变为四次三角曲面片．对应三角形 T 的分割及网点标号如图 2 所示．记 U_i 点对应的 Bézier 点为 c_i，$i = 1, 2, \cdots, 109$，它们可利用升阶公式由点集 $\{b_i\}_{i=1}^{64}$ 求得．

记使插值曲面满足所需光滑性条件的 Bézier 点集为 $\{a_i\}_{i=1}^{109}$．显然有

$$a_i = c_i,\ i = 1, 2, \cdots, 31. \tag{2.2.1}$$

设三角形 $T = \langle T_1, T_2, T_3 \rangle \in \Delta \backslash \partial\Delta$．记三角形 $\hat{T} = \langle \hat{T}_1, \hat{T}_2, \hat{T}_3 \rangle \in \Delta$ 且 $\hat{T} \cap T = \overrightarrow{T_1T_2}$，$\hat{T}_1 = T_1$，$\hat{T}_3 = T_2$．记

$$\begin{cases} \eta = -A_T/A_{\hat{T}},\ d = \text{area}(a_{74}, a_4, \hat{a}_{76})/\text{area}(a_4, a_{23}, \hat{a}_{76}),\ \eta_3 = \text{area}(a_{29}, a_1, \hat{a}_{29})/ \\ \quad \text{area}(b_{11}, a_1, \hat{a}_{2?}),\ \delta_1 = \text{area}(a_{30}, b_{12}, \hat{a}_{31})/\text{area}(a_2, b_{12}, \hat{a}_{31}), \\ \eta_1 = [13(1-\eta) + 9d]/16,\ \eta_2 = 3(1 - \eta - 3d)/16,\ \eta_4 = 1 - \eta - \eta_3, \\ \quad \delta_2 = 1 - \eta - \delta_1,\ \delta_3 = 3(3d + 1 - \eta)/16,\ \delta_4 = [13(1-\eta) - 9d]/16. \end{cases} \tag{2.2.2}$$

取

$$a_{74} = a_4 - A_T \cdot (\boldsymbol{n}_4 \times \boldsymbol{\gamma})/s + t_1\boldsymbol{\gamma},\ \hat{a}_{76} = a_4 + A_{\hat{T}} \cdot (\boldsymbol{n}_4 \times \boldsymbol{\gamma})/s + t_2\boldsymbol{\gamma}, \tag{2.2.3}$$

其中 $\boldsymbol{\beta} = \overrightarrow{a_4a_{23}}$，$\boldsymbol{\gamma} = \boldsymbol{\beta}/\|\boldsymbol{\beta}\|$，$s = 48 \cdot \|\boldsymbol{n}_4 \times \boldsymbol{\beta}\|$，$t_1, t_2$ 满足等式

$$t_1 - \eta \cdot t_2 = 8 \cdot \|\boldsymbol{\beta}\| \cdot (\delta_2 - \eta_3)/9. \tag{2.2.4}$$

从而下列向量线性方程组可求解

$$\begin{cases} a_{62} + a_{63} = (a_{17} + a_{18} + 6a_{74})/4, \\ a_{63} - \eta\hat{a}_{66} = [(\delta_1 + 2\delta_4)b_{18} + \delta_2 a_4 + 2\delta_3 b_{12}]/3, \\ \hat{a}_{66} + \hat{a}_{67} = (a_{17} + a_{18} + 6\hat{a}_{76})/4, \\ a_{62} - \eta\hat{a}_{67} = [(2\eta_1 + \eta_4)b_{17} + 2\eta_2 b_{11} + \eta_3 a_4]/3, \end{cases} \tag{2.2.5}$$

但其解不唯一．令

$$a_{62} = c_{62} + Q_T, \tag{2.2.6}$$

便可求出 $a_{63}, \hat{a}_{66}, \hat{a}_{67}$ 各点，Q_T 点为形状参量．

若 $\overrightarrow{T_1T_2} \in \partial\Delta$，取 $a_{74} = c_{74}$，$a_i = c_i$，$i = 62, 63$．

用同样的方法可求点 a_i，$i = 64, 65, 66, 67, 75, 76$．

由光滑性条件，得

$$a_i = \frac{3}{4} a_j + \frac{1}{4} a_{i-15},\ j = \text{int}(i/2) + 55,\ i = 38, 39, \cdots, 43. \tag{2.2.7}$$

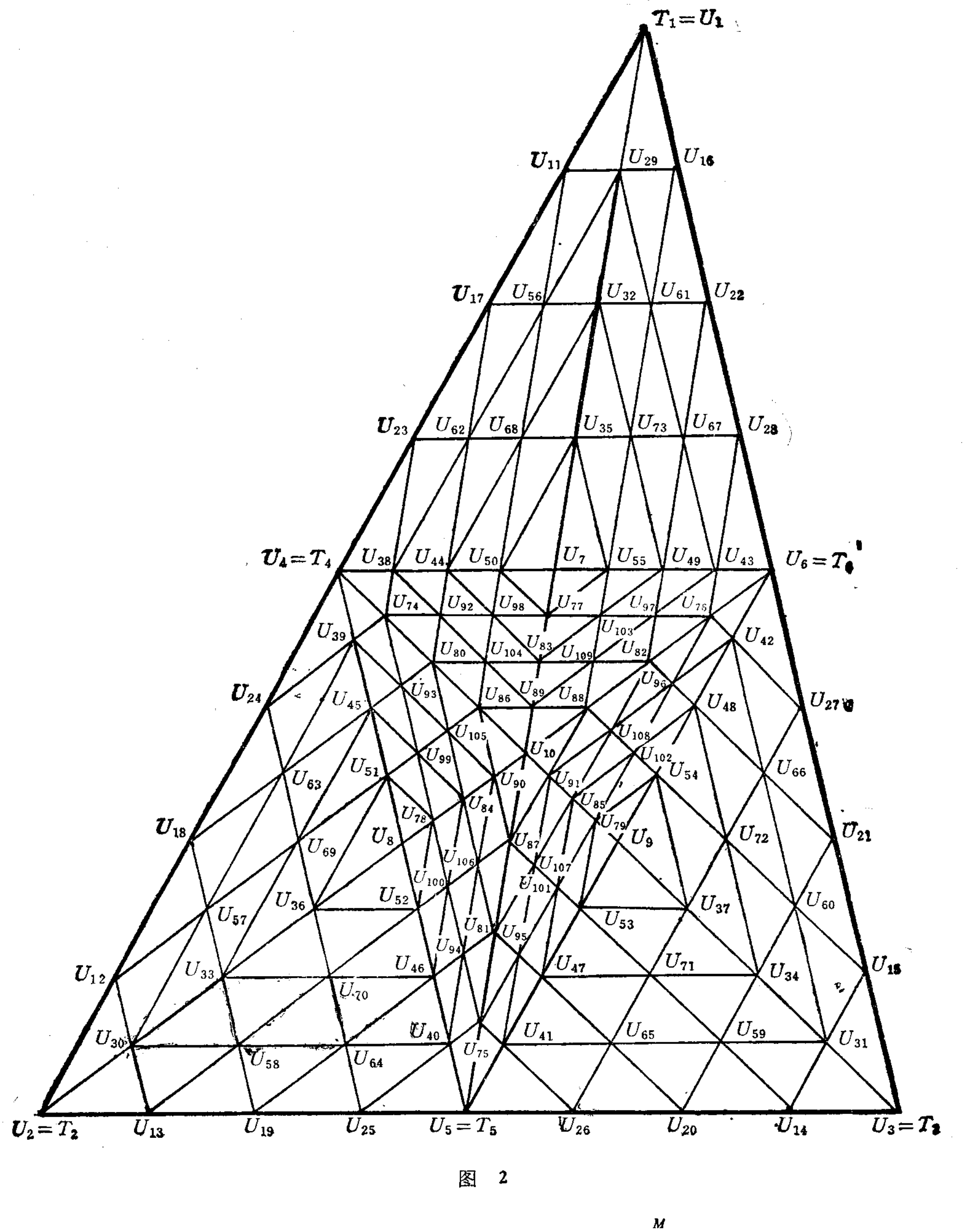

图 2

设 $T^{(k)}\in\Delta$ (其中 $T^{(1)}=T$), $k=1,2,\cdots,M$, 且 $\bigcap_{k=1}^{M}T^{(k)}=T_1$, $T_3^{(k)}=T_2^{(k+1)}$, $k=1,2,\cdots,M$. 当点 $T_1\notin\partial\Delta$ 时,规定 $T^{(M+1)}=T^{(1)}$. 这时由于M为奇数, 从而可求得

$$\begin{cases} a_{61}^{(M)}=\left[R^{(M)}+Q^{(M)}+\sum_{i=1}^{M-1}\left(\prod_{j=i+1}^{M}\eta^{(j)}\right)(R^{(i)}+Q^{(i)})\right]\Big/2, \\ a_{56}^{(k)}=a_{61}^{(k)}-Q^{(k)},\ k=M,M-1,\cdots,1, \\ a_{61}^{(k-1)}=(a_{56}^{(k)}-R^{(k)})/\eta^{(k)},\ k=M,\ M-1,\cdots,2, \end{cases} \tag{2.2.8}$$

其中

$$\begin{cases} R^{(k)} = [4(\eta_1^{(k)} + 2\eta_4^{(k)})a_{11}^{(k)} + 2\eta_3^{(k)}(4a_{23}^{(k)} - a_4^{(k)}) + (3\eta_2^{(k)} - \eta_1^{(k)} - 2\eta_4^{(k)})a_1]/9, \\ Q^{(k)} = (a_{22}^{(k)} - a_{17}^{(k)})/2, \ k = 1,2,\cdots,M, \end{cases}$$

而 $\eta^{(k)}$, $\eta_i^{(k)}$, $i = 1,2,3,4$, 可由(2.2.2)式求得。

当点 $T_1 \in \partial\Delta$ 时，取 $a_{56}^{(1)} = c_{56}^{(1)}$，从而有

$$\begin{cases} a_{61}^{(k)} = a_{56}^{(k)} + Q^{(k)}, \ k = 1,2,3,\cdots,M, \\ a_{56}^{(k)} = R^{(k)} + \eta^{(k)} \cdot a_{61}^{(k-1)}, \ k = 2, 3,\cdots,M. \end{cases} \tag{2.2.9}$$

采用同样的方法可求出 a_i, $i = 57,58,59,60$ 各点。取

$$a_i = c_i, \ i = 35,36,37,80,81,82. \tag{2.2.10}$$

最后由光滑性条件，得

$$a_i = (a_j + a_{j+1})/2, \ j = 2i - 9 + 6 \cdot \mathrm{int}\left(\frac{32}{i}\right), \ i = 32,33,34, \tag{2.2.11}$$

$$\begin{cases} a_i = a_{35} + (-1)^i(a_{62} - a_{67})/4, \ i = 68,73, \\ a_j = a_l + (-1)^{j+1}(a_k - a_{k+1})/4, \ l = 36 + \mathrm{int}(j/71), \ k = 2l - 9, \\ \qquad j = 69,70,71,72, \end{cases} \tag{2.2.12}$$

$$a_i = \frac{3}{4}a_j - \frac{1}{12}a_{i-75} + \frac{1}{3}a_{i-30}, \ j = 80 + \mathrm{int}\left(\frac{i-92}{2}\right), \ i = 92,93,\cdots,97, \tag{2.2.13}$$

$$a_i = (3a_{i+48} + a_{i+18})/4, \ i = 44,45,\cdots,49, \tag{2.2.14}$$

$$\begin{cases} a_i = a_7 + (-1)^i(a_{44} - a_{49})/4, \ i = 50,55, \\ a_j = a_k + (-1)^j(a_{l+1} - a_l)/4, \ k = 8 + \mathrm{int}\left(\frac{j-51}{2}\right), \ l = 2k + 29, \\ \qquad j = 51,52,53,54. \end{cases} \tag{2.2.15}$$

$$a_i = (12 \cdot a_{i-6} + a_{i-48} - 4 \cdot a_{i-36})/9, \ i = 104,105,\cdots,109, \tag{2.2.16}$$

$$a_i = (4a_{i-48} - a_{i-30})/3, \ i = 98,99,\cdots,103, \tag{2.2.17}$$

$$a_i = [2(a_{2i-68} + a_{2i-67}) - a_{i-6}]/3, \ i = 86,87,88, \tag{2.2.18}$$

$$a_{77} = (a_{103} + a_{98})/2, \ a_i = (a_{2i-57} + a_{2i-56})/2, \ i = 78,79, \tag{2.2.19}$$

$$a_{83} = (a_{109} + a_{104})/2, \ a_i = (a_{2i-63} + a_{2i-62})/2, \ i = 84,85, \tag{2.2.20}$$

$$a_{89} = (a_{88} + a_{86})/2, \ a_i = (a_{i-4} + a_{i-3})/2, \ i = 90,91, \tag{2.2.21}$$

$$a_{10} = (a_{86} + a_{87} + a_{88})/3. \tag{2.2.22}$$

至此，由上面各式确定的点集 $\{a_i\}_{i=1}^{109}$ 满足所要求的光滑性条件。对 Δ 中的每个三角形 T 都求出相应的 Bézier 点集 $\{a_i^{(T)}\}_{i=1}^{109}$，则由所有的 Bézier 点确定的曲面是整体 VC^1，局部 C^2 光滑的插值曲面。

三、附 记

在本文确定插值曲面的 Bézier 点的方法中，(2.1.5)式的 q_j, $j = 4,5,6$; (2.2.4) 式的 t_1 或 t_2; (2.2.6) 式的 Q_T 点作为参量用于调整曲面某一部分的形状。

显然本文给出的插值曲面构造方法是局部的.

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