向量值三重分叉连分式插值的算法 *1)

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AN ALGORITHM FOR VECTOR VALUED INTERPOLANTS BY TRIPLE BRANCHED CONTINUED FRACTIONS

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Abstract

A kind of triple branched continued fractions is defined by making use of Samelson inverse and Thiele-type reciprocal difference quotient. A recursive algorithm is constructed and a numerical example is given.

§1. 引　　言

设 $X_l=\{x_i|i=0, 1, \cdots, l\}\subset R$, $Y_m=\{y_j|j=0, 1, \cdots, m\}\subset R$, $Z_n=\{z_k|k=0, 1, \cdots, n\}\subset R$, $\Pi^{l,m,n}=X_l\times Y_m\times Z_n\subset R^3$ 为立体点阵. 在每个点$(x_i, y_j, z_k)\in\Pi^{l,m,n}$处给定 d 维插值向量 $\boldsymbol{v}_{ijk}$, 它们构成的向量集称为立体向量网, 并用 $V^{l,m,n}$ 记之.

任一 d 维复向量 $\boldsymbol{v}\in\mathbf{C}^d$ 的 Samelson 逆定义为

$$\boldsymbol{v}^{-1}=\boldsymbol{v}^*/|\boldsymbol{v}|^2, \tag{1.1}$$

其中 $\boldsymbol{v}^*$是 $\boldsymbol{v}$ 的复共轭, $|\boldsymbol{v}|$ 表示 $\boldsymbol{v}$ 的范数.

定义 1. 若三元向量值有理函数 $\boldsymbol{R}(x, y, z)$ 可表示为

$$\boldsymbol{R}(x, y, z)=\boldsymbol{a}_0(y, z)+\frac{x-x_0|}{|\boldsymbol{a}_1(y, z)}+\cdots+\frac{x-x_{l-1}|}{|\boldsymbol{a}_l(y, z)}, \tag{1.2}$$

其中

$$\boldsymbol{a}_p(y, z)=\boldsymbol{b}_{p_0}(z)+\frac{y-y_0|}{|\boldsymbol{b}_{p_1}(z)}+\cdots+\frac{y-y_{m-1}|}{|\boldsymbol{b}_{pm}(z)}, \ p=0, 1, \cdots, l \tag{1.3}$$

而

$$\boldsymbol{b}_{pq}(z)=\boldsymbol{c}_{pq0}+\frac{z-z_0|}{|\boldsymbol{c}_{pq1}}+\cdots+\frac{z-z_{n-1}|}{|\boldsymbol{c}_{pqn}}, \ q=0, 1, \cdots, m, \tag{1.4}$$

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则称 $\boldsymbol{R}(x, y, z)$ 为向量值三重分叉连分式，$\boldsymbol{a}_p(y, z)(p=0, 1, \cdots, l)$ 称作是 $\boldsymbol{R}(x, y, z)$ 的第二次向量分叉，$\boldsymbol{b}_{pq}(z)(q=0, 1, \cdots, m)$ 称作是 $\boldsymbol{a}_p(y, z)$ 的第二次向量分叉或 $\boldsymbol{R}(x, y, z)$ 的第三次向量分叉.

§2. 分 层 算 法

在给定立体点阵 $\Pi^{l, m, n}$ 和立体向量网 $V^{l, m, n}$ 的前提下，如何计算向量系数 $\boldsymbol{c}_{pqr}(p=0, 1, \cdots, l;\ q=0, 1, \cdots, m;\ r=0, 1, \cdots, n)$ 使得三重分叉连分式 $\boldsymbol{R}(x, y, z)$ 在立体点阵 $\Pi^{l, m, n}$ 上插值立体向量网 $V^{l, m, n}$，即

$$\boldsymbol{R}(x_i, y_j, z_k)=\boldsymbol{v}_{ijk};\ i=0, 1, \cdots, l;\ j=0, 1, \cdots, m;\ k=0, 1, \cdots, n, \tag{2.1}$$

是一个十分有趣的问题. 下面我们给出一种分层式的递推算法：

第一步. 初始化

对 $i=0, 1, \cdots, l;\ j=0, 1, \cdots, m;\ k=0, 1, \cdots, n$，令

$$\boldsymbol{D}_{i, j, k}^{(0, 0, 0)}=\boldsymbol{v}_{ijk}. \tag{2.2}$$

第二步. x 方向 Samelson 逆变换

对 $j=0, 1, \cdots, m;\ k=0, 1, \cdots, n;\ p=1, 2, \cdots, l;\ i=p, p+1, \cdots, l$，令

$$\boldsymbol{D}_{i, j, k}^{(p, 0, 0)}=\frac{x_i-x_{p-1}}{\boldsymbol{D}_{i, j, k}^{(p-1, 0, 0)}-\boldsymbol{D}_{p-1, j, k}^{(p-1, 0, 0)}}. \tag{2.3}$$

第三步. y 方向 Samelson 逆变换

对 $i=0, 1, \cdots, l;\ k=0, 1, \cdots, n;\ q=1, 2, \cdots, m;\ j=q, q+1, \cdots, m$，令

$$\boldsymbol{D}_{i, j, k}^{(i, q, 0)}=\frac{y_j-y_{q-1}}{\boldsymbol{D}_{i, j, k}^{(i, q-1, 0)}-\boldsymbol{D}_{i, q-1, k}^{(i, q-1, 0)}}. \tag{2.4}$$

第四步. z 方向 Samelson 逆变换

对 $i=0, 1, \cdots, l;\ j=0, 1, \cdots, m;\ r=1, 2, \cdots, n,\ k=r, r+1, \cdots, n$，令

$$\boldsymbol{D}_{i, j, k}^{(i, j, r)}=\frac{z_k-z_{r-1}}{\boldsymbol{D}_{i, j, k}^{(i, j, r-1)}-\boldsymbol{D}_{i, j, r-1}^{(i, j, r-1)}}. \tag{2.5}$$

定理 1. 令

$$\boldsymbol{c}_{pqr}=\boldsymbol{D}_{p, q, r}^{(p, q, r)},\ p=0, 1, \cdots, l;\ q=0, 1, \cdots, m;\ r=0, 1, \cdots, n,$$

则由(1.2)—(1.4)确定的向量值三重分叉连分式 $\boldsymbol{R}(x, y, z)$ 在立体点阵 $\Pi^{l, m, n}$ 上插值立体向量网 $V^{l, m, n}$.

证明：由(1.4)和(2.5)知

$$\boldsymbol{b}_{pq}(z_k)=\boldsymbol{D}_{p, q, 0}^{(p, q, 0)}+\frac{z_k-z_0\,|}{|\,\boldsymbol{D}_{p, q, 1}^{(p, q, 1)}}+\cdots+\frac{z_k-z_{k-1}\,|}{|\,\boldsymbol{D}_{p, q, k}^{(p, q, k)}}=\boldsymbol{D}_{p, q, k}^{(p, q, 0)}$$

由此及(1.3)和(2.4)得

$$\begin{aligned}\boldsymbol{a}_p(y_j, z_k)&=\boldsymbol{b}_{p0}(z_k)+\frac{y_j-y_0\,|}{|\,\boldsymbol{b}_{p1}(z_k)}+\cdots+\frac{y_j-y_{j-1}\,|}{|\,\boldsymbol{b}_{pj}(z_k)}\\&=\boldsymbol{D}_{p, 0, k}^{(p, 0, 0)}+\frac{y_j-y_0\,|}{|\,\boldsymbol{D}_{p, 1, k}^{(p, 1, 0)}}+\cdots+\frac{y_j-y_{j-1}\,|}{|\,\boldsymbol{D}_{p, j, k}^{(p, j, 0)}}=\boldsymbol{D}_{p, j, k}^{(p, 0, 0)}\end{aligned}$$

最后由(1.2)、(2.3)和(2.2)有

$$\begin{aligned}\boldsymbol{R}(x_i,\ y_j,\ z_k)&=\boldsymbol{a}_0(y_j,\ z_k)+\frac{x_i-x_0\ |}{|\ \boldsymbol{a}_1(y_j,\ z_k)}+\cdots+\frac{x_i-x_{i-1}\ |}{|\ \boldsymbol{a}_i(y_j,\ z_k)}\\&=\boldsymbol{D}_{0,\ j,\ k}^{(0,\ 0,\ 0)}+\frac{x_i-x_0\ |}{|\ \boldsymbol{D}_{1,\ j,\ k}^{(1,\ 0,\ 0)}}+\cdots+\frac{x_i-x_{i-1}\ |}{|\ \boldsymbol{D}_{i,\ j,\ k}^{(i,\ 0,\ 0)}}\\&=\boldsymbol{D}_{i,\ j,\ k}^{(0,\ 0,\ 0)}=\boldsymbol{v}_{ijk}.\end{aligned}$$

定理1证毕.

由(2.3)—(2.5)不难得到下述定理:

定理2. 设给定$\Pi^{l,\ m,\ n}$和$V^{l,\ m,\ n}$，则确定形如(1.2)—(1.4)的向量值三重分叉连分式须求$(l+1)(m+1)(n+1)(l+m+n)\ /2$次Samelson逆变换.

§3. 数　值　例

设$l=m=n=1$，$d=3$时的$\Pi^{1,\ 1,\ 1}$和$V^{1,\ 1,\ 1}$由表1给出.

表1

$x_0=1$	$\boldsymbol{v}_{0jk}$	$y_0=1$	$y_1=2$
	$z_0=1$	(0, 0, 0)	(0, 1, 0)
	$z_1=2$	(0, 0, 1)	(1, 0, 1)
$x_1=2$	$\boldsymbol{v}_{1jk}$	$y_0=1$	$y_1=2$
	$z_0=1$	(1, 0, 0)	(0, 1, 1)
	$z_1=2$	(0, 0, 2)	$\left(1,\ 0,\ \frac{4}{3}\right)$

1) 初始化

$$\boldsymbol{D}_{0,0,0}^{(0,0,0)}=(0,\ 0,\ 0),\ \boldsymbol{D}_{0,1,0}^{(0,0,0)}=(0,\ 1,\ 0),\ \boldsymbol{D}_{0,0,1}^{(0,0,0)}=(0,\ 0,\ 1),$$
$$\boldsymbol{D}_{0,1,1}^{(0,0,0)}=(1,\ 0,\ 1),\ \boldsymbol{D}_{1,0,0}^{(0,0,0)}=(1,\ 0,\ 0),\ \boldsymbol{D}_{1,1,0}^{(0,0,0)}=(0,\ 1,\ 1),$$
$$\boldsymbol{D}_{1,0,1}^{(0,0,0)}=(0,\ 0,\ 2),\ \boldsymbol{D}_{1,1,1}^{(0,0,0)}=\left(1,\ 0,\ \frac{4}{3}\right).$$

2) x方向递推

$$\boldsymbol{D}_{0,0,0}^{(0,0,0)}=(0,\ 0,\ 0),\ \boldsymbol{D}_{0,1,0}^{(0,0,0)}=(0,\ 1,\ 0),\ \boldsymbol{D}_{0,0,1}^{(0,0,0)}=(0,\ 0,\ 1),\ \boldsymbol{D}_{0,1,1}^{(0,0,0)}=(1,\ 0,\ 1),$$
$$\boldsymbol{D}_{1,0,0}^{(1,0,0)}=(1,\ 0,\ 0),\ \boldsymbol{D}_{1,1,0}^{(1,0,0)}=(0,\ 0,\ 1),\ \boldsymbol{D}_{1,0,1}^{(1,0,0)}=(0,\ 0,\ 1),\ \boldsymbol{D}_{1,1,1}^{(1,0,0)}=(0,\ 0,\ 3).$$

3) y方向递推

$$\boldsymbol{D}_{0,0,0}^{(0,0,0)}=(0,\ 0,\ 0),\ \boldsymbol{D}_{0,1,0}^{(0,1,0)}=(0,\ 1,\ 0),\ \boldsymbol{D}_{0,0,1}^{(0,0,0)}=(0,\ 0,\ 1),\ \boldsymbol{D}_{0,1,1}^{(0,1,0)}=(1,\ 0,\ 0),$$

$$\boldsymbol{D}_{1,0,0}^{(1,0,0)}=(1,\ 0,\ 0),\ \boldsymbol{D}_{1,1,0}^{(1,1,0)}=\left(-\frac{1}{2},\ 0,\ \frac{1}{2}\right),\ \boldsymbol{D}_{1,0,1}^{(1,0,0)}=(0,\ 0,\ 1),\ \boldsymbol{D}_{1,1,1}^{(1,1,0)}=\left(0,\ 0,\ \frac{1}{2}\right).$$

4) z方向递推

$$\boldsymbol{D}_{0,0,0}^{(0,0,0)}=(0,\ 0,\ 0),\ \boldsymbol{D}_{0,1,0}^{(0,1,0)}=(0,\ 1,\ 0),\ \boldsymbol{D}_{0,0,1}^{(0,0,1)}=(0,\ 0,\ 1),\ \boldsymbol{D}_{0,1,1}^{(0,1,1)}=\left(\frac{1}{2},\ -\frac{1}{2},\ 0\right),$$

$D_{1,0,0}^{(1,0,0)}=(1,0,0)$，$D_{1,1,0}^{(1,1,0)}=\left(-\frac{1}{2},0,\frac{1}{2}\right)$，$D_{1,0,1}^{(1,0,1)}=\left(-\frac{1}{2},0,\frac{1}{2}\right)$，$D_{1,1,1}^{(1,1,1)}=(2,0,0)$，

于是

$$a_0(y,z)=(0,0,0)+\frac{z-1}{(0,0,1)}+\frac{y-1}{(0,1,0)+\cfrac{z-1}{\left(\frac{1}{2},-\frac{1}{2},0\right)}}$$

$$=\frac{((y-1)(z-1),(y-1)(2-z),(z-1)(2z^2-6z+5))}{2z^2-6z+5},$$

$$a_1(y,z)=(1,0,0)+\frac{z-1}{\left(-\frac{1}{2},0,\frac{1}{2}\right)}+\frac{y-1}{\left(-\frac{1}{2},0,\frac{1}{2}\right)+\cfrac{z-1}{(2,0,0)}}$$

$$=\frac{((2-z)(z^2-4z-2y+7),0,2(y-1)+(z-1)(z^2-4z+5))}{z^2-4z+5},$$

从而

$$R(x,y,z)=\frac{((y-1)(z-1),(y-1)(2-z),(z-1)(2z^2-6z+5))}{2z^2-6z+5}$$
$$+\frac{(x-1)((2-z)(z^2-4z-2y+7),0,2(y-1)+(z-1)(z^2-4z+5))}{(z^2-4z+5)(2z^2-6z+5)+4(1-y)(z^2-5z-y+6)}$$

容易验证 $R(x_i,y_j,z_k)=v_{ijk}$，$i,j,k=0,1$.

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