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A GREY SYSTEM MODEL FOR PREDICTING TREND CHANGE OF URBAN WASTE WATER LOAD

Xia Jun
Hu Baoqing
Institute of Hydrology and Water Environmental Science,
Wuhan University of Hydraulic & Electric Engineering, China

The present paper introduces a Grey System Prediction Model (GSPM) by incorporating concepts of grey numbers within a grey differential equation framework as a means for predicting trends in urban waste water load under uncertainty. A new operation law of grey numbers was developed that overcomes some of disadvantages of present interval operations. In the light of these operation principles, the original grey differential equation can be reduced to two equivalent sub-models. One is whitening or averaged value system equation that is equivalent to the GM(1,1) prediction model in grey system theory. Another is a grey radius equation, expressing system uncertainty. The GSPM approach improves upon previous prediction models by allowing uncertainty information to be directly communicated into the grey series modeling and prediction. The method also does not lead to more complicated models, and is applicable to practical problems. The GSPM approach has been applied to trend change prediction of waste water load in Luo Yang City, He Nan Province, China. Preliminary examinations indicate that reasonable solutions can be generated. Besides trend change information, the grey system model could provide the degree of uncertainty of model parameters and output variables.

INTRODUCTION

In China, urban development is facing major problems in managing its water resources. It has been observed that water may very soon become a limiting factor in numerous development efforts. On the other hand, growing water quality problems are inviting innovative management approaches, such as those that will insure joint consideration of economic, ecological and social issues.

As far as urban water environmental management is concerned, we must understand trends in a city's economic development and waste water loading for the future. However, these issues involve much uncertainty and variable factors. One potential approach for mitigating this problem is through the introduction of the concepts of grey systems and grey prediction models (Deng, 1982, 1988; Xia, 1990, 1992, 1995; Huang *et al*. 1990).

In the present paper, a Grey System Prediction Model (GSPM) applied to trend prediction of urban waste water load over a long term is developed. The case study of the Luo Yang City waste water load located along the Yellow River in He Nan Province, China, is given as a preliminary example of the application of this method.

GREY SYSTEM AND GREY INTERVAL OPERATIONS

A grey system is defined as a system containing unknown information presented by grey numbers and grey variables (Deng, 1982, 1988; Xia, 1990, 1992, 1995). The concept of a grey system is shown in the Figure 1.

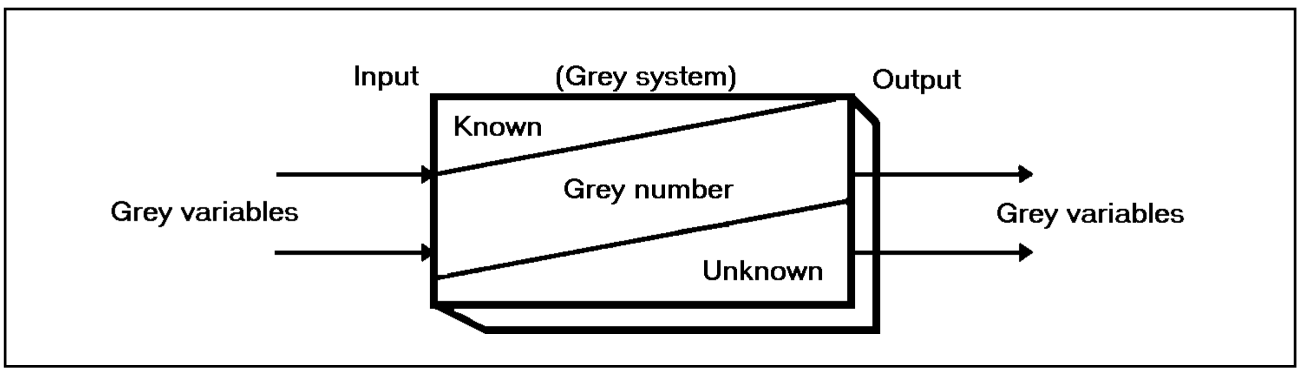

Figure 1. The concept of a grey system.

The grey number can be defined as a number with non-precise information. For example, when a model uses an observed data set on urban waste water load, there will be a numerical interval accompanying it. This numerical interval will contain the accuracy and the other sources of uncertainty that are associated with the observed values in the data set. As one of grey system approaches, this numerical interval could be taken as a special grey number, $X_g(\otimes)$, with bound values X_d and X_u , and a whitening value, X_o:

$$X_g(\otimes) = [\, X_d, X_o , X_u \,]$$

where X_d is the lower limit and X_u is upper limit.

To make it easy for grey number operation, the whitening value could be defined as a center point, i.e.,$X_o = (X_d + X_u)/2$. The uncertainty of the numerical interval could be expressed by a grey radius, $X = (X_u - X_d) / 2$. Therefore, the grey number,$X_g(\otimes)$, is also formulated in a comprehensive way as

$$X_g(\otimes) = < X_o , \delta X > \qquad (1)$$

At present, there are several methods to make numerical interval operation and functional mapping such as the Hop and Jump approach or interval calculation law (Moore,1966; Viertl,1990). However, these methods do not directly obtain the input-output result when input is a numerical interval variable. Moreover, existing interval operation laws have some inadequate properties. For example, if two intervals are exactly equal to each other, their difference is not yet equal to zero . These operation laws do not match with reality on water environmental observations. To overcome these difficulties, a grey number operation law has been developed. Let two grey numbers be $<X_o, \delta X>$ and $<Y_o, \delta Y>$. New basic operation laws could be derived as follows:

$$< X_o, \delta X > + < Y_o, \delta Y > = < X_o + Y_o, \delta X + \delta Y > \tag{2}$$

$$< X_o, \delta X > - < Y_o, \delta Y > = < X_o - Y_o, |\delta X - \delta Y| > \tag{3}$$

$$< X_o, \delta X > \times < Y_o, \delta Y > = < X_o Y_o, |X_o|\delta Y + \delta X|Y_o| + \delta X \delta Y > \tag{4}$$

$$< X_o, \delta X > \div < Y_o, \delta Y > = < X_o, \delta X > \times 1/ < Y_o, \delta Y >, 0 \notin < Y_o, \delta Y > \tag{5}$$

where:

$$1 / < Y_o, \delta Y > = < Y_o/(Y_o^2 - \delta Y^2), \delta Y / (Y_o^2 - \delta Y^2) >, 0 \notin < Y_o, \delta Y >$$

For instance, suppose two grey numbers $\otimes_1 = [9, 15] = <12, 3>$, $\otimes_2 = [3, 7] = <5, 2>$, then their basic operations could be shown as

$$<12, 3> + <5, 2> = <17, 5> \quad ===> \quad [12, 22]$$

$$<12, 3> - <5, 2> = <7, 1> \quad ===> \quad [6, 8]$$

$$<12, 3> \times <5, 2> = <60, 45> \quad ===> \quad [15, 105]$$

$$<12, 3> \div <5, 2> = <2.4, 1.5> \quad ===> \quad [2.857, 1.142]$$

$$<5, 2> - <5, 2> = <0, 0> \quad ===> \quad [0, 0]$$

It should be mentioned that these operational laws were proven from basic closure properties. Moreover, the operation laws could be easily extended to grey system modeling, interval matrix formulation, interval linear programming / nonlinear programming and dynamic differential equations.

FORMULATION OF A GREY SYSTEM PREDICTION MODEL

The grey system approach creates a model of the data from a minimum and maximum value, which can also be used in a prediction model to estimate and extrapolate a series of trend values from a series of observations.

Let us consider a data set of grey series $X(\otimes)$ with n elements corresponding to n time periods, $X^{(0)} = \{X^{(0)}(i); i=1,2,...,n\}$, where $X^{(0)}(i)$ is the ith element corresponding to period i. The problem under consideration, for the Grey System Prediction Model (GSPM), is the prediction of $X^{(0)}(i)$ for $i >$ n when standard statistical approaches are not applicable. Uncertainty information could be characterized by introducing a grey interval variable, $\delta X^{(1)}$, and a grey model parameter set. A GSPM with optimal parameters is introduced which can effectively approximate the uncertainty existing in $X^{(0)}$.

Accumulated Generating Operation and Its Inverse

To find a linkage of a discrete grey series with a continuous differential equation, the concepts of accumulated generating operation (AGO) and it's inverse (IAGO) are required for the GSPM. The rth AGO of X is defined as

$$X^{(r)}(k) = \text{AGO}\,(X^{(r)}(k)) = X^{(r)}(k\text{-}1) + X^{(r\text{-}1)}(k), \quad k \notin [1, n] \tag{6}$$

where $r = 1,2,...,m$; The rth IAGO of X is given by

$$a^{(r\text{-}1)}(X(k)) = \text{IAGO}(X^{(r)}(k)) = X^{(r)}(k) - X^{(r)}(k\text{-}1) \tag{7}$$

The purpose of the AGO is to form a model that can provide more available information about trend change, data filtering and grey prediction. In application, we usually use the first order accumulating generation.

Grey System Prediction Model -- GSPM(1,1) Approach

If a generating series based on the AGO satisfies the condition of discrete smooth (Deng, 1988), the dynamic characteristics of a discrete grey series could be represented in terms of a grey differential equation. For the prediction of urban waste load, the first order grey differential model could be written as

$$\frac{dX^{(1)}(\otimes)}{dt} + a(\otimes)X^{(1)}(\otimes) = u(\otimes) \tag{8}$$

where: $a(\otimes)$ and $u(\otimes)$ are two grey parameters, $X^{(1)}(\otimes)$ is grey system variable.

To obtain two operational models linked directly with a whitening prediction equation, we rewrite equation (8) as follows:

$$\frac{d < X^{(1)}, \delta X^{(1)} >}{dt} + a, \delta a >< X^{(1)}, \delta X^{(1)} > = < u, \delta u > \tag{9}$$

where $(X^{(1)}, a, u)$ is a set of whitening values of the grey system model and $(\delta X^{(1)}, \delta a, \delta u)$ is a set of grey radius.

Considering the operation law of grey numbers from equation (2) to equation (5), the grey system model (9) could be reduced as two systems equivalent equations. One is its whitening equation relative to a whitening value X_o, given by

$$\frac{dX^{(1)}}{dt} + aX^{(1)} = u \tag{10}$$

Another is its grey interval radius equation expressing system uncertainty, i.e.,

$$\frac{d\delta X^{(1)}}{dt} + (|a| + \delta a)\delta X^{(1)} = \delta u - \delta a\left|X^{(1)}\right| \tag{11}$$

The solution of these grey prediction models can be found as

$$\begin{aligned} X^{(1)}(t+1 &= \left(X^{(0)}\ 1 - \frac{u}{a} \right) e^{-at} + \frac{u}{a} \\ X^{(0)}\ t+1 &= X^{(1)}\ t+1 - X^{(1)}\ t \end{aligned} \tag{12}$$

and

$$\delta X^{(1)}(t+1) = e^{-Et}\left[\frac{F}{E}\left(e^{Et}-1\right)+\frac{G}{E-a}\left(e^{(E-a)t}-1\right)+\delta XH(0)(1)\right]$$
$$\delta X^{(0)}(t1) = \delta X^{(1)}(t+1) - \delta X^{(1)}(t) \tag{13}$$

where: $E = |\,a\,| + \delta a$, $F = \delta u - \delta a \times u/a$, $G = -a(X^{(0)}(1) - u/a)$.

Equation (10) is a whitening prediction model that is also called the GM(1,1) model in the present application of grey prediction methods (Deng, 1988). Thus, model (9) gives a link of the present GM(1,1) with a grey interval model, equation (8). It means that equation (8) provides a comprehensive representation of a grey system. On the other hand, their contribution can be easily analyzed by two sub-models (10) and (11). Information on trend change is provided by the whitening equation (10).

Uncertainty can be analyzed by the grey radius model (11). Define uncertain degree of the ith grey parameter as a ratio of grey interval width to its whitening value, then system uncertainty could be approximately measured by their averaged values, i.e.,

$$U_i\% = \left(\frac{2\delta\theta_i}{|\theta_i|}\right)\times 100\%$$

$$U\% = \frac{1}{n}\sum_{i=1}^{n} U_i\%$$

Moreover, if the model parameter set, $(a(\otimes), u(\otimes))$, is variable or controllable in the objective year, such a GSPM is called a grey prediction model with time-variant parameters.

PARAMETER ESTIMATION OF THE GSPM MODEL

Because of the properties of AGO and IAGO, the concept of grey derivation is introduced as

$$\frac{dX^{(1)}}{dt} \approx a^{(1)}\left(X(k+1)\right) \tag{14}$$

Define the scenario value $X^{(1)}$ in equation (10) as $G[X^{(1)}]$. On the basis of the elements of multiple information utilization, $G[X^{(1)}]$ is taken as

$$G\left[X^{1}(k+1)\right] = \lambda X^{(1)}(k+1) + (1-\lambda)X^{(1)}(k) \tag{15}$$

where λ is an optimal weight coefficient, $\lambda \in [0, 1]$. In many cases, λ is approximately taken as 0.5. For a trend change prediction, optimal weight can be determined by data filter principles:

$$f(\lambda) = \Sigma\left[X^{(0)}(k+1) - X^{(0)}(k+1/k)\right] = 0 \tag{16}$$

So, we can convert equation (10) to

$$\alpha^{(1)}\left(X(k+1)\right)+aG\left[X^{(1)}\right]=u,k\in\left[1,n-1\right] \tag{17}$$

Let:

$$\theta=\left[a,u\right]T$$

$$Y=\left[X^{(0)}(2),X^{(0)}(3),\ldots,X^{(0)}(n)\right]^{T},$$

and

$$A=\begin{bmatrix}-G\left[X^{(1)}(2)\right] & 1\\ -G\left[X^{(1)}(3)\right] & 1\\ & \vdots\\ -G\left[X^{(1)}(n)\right] & 1\end{bmatrix}$$

The equation (17) could be written as a matrix formulation ,**Y= A** θ. Thus, parameter vector can be obtained by applying the least-squares criterion, given by

$$\theta=(\mathbf{A}^{T}\mathbf{A})^{-1}\mathbf{A}^{T}\mathbf{Y} \tag{18}$$

Hence, $X^{(1)}$ (k+1) can be obtained by solving (12). Thus, from the definition of the IAGO, $X^{(0)}$ (k+1) can be obtained from $X^{(1)}$ (k+1). Moreover, when $k > n$-1, the obtained $X^{(0)}$ (k+1) provides a trend prediction of the X value in a future period k+1.

In the same way, we could evaluate the parameter of grey interval model in equation (11). Evidently, the identification of grey interval radius, (δa, δu), is not only a function of output grey interval $\delta X^{(0)}(t)$, but also has a relation to the whitening equation (10).

APPLICATION

An example of the application of the method described in this paper is the trend change prediction of urban waste load for Luo Yang City in He Nan Province, China. The Luo Yang City is located in a tributary of the Yellow River (see Figure 2). In 1993 the population was about 791,900. It is one of the seven largest ancient capitals in China. There are many scenic and historic sites in the city.

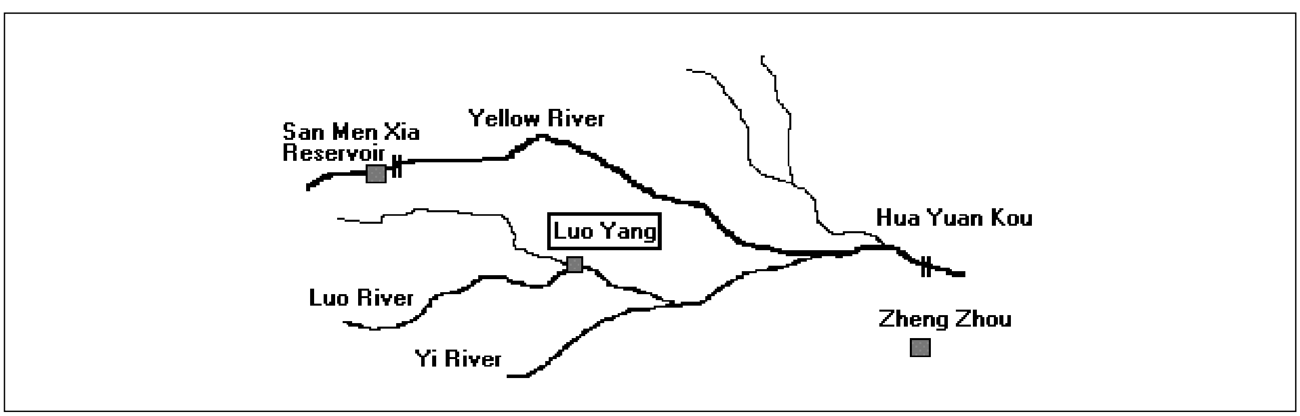

Figure 2. Location of Luo Yang City in middle stream of Yellow River.

Luo Yang is also a modern industrial city with petroleum, chemical industry, metallurgy and building materials. Along with the city's economic development, the problem of waste water discharge has become serious. The city's waste water mainly results from industrial water and domestic waste water generated in residential areas. Industrial activities contribute significantly to water pollution. The nature of pollutants from an industrial site is a function of the type of industry. By investigation, it was found that most of the waste water materials come from organic wastes, oil and other petroleum products, and heavy metals. The population density also affects waste production because more people generate more waste. The waste water generated in residential areas depends on the different standards of living.

To control water pollution and protect the city's water resources, one of the very important issues is to predict the trend change of waste load. Data sets applied to this purpose were collected based on recent year's investigation and water environmental monitoring. They included population (P) and total industrial output value (TIOV) from 1985 to 1993. However, the data set of waste water load is quite limited. Only five years of data are available from 1985 to 1993. They are industrial waste load (IWL), living waste load (LWL) and chemical oxygen demand (COD). Owing to data limitations and problems of non-precise data, the Grey System Prediction Model (GSPM) was used.

The main purposes of this investigation are twofold. One is to predict a scenario value of population and total industrial output value in the 2000 objective year. The second is to predict the trend change of waste water load where the development of Luo Yang and environmental planning in the long-term were approximately considered in the variable parameters a(⊗) and u(⊗). The uncertainty of predicting waste load will be analyzed by grey parameters or interval variant ratio of load series. The results of the population (P) modeling and the total industrial output value (TIOV) series modeling, based on the GSPM approach, are shown in Figures 3 and 4. The results of the grey

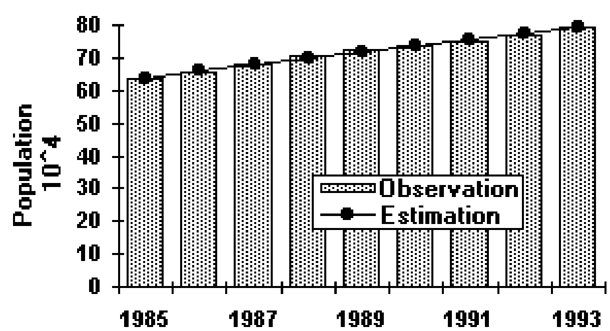

Figure 3. Population series simulation by the GSPM approach for Luo Yang City.

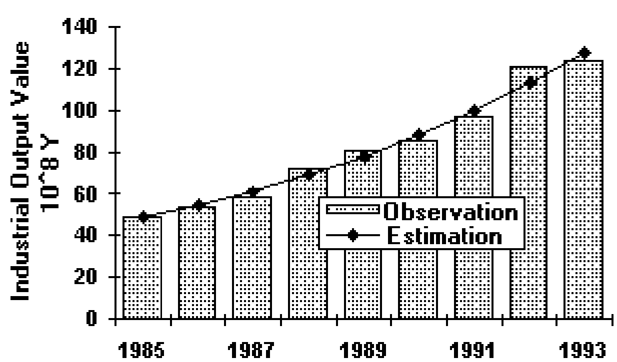

Figure 4. Total industrial output values simulation by the GSPM approach for Luo Yang City.

model identification and uncertainty analysis are listed in Table 1. Trend changes and grey predictions with high, middle and low values on both the impact factors and waste water load for Luo Yang City are summarized in Table 2.

Table 1. The Identification of GSPM Model for Luo Yang City from 1985 to 1993

Items	**Grey Model Parameters**		**Measure of Uncertainty**		
	$a(\otimes) = < a_0, \delta a >$	$u(\otimes) = < u_0, \delta u >$	$U_{a\%}$	$U_{u\%}$	%
Population	<-0.02561 , 0.0003 >	<63.8514 , 0.38 >	2.3	1.2	1.8
Total Industrial Output Value	<-0.12208 , 0.0041 >	<45.1845 , 4.56 >	6.7	20.2	13.4
Industrial Waste Load	<-0.0511 , 0.0075 >	<11339.9100 , 1361.843 >	29.3	24.0	26.7
COD Load	<-0.0136 , 0.0023 >	< 8041.2510 , 144.05 >	33.8	3.6	18.7

Table 2. Trends in Waste Water Load for Luo Yang City Predicted by the GSPM

Items		**Years**			**%**
		1985	1990	2000	
Total Industrial Output Value (10^8 *Y*)	High Middle Low	48.34	86.16	254 241 228	10.94 9.41 7.88
City's Population (10^4)	Middle	63.62	73.87	94.98	2.63
Industrial Waste Load (10^4 m^3/a)	High Middle Low	8039	8515	18381 17979 17580	8.00 7.76 7.52
Living Waste Load (10^4 m^3/a)	High Middle Low	4005	5534	7949 7672 7394	3.69 3.32 2.94
COD Load (t/a)	High Middle Low	11871	14425	28956 28807 27711	7.22 7.16 6.75

The results and discussions are given as follows:

(a) Averaged growth rate of population in the Luo Yang City is about 2.5% during the period of 1985 to1993. The rate of increase for the total industrial output value is 12.21%. Therefore, economical development of Luo Yang City is rather fast in the last ten years. This is one of important reasons for increasing waste water load.

(b) Comparisons of calculating the value of the grey system model with observed values indicate that the GSPM could give quite good simulations for predicted variables in the calibration period (see Figure 3). Besides whitening values, moreover, the grey system model could give uncertainty information applied to grey system prediction. Identifications of grey parameters show that the degree of uncertainty for both the industrial waste load and the COD load is more significant that that of population prediction (see Table 1). Thus, these uncertainties need to be considered in the objective year's predictions.

(c) Using a time-variable GSPM model and referencing planning information of the Luo Yang City development in the year 2000, we predict trends in main impact factors and the city's waste water. By the year 2000, the total industrial output value of Luo Yang City will reach 241 billion yuan. Average rate of increase is about 9.41 %. The population in the urban area will increase to 949,800 in 2000. Therefore, it is certain that waste water load will increase in the future. The predictions also indicate that by the objective year 2000, industrial waste water load will increase from the present 8.5 million cubic meters per year to 17.9 million cubic meters per year. Domestic waste water will increase to 7.7 million cubic meters per year. The COD load will also reach 28,807 tons per year. Thus, water pollution will be one of the most urgent problems in sustainable development in urban regions.

To improve water environmental quality in the city, both engineering and non-engineering measures on controlling waste water discharge must be considered. Preliminary research shows that if we expect to improve significantly the water quality of Luo Yang City in 2000, the discharges of COD load need to be at least controlled to 4,947 tons per year. The cutting down amount of waste load is about 23,680 ton per year. Thus, it is necessary to build waste water treatment factories and carry out policy actions that focus on issues of water quality.

CONCLUDING REMARKS

(a) Along with economic development and increase of population, the discharge of waste water load will be one of the most important of environmental problems in urban areas. The investigation from Luo Yang City has shown that in the past ten years the average rate of increase of industrial waste water load reaches 1.16%. The growth rate of domestic waste water load in the city arrives at 3.97%. That of COD load is about 3.97%.

(b) A GSPM method was developed and applied to trend predictions of Luo Yang city's waste water load. The method allows uncertain information to be effectively communicated into the modeling and prediction process by introducing grey interval variable and grey parameters into dynamic differential equations. The solutions could be realized by two equivalent system equations. The method also does not lead to more complicated models, and is applicable to practical problems.

(c) The results of the case study for Luo Yang City indicate that in pace with urban economic development and population movement from rural areas to cities, the developing trends in waste water load will continue to increase for years. By the year 2000, the total industrial output value of Luo Yang City will reach 241 billion yuan. The population in the urban area will increase to 949,800. As a result, the industrial waste water load will increase to 17.9 million cubic meters per year. Domestic waste water will come up to 7.7 million cubic meters per year. The COD load will also reach 28,807 ton per year. On the other hand, grey system identifications show that the degree of uncertainty of waste load prediction is significantly larger than that of population and total industrial output value. Therefore, the risk to the urban water environment needs to be further analyzed.

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Professor Xia Jun
Faculty of Hydrology, Department of River Engineering
Wuhan University of Hydraulic and Electric Engineering
Wuhan
430072 China
e-mail: jxia@sun20.wuhee.edu.cn