The Effect of Wave Breaking on the Wave Energy Spectrum

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ABSTRACT

The effect of wave breaking on the wave energy spectral shape is examined. The Stokes wave-breaking criterion is first extended to random waves and a breaking wave model is established in which the elevation of breaking waves is expressed in terms of that of the original ideal waves, which are assumed to be stationary and Gaussian. Based on this model, a simple but approximate expression for the spectrum of breaking waves is derived and applied to the case in which a deep water unidirectional wave train enters a region of adverse current steady in time and uniformly distributed in depth.

1. Introduction

Many analytical forms of the wave energy spectrum have been proposed. However, all of these spectra are for specific conditions. For example, the Pierson–Moskowitz spectrum is for fully developed seas, the JONSWAP spectrum is for fetch-limited developing seas and the Wallops spectrum (Huang et al., 1981) is derived based on wave dynamics but does not include the possibility of wave breaking.

When conditions differ from those for which the spectra are intended, or as the waves move into regions where the conditions are changed, the wave spectra naturally undergo corresponding changes. This happens, for example, when the waves propagate from deep to shallow water, or when they encounter current (Huang et al., 1972). The usual method to calculate the wave spectrum in these circumstances is to use the classical energy conservation equation. It is known, however, that when the waves reach the surf zone or when they meet an adverse current, vigorous wave breaking may take place. It is therefore necessary to devise a method to calculate the spectrum of the waves which takes into account the effect of wave breaking.

To account for the effect of wave breaking on the wave spectrum in the presence of an adverse current, Hedges et al. (1979) applied the equilibrium range spectrum [Phillips, 1980; see (29)] to limit the ordinates of the spectrum. However, it is known that the equilibrium range spectrum only applies to frequencies much higher than that corresponding to the peak of the wave spectrum and therefore cannot be extended to frequencies in the vicinity of the peak frequency (Yuan et al., 1986).

For the purpose of obtaining the energy-containing portion of the spectrum of breaking waves, we propose a model for deep water random breaking waves based on the Stokes wave-breaking criterion, wherein the elevation of the breaking wave is expressed as a function of that of the original ideal waves. By assuming that the latter are Gaussian and stationary, an approximate but simple expression is derived for the spectrum of the breaking waves. This expression is then applied to the case in which a unidirectional deep water random wave train propagates into an adverse current which is steady in time and uniformly distributed in depth.

2. Breaking wave model

Stokes showed that when the vertical downward acceleration at the wave crest reaches a value of $0.5g$ (g being the gravitational acceleration), the wave breaks and the amplitude, a, of the original ideal wave of frequency ω, is reduced to

$$a_b = \frac{0.5g}{\omega^2} = a\frac{0.5g}{a\omega^2} \tag{1}$$

according to the ratio of $0.5g$ and the magnitude of the acceleration of the original ideal wave at the crest.

Longuet-Higgins (1969) applied this wave-breaking criterion to a narrow-band wave train in which the breaking wave amplitude is taken as

$$a_b = a\frac{0.5g}{a\bar{\omega}^2} \tag{2}$$

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where $\bar{\omega}$ is the characteristic frequency of the narrow-band ideal wave train.

In order to obtain the spectrum of the breaking waves, it is necessary to express the elevation of the breaking waves in terms of that of the original ideal waves. For this purpose it is not unreasonable to assume (Phillips, 1980) that when the downward local acceleration on the wave surface reaches a certain fraction (usually 0.4 to 0.5, Ochi and Tsai, 1983) of the gravitational acceleration, local wave breaking occurs. Referring to Fig. 1, let $\zeta(t)$ and $\zeta_b(t)$ represent respectively the elevations of the ideal and breaking waves as functions of time t. Wave breaking may occur at points such as A and B where $\ddot{\zeta}(t) < 0$. However, for waves whose spectrum is reasonably narrow, the probability of the occurrence of negative peaks such as at point B is rather small so that for convenience, we shall assume that wave breaking only occurs at points such as A where $\zeta(t) > 0$ and when $\ddot{\zeta}(t) < -Kg$ ($K = 0.4$ in this study); the breaking wave elevation is given by

$$\zeta_b(t) = -\zeta(t)\frac{Kg}{\ddot{\zeta}(t)} \qquad (3)$$

which is merely a restatement of (1) or (2). That is, when the wave breaks, the local wave elevation is reduced according to the ratio of Kg and the local acceleration of the ideal wave.

Based on the above consideration and noting that no wave breaking occurs when $\ddot{\zeta}(t) > -Kg$, in which case $\zeta(t)$ remains unchanged, $\zeta_b(t)$ may be written as

$$\zeta_b = -\frac{Kg}{\ddot{\zeta}}\zeta H(-\ddot{\zeta} - Kg)H(\zeta) + \zeta H(\ddot{\zeta} + Kg) \qquad (4)$$

where $H(\cdot)$ is the Heaviside unit step function and, for convenience, the variable t in $\zeta_b(t)$, $\zeta(t)$ and $\ddot{\zeta}(t)$ is omitted. In (4), the first term corresponds to the situation when wave breaking occurs at the points such as A in Fig. 1 and the second term merely states that $\zeta(t)$ remains unchanged as long as $\ddot{\zeta}(t) > -Kg$.

From (4), it is seen that the breaking wave elevation, ζ_b, is a nonlinear function of ζ and $\ddot{\zeta}$, respectively the elevation and surface acceleration of the ideal waves which are assumed to be stationary and jointly Gaussian with zero mean values. The determination of the autocorrelation function and hence the spectrum of ζ_b can therefore be achieved in a straightforward manner (Papoulis, 1965).

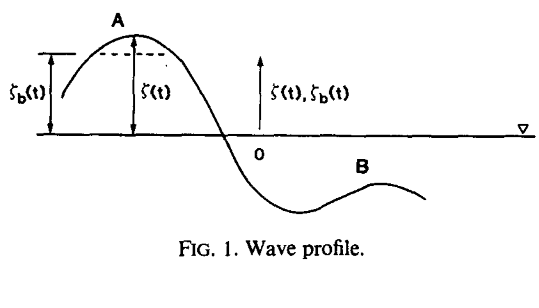

FIG. 1. Wave profile.

3. Spectrum of ζ_b

To obtain the spectrum of ζ_b, we first form its autocorrelation function. For convenience, let the subscripts 1 and 2 refer to quantities evaluated at the time instants $t_1 = t + \tau$ and $t_2 = t$, respectively. Furthermore, let $H = H(\zeta)$, $H''_+ = H(\ddot{\zeta} + Kg)$ and $H''_- = H(-\ddot{\zeta} - Kg)$. By anticipating that ζ_b is stationary, the autocorrelation function of ζ_b, denoted $R_b(\tau)$, is, from (4)

$$R_b(\tau) = (Kg)^2 E\left[\frac{\zeta_1\zeta_2}{\ddot{\zeta}_1\ddot{\zeta}_2}H''_{1-}H''_{2-}H_1H_2\right]$$

$$-2KgE\left[\frac{\zeta_1\zeta_2}{\ddot{\zeta}_1}H''_{1-}H''_{2+}H_1\right] + E[\zeta_1\zeta_2H''_{1+}H''_{2+}] \qquad (5)$$

where $E[\cdot]$ denotes expected value of the quantity enclosed in the brackets. The expected values in (5) involve the random variables ζ_1, ζ_2, $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$, which are jointly Gaussian with zero mean values. These expected values can all be obtained although the task is rather tedious. In the Appendix, the last expected value of (5) is evaluated to illustrate the techniques employed in obtaining these expected values. The resulting autocorrelation function is a nonlinear function of the correlation functions $r_{12}(\tau) = E[\zeta_1\zeta_2]$, $r_{12}^{(2)}(\tau) = E[\zeta_1\ddot{\zeta}_2]$ and $r_{12}^{(4)}(\tau) = E[\ddot{\zeta}_1\ddot{\zeta}_2]$ of the original wave elevation ζ and surface acceleration $\ddot{\zeta}$. If we denote

$$r = r_{12}(0) = \int S(\omega)d\omega \qquad (6)$$

$$r^{(2)} = r_{12}^{(2)}(0) = -\int \omega^2 S(\omega)d\omega \qquad (7)$$

$$r^{(4)} = r_{12}^{(4)}(0) = \int \omega^4 S(\omega)d\omega \qquad (8)$$

where $S(\omega)$ is the spectrum of the original ideal waves, then the magnitudes of the corresponding correlation coefficient functions $r_{12}(\tau)/r$, $r_{12}^{(2)}(\tau)/r^{(2)}$ and $r_{12}^{(4)}(\tau)/r^{(4)}$ are all less than unity. The autocorrelation function $R_b(\tau)$, viewed as a function of the above three correlation coefficient functions, may be expanded by Taylor series. By retaining only the zeroth and first order terms of the series, it may be verified that the zeroth order term is equal to the square of the expected value, $E[\zeta_b]$, of ζ_b. The first order, approximate, autocovariance function

$$K_b(\tau) = R_b(\tau) - E^2[\zeta_b] \qquad (9)$$

is therefore a linear function of $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$ and $r_{12}^{(4)}(\tau)$, the Fourier transforms of which are respectively $S(\omega)$, $-\omega^2 S(\omega)$ and $\omega^4 S(\omega)$. Thus, by taking the Fourier transform of (9), we have the approximate spectrum of the breaking waves simply related to $S(\omega)$ as

$$S_b(\omega) = F(\omega)S(\omega) \qquad (10)$$

in which

$$F(\omega)=A_1^2\left(\frac{\omega^2}{\omega_1^2}-1\right)^2 \tag{11}$$

is a fourth-order polynomial function of ω and may be looked upon as a filter function which accounts for the effects of wave breaking on the spectrum $S(\omega)$ of the ideal waves.

In (11),

$$\omega_1^2=\left|\frac{A_1}{A_2}\right|\left|\frac{r^{(4)}}{r^{(2)}}\right| \tag{12}$$

$$A_1=\frac{\beta N}{\sqrt{2\pi}}+Q(-\beta)>0 \tag{13}$$

$$A_2=\frac{\beta N}{(2\pi)^{1/2}}-\beta Z(\beta)Q(-\lambda)-\frac{\beta Q(\gamma)}{[2\pi(1-\epsilon^2)]^{1/2}}+\beta Z(\beta). \tag{14}$$

The quantities in (13) and (14) are given in the following:

$$Z(x)=\exp\left(-\frac{x^2}{2}\right)\Big/(2\pi)^{1/2} \tag{15}$$

$$Q(x)=\int_x^\infty Z(y)dy \tag{16}$$

$$\epsilon^2=1-\frac{(r^{(2)})^2}{rr^{(4)}} \tag{17}$$

$$\beta=Kg/(r^{(4)})^{1/2} \tag{18}$$

$$\gamma=\beta/\epsilon \tag{19}$$

$$\lambda=\frac{\beta(1-\epsilon^2)^{1/2}}{\epsilon} \tag{20}$$

$$N=(2\pi)^{1/2}\int_\beta^\infty \frac{Z(x)}{x}Q\left(-\frac{(1-\epsilon^2)^{1/2}}{\epsilon}x\right)dx>0. \tag{21}$$

It is seen that the breaking wave spectrum $S_b(\omega)$ depends only on the zeroth, second and fourth moments of the spectrum, $S(\omega)$, of the ideal waves. The quantity $0<\epsilon<1$ is recognized as the bandwidth parameter (Cartwright and Longuet-Higgins, 1956). The quantity β, which is the ratio of Kg and the standard deviation $(r^{(4)})^{1/2}$ of the zero-mean surface acceleration of the ideal waves, is a measure of the extent of wave breaking and may be given a rough estimate. Referring to Fig. 1, let us assume that the local acceleration in those portions of the surface where $|\ddot{\zeta}|$ reaches or exceeds the value Kg remains equal to Kg, but on the rest of the surface, the local acceleration is equal to zero. The standard deviation $(r^{(4)})^{1/2}$ of $\ddot{\zeta}$ is therefore roughly equal to $Kg(A_B/A)^{1/2}$ so that $\beta = 1/(A_B/A)^{1/2}$ where A_B and A are respectively the area of wave surface with $|\ddot{\zeta}|>Kg$ and the total surface area. The ratio A_B/A is normally a small quantity so that β is expected to be rather larger than unity.

From (13), it is obvious that $A_1>0$. By employing variously the series representation and asymptotic behavior of $Q(\cdot)$ for large values of its argument (Abramowitz and Stegun, 1968), it may be shown that $A_2>0$ and $A_1/A_2 \gg 1$. Furthermore, since $(|r^{(4)}/r^{(2)}|)^{1/2} > (|r^{(2)}/r|)^{1/2}$, and the latter quantity is in fact the characteristic frequency $\bar{\omega}$, it is seen from (12) that $\omega_1 \gg \bar{\omega}$. The filter function, $F(\omega)$, is a monotonically decreasing function of ω for $0<\omega<\omega_1$, decreasing from $F(0)=A_1^2$ to $F(\omega_1)=0$. Beyond $\omega=\omega_1$, $F(\omega)$ increases indefinitely. The range of frequency of wind waves of practical interest, however, is usually limited to within $0<\omega<\omega_1$ ($\gg\bar{\omega}$). The manner in which $F(\omega)$ varies with β, a measure of the sea state or the extent of wave breaking, may be seen by taking the derivative of A_1 with respect to β. It may be verified that A_1 is a monotonically increasing function of β and that A_1 approaches unity as β approaches infinity. This means that in mild seas, β and ω_1 are rather large so that $F(\omega)\approx 1$ for $0<\omega<\omega_1$ and, therefore, $S_b(\omega)\approx S(\omega)$; no wave breaking takes place and the original wave spectrum is unchanged. In high seas, on the other hand, $A_1<1$ and so is $F(\omega)$ for $0<\omega<\omega_1$. Thus, the original wave spectrum is reduced as a consequence of wave breaking, as expected.

4. Wave-current interaction

Although wind waves always have components propagating in various directions, for simplicity, let us consider a unidirectional deep water linear wave train entering a region of steady current whose flow velocity, U, considered positive in the direction of the wave, is uniformly distributed with depth. For each wave component, the apparent wave frequency, ω_a, in the stationary frame of reference is related to the relative or intrinsic wave frequency, ω_r, in the frame of reference moving with the current as

$$\omega_a=\omega_r+kU=\omega_r+\frac{\omega_r^2U}{g} \tag{22}$$

where $k=\omega_r^2/g$ is the wavenumber.

Ignoring wave breaking and using the energy balance (Huang et al., 1972) or the conservation of wave action (Hedges et al., 1979), it was shown that the wave spectrum, $S(\omega_a)$, under the influence of current, is related to $S_0(\omega_a)$, the spectrum in quiescent water without current, as

$$S(\omega_a)=\frac{S_0(\omega_a)}{[1+(\omega_rU/g)]^2[1+(2\omega_rU/g)]} \tag{23}$$

where ω_r is to be expressed in terms of ω_a according to (22). Here we note that although for each ω_a there can be two ω_r, for waves generated outside the current region, only one value of ω_r can be used (see Peregrine, 1976, and Hedges et al., 1979).

In the relative frame of reference, the wave spectrum,

$\bar{S}(\omega_r)$, can be obtained from (23) by changing the frame of reference (Hedges et al., 1979) as

$$\bar{S}(\omega_r) = S(\omega_a)\frac{d\omega_a}{d\omega_r} = S(\omega_a)\left(1 + \frac{2\omega_r U}{g}\right) = \frac{S_0(\omega_a)}{(1 + \omega_r U/g)^2} \tag{24}$$

where ω_a in $S_0(\cdot)$ is given by (22). From (23), it is seen that in an adverse current, where $U < 0$, the wave components with $\omega_a > g/4|U|$ or, equivalently, with $\omega_r > g/2|U|$, cannot exist.

An adverse current feeds energy into the wave system giving rise to an increase in spectral ordinates, as can be seen from (23) and (24), resulting in wave breaking. To account for the effect of wave breaking on the wave spectrum, (10) may be applied directly to $\bar{S}(\omega_r)$ in (24) so that, in the relative reference frame, the breaking wave spectrum $\bar{S}_b(\omega_r)$ is

$$\bar{S}_b(\omega_r) = F(\omega_r)\bar{S}(\omega_r). \tag{25}$$

In (25), $F(\cdot)$ is given by (11) where the quantities A_1 and ω_1 are determined from (12) to (21) and dependent only on the zero, second, and fourth spectral moments of $\bar{S}(\cdot)$, the wave spectrum under the influence of current in the relative reference frame given by (24), but without considering wave breaking. These spectral moments are carried out numerically for $0 < \omega_r < \omega_e = g/2|U|$ for $U < 0$ since the wave components with frequencies $\omega_r > \omega_e$ cannot enter the current area. The upper limit of ω_r in (25) is the lesser of ω_e and ω_1 since the argument of $F(\cdot)$ must not exceed ω_1.

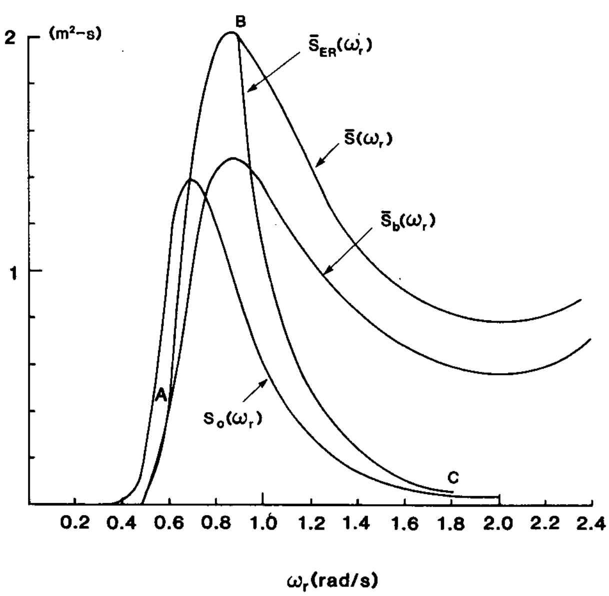

FIG. 2. Spectra $S_0(\omega_r)$, $\bar{S}(\omega_r)$, $\bar{S}_b(\omega_r)$ and $\bar{S}_{ER}(\omega_r)$ in relative reference frame for $h_s = 3.3$ m and $U = -2$ m s^{-1}.

In the stationary frame of reference, the breaking wave spectrum is given by

$$S_b(\omega_a) = \bar{S}_b(\omega_r)\frac{d\omega_r}{d\omega_a} = \frac{\bar{S}_b(\omega_r)}{1 + (2\omega_r U/g)} = \frac{\bar{S}_b(\omega_r)}{[1 + (4\omega_a U/g)]^{1/2}} \tag{26}$$

where ω_r in $\bar{S}_b(\omega_r)$ is to be expressed in terms of ω_a via (22) and the upper limit of ω_a is similarly determined from that of ω_r.

5. Results and discussion

To illustrate the use of (25) and (26) to examine the effect of wave breaking on a wave spectrum in an opposing current, and to compare our results with those of Hedges, et al. (1979), the Pierson–Moskowitz spectrum

$$S_0(\omega) = \frac{\alpha_1 g^2}{\omega^5}\exp\left[-\alpha_2\left(\frac{g}{W\omega}\right)^4\right] \tag{27}$$

is used for waves in quiescent water. The quantities $\alpha_1 = 0.0081$ and $\alpha_2 = 0.74$ are numerical constants and W is the wind speed.

For $W = 12.44$ m s^{-1} corresponding to significant wave height

$$h_s = 4(r)^{1/2} = \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2}\frac{2W^2}{g} = 3.3\ \text{m}, \tag{28}$$

$S_0(\cdot)$ and $\bar{S}(\omega_r)$ in (24) are plotted in Fig. 2 for $U = -2$ m s^{-1} where $\bar{S}(\omega_r)$ is terminated at a frequency slightly smaller than $\omega_r = g/2|U| = 2.5$ rad s^{-1}. The breaking wave spectrum, $\bar{S}_b(\omega_r)$, is computed using $K = 0.4$, also terminating at $\omega_r \approx 2.4$ rad s^{-1}.

In the stationary reference frame, the breaking wave spectrum, $S_b(\omega_a)$, is computed from (26) based on the $\bar{S}_b(\omega_r)$ in Fig. 2. This $S_b(\omega_a)$, together with $S_0(\omega_a)$ and, $S(\omega_a)$ in (23), are plotted in Fig. 3. Both $S(\omega_a)$ and $S_b(\omega_a)$ terminate at a frequency slightly smaller than $\omega_a = g/4|U| = 1.25$ rad s^{-1}. As ω_a approaches this cutoff frequency, the spectral ordinates of $S(\omega_a)$ and $S_b(\omega_a)$ begin to grow indefinitely. This phenomenon is expected because of the singularity in (23) and (26) at $\omega_a = g/4|U|$ as a result of the transformation from ω_r to ω_a-space.

Figure 2 also contains an equilibrium range spectrum

$$\bar{S}_{ER}(\omega_r) = \frac{\alpha g^2}{\omega_r^5} \tag{29}$$

in which α is a numerical constant ranging between 0.008 and 0.015. Using $\alpha = 0.0117$, Hedges et al. (1979) proposed to limit the spectral ordinates of $\bar{S}(\omega_r)$ by $\bar{S}_{ER}(\omega_r)$ to take into account the effect of wave breaking on $\bar{S}(\omega_r)$ in an opposing current. It is seen that the resulting spectrum, curve ABC in Fig. 2, is somewhat different from the $\bar{S}_b(\omega_r)$ obtained in this study.

Since the equilibrium range spectrum is meant for waves of frequencies much higher than that corresponding to the peak of the spectrum, the expression

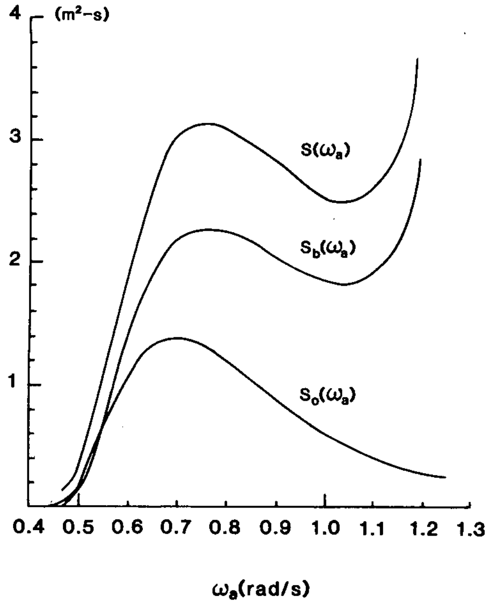

FIG. 3. Spectra $S_0(\omega_a)$, $S(\omega_a)$ and $S_b(\omega_a)$ in stationary reference frame for h_s = 3.3 m and $U = -2$ m s^{-1}.

derived in this study offers a means to compute the energy containing part of the spectrum of breaking waves. Within this range of frequencies it is known that wave breaking tends to broaden the spectrum and hence reduce the magnitude of the spectral slope; this is in fact what is observed in Figs. 2 and 3.

We include in our discussion the cases where the waves propagate into a region of a following current and where no current is present. Using the Pierson–Moskowitz spectrum as the original wave spectrum $S_0(\omega)$ in (24) with wind speed W = 12.44 m s^{-1} (or significant wave height h_s = 3 m) and current speed U = 2 m s^{-1}, our results show that the spectrum $\bar{S}_b(\omega_r)$ in (25) is practically the same as $\bar{S}(\omega_r)$, the spectrum of the waves on the following current but without considering wave breaking. This result is expected since, on a following current, the wave is lengthened, the height is reduced, the sea becomes calmer and the probability of wave breaking is correspondingly lessened.

For the case where there is no current, the Pierson–Moskowitz spectrum is again used for the spectrum $S(\omega)$ in (10) of the original waves and $S_b(\omega)$ in (10) is computed for various values of wind speed. For the Pierson–Moskowitz spectrum it is known that the fourth spectral moment does not exist. By using a cutoff frequency equal to twice the spectral peak frequency, we found that the difference between $S_b(\omega)$ and $S(\omega)$ is imperceptibly small. The explanation of this result rests on the concept of the significant slope defined as

$$\S = \frac{r^{1/2}}{\lambda_p}, \tag{30}$$

the ratio of $r^{1/2}$, the standard deviation [see (6)] of wave elevation and λ_p, the length of the wave component whose frequency is that of the peak of the spectrum.

The significant slope is a measure of the steepness of the waves and was first introduced by Huang et al. (1981) in connection with the development of the deep-water Wallops spectrum which is established based on wave dynamics according to the Stokes wave theory where no consideration is given to wave breaking. It is concerned with the energy containing part of the wave spectrum and takes the form

$$S(\omega) = \frac{\alpha_3 g^2}{\omega^m \omega_0^{5-m}} \exp\left[-\frac{m}{4}\left(\frac{\omega_0}{\omega}\right)^4\right] \tag{31}$$

where ω_0 is the frequency corresponding to the spectral peak,

$$m = \left|\frac{\log(\sqrt{2}\pi\S)^2}{\log 2}\right| \tag{32}$$

is the magnitude of the slope of the spectrum (on log-log scale) in the frequency range beyond the spectral peak, and

$$\alpha_3 = \frac{(2\pi\S)^2 m^{(m-1)/4}}{\Gamma[(m-1)/4]4^{(m-5)/4}} \tag{33}$$

is a nondimensional coefficient, $\Gamma(\cdot)$ being the gamma function.

The specification of the Wallops spectrum is seen to depend on two parameters, ω_0 and §. In an earlier paper (Tung and Huang, 1987) an example was given to show the effect of wave breaking on the Wallops spectrum for a specific set of values of § and ω_0. To examine the manner in which the significant slope affects wave breaking, we use the Wallops spectrum as $S(\omega)$ in (10) and compute $S_b(\omega)$ for ω_0 = 3 rad s^{-1} and various values of § ranging from § = 0.01 to 0.025 (under field conditions, § rarely exceeds 0.025). Our results show that the higher the value of §, the more $S_b(\omega)$ deviates from $S(\omega)$ but for § < 0.015, little difference between the two spectra is observed indicating that wave breaking is directly related to the value of §. For the Pierson–Moskowitz spectrum, it may be verified that § $= (0.8\alpha_1)^{1/2}/4\pi$ = 0.0064 which is a constant and is less than § = 0.015. This means that little wave breaking can be expected and the two spectra $S(\omega)$ and $S_b(\omega)$ remain practically the same regardless of the wind speed. It is interesting to note here that for a narrow-band wave train, the wave breaking parameter $\beta = Kg/(r^{(4)})^{1/2}$ in (18) is approximately equal to $K/2\pi\S$ so that the significant slope also serves as a direct albeit approximate measure of the amount of wave breaking.

6. Concluding remarks

In this paper, a model for the elevation of breaking waves is established based on which an expression for the breaking wave spectrum is derived. This enables us to examine the behavior of the breaking wave spectrum with or without the presence of current. Of special interest is the case when a wave train encounters an adverse current resulting in vigorous wave breaking, a phenomenon unaccounted for by the equation of energy balance. The method is approximate in that 1) the wave breaking model is heuristic, 2) the assumption that there exists an original ideal wave train is a simplification, 3) the higher-order terms in the expression for the breaking wave spectrum are ignored and, most importantly, 4) only a first-order solution of wave current interaction is employed and the interaction among wave components and the possibility of generation of higher-frequency waves by breaking waves are all not considered. As a result, the range of frequencies in which the breaking wave spectrum is applied should be restricted to the energy-containing part of the spectrum. The accuracy of the spectrum obtained in this study must be viewed in the light of the assumptions introduced and the success of the method must await further experimental verification.

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APPENDIX

Derivation of the Third Term in (5)

To obtain $E[\zeta_1\zeta_2 H''_{1+}H''_{2+}]$, we make use of the concept of conditional probability and conditional expectation to reduce the number of random variables. Thus, (Papoulis, 1965)

$$E[\zeta_1\zeta_2 H''_{1+}H''_{2+}] = E[H''_{1+}H''_{2+}E(\zeta_1\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2)] \quad \text{(A1)}$$

where

$$E(\zeta_1\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2) = \int_{-\infty}^{\infty}\int \zeta_1\zeta_2 f_{\zeta_1,\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2}(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2 \quad \text{(A2)}$$

is the conditional expected value of $\zeta_1\zeta_2$ given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$ and

$$f_{\zeta_1,\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2}(\zeta_1,\zeta_2) = \frac{1}{2\pi(1-\rho_{12}^2)^{1/2}\sigma_1\sigma_2} \times \exp\left\{-\frac{1}{2(1-\rho_{12}^2)}\left[\left(\frac{\zeta_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{\zeta_2-\mu_2}{\sigma_2}\right)^2 - 2\rho_{12}\left(\frac{\zeta_1-\mu_1}{\sigma_1}\right)\left(\frac{\zeta_2-\mu_2}{\sigma_2}\right)\right]\right\} \quad \text{(A3)}$$

is the jointly Gaussian conditional probability density function of ζ_1 and ζ_2 given $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$.

The quantities μ_1 and μ_2 are the conditional mean value functions of ζ_1 and ζ_2, $\sigma_1{}^2$ and $\sigma_2{}^2$ are the conditional variance functions of ζ_1 and ζ_2 and ρ_{12} is the conditional covariance coefficient function of ζ_1 and ζ_2. These five parameters may all be determined using the linear mean-square estimation technique (Papoulis, 1965). That is,

$$\mu_1 = a_1\ddot{\zeta}_1 + b_1\ddot{\zeta}_2 \quad \text{(A4)}$$

$$\mu_2 = a_2\ddot{\zeta}_1 + b_2\ddot{\zeta}_2 \quad \text{(A5)}$$

where a_1, b_1, a_2 and b_2 are determined based on the condition that $(\zeta_1 - \mu_1)$ and $(\zeta_2 - \mu_2)$ are orthogonal to and hence independent of $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$ giving

$$a_1 = [r^{(2)}r^{(4)} - r_{12}^{(2)}r_{12}^{(4)}]/\Delta = b_2 \quad \text{(A6)}$$

$$b_1 = [r_{12}^{(2)}r^{(4)} - r^{(2)}r_{12}^{(4)}]/\Delta = a_2 \quad \text{(A7)}$$

$$\Delta = (r^{(4)})^2 - (r_{12}^{(4)})^2. \quad \text{(A8)}$$

The same orthogonality properties lead to

$$\sigma_1{}^2 = E[(\zeta_1-\mu_1)^2|\ddot{\zeta}_1,\ddot{\zeta}_2] = E[\zeta_1-\mu_1)^2] = r - a_1 r^{(2)} - b_1 r_{12}^{(2)} \quad \text{(A9)}$$

which may be shown to be the same as $\sigma_2{}^2$ and

$$\rho_{12} = E[(\zeta_1-\mu_1)(\zeta_2-\mu_2)|\ddot{\zeta}_1,\ddot{\zeta}_2]/\sigma_1\sigma_2 = E[(\zeta_1-\mu_1)(\zeta_2-\mu_2)]/\sigma_1\sigma_2 = [r_{12} - a_1 r_{12}^{(2)} - b_1 r^{(2)}]/\sigma_1\sigma_2. \quad \text{(A10)}$$

In the above, $r = r_{12}(0)$, $r^{(2)} = r_{12}^{(2)}(0)$ and $r^{(4)} = r_{12}^{(4)}(0)$ are given in terms of $S(\omega)$ as indicated in (6), (7) and (8). The argument τ, the time lag, in $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$ and $r_{12}^{(4)}(\tau)$ is omitted for brevity. The quantities μ_1, μ_2, σ_1, σ_2 and ρ_{12} are all functions of $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$ and $r_{12}^{(4)}(\tau)$ and hence are functions of τ.

The conditional expected value $E[\zeta_1\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2]$ is seen to be the conditional correlation function of ζ_1 and ζ_2 and is therefore by definition given by

$$E[\zeta_1\zeta_2|\ddot{\zeta}_1,\ddot{\zeta}_2] = \mu_1\mu_2 + \rho_{12}\sigma_1\sigma_2 = (a_1{}^2+b_1{}^2)\ddot{\zeta}_1\ddot{\zeta}_2 + a_1b_1(\ddot{\zeta}_1{}^2+\ddot{\zeta}_2{}^2) + \rho_{12}\sigma_1\sigma_2. \quad \text{(A11)}$$

The expected value sought is therefore, from (A1)

$$E[\zeta_1\zeta_2 H''_{1+}H''_{2+}] = \int_{-kg}^{\infty}\int [(a_1{}^2+b_1{}^2)\ddot{\zeta}_1\ddot{\zeta}_2 + a_1b_1(\ddot{\zeta}_1{}^2+\ddot{\zeta}_2{}^2) + \rho_{12}\sigma_1\sigma_2] f_{\ddot{\zeta}_1,\ddot{\zeta}_2}(\ddot{\zeta}_1,\ddot{\zeta}_2)d\ddot{\zeta}_1 d\ddot{\zeta}_2 \quad \text{(A12)}$$

where $f_{\ddot{\zeta}_1,\ddot{\zeta}_2}(\ddot{\zeta}_1,\ddot{\zeta}_2)$ is the joint Gaussian probability density function of the zero-mean random variables $\ddot{\zeta}_1$ and $\ddot{\zeta}_2$ whose variances are $E[\ddot{\zeta}_1{}^2] = E[\ddot{\zeta}_2{}^2] = r^{(4)}$ and whose correlation coefficient function is $\rho_{12}^{(4)}(\tau) = E[\ddot{\zeta}_1\ddot{\zeta}_2]/r^{(4)} = r_{12}^{(4)}/r^{(4)}$.

The above integrals may all be carried out giving

$$E[\zeta_1\zeta_2 H''_{1+}H''_{2+}] = (a_1{}^2+b_1{}^2)F_1 + 2a_1b_1F_2 + \rho_{12}\sigma_1\sigma_2F_3 \quad \text{(A13)}$$

where

$$F_1 = r^{(4)}\{[1-(\rho_{12}^{(4)})^2]^{1/2}Z(\beta)Z(\Omega) - 2\beta\rho_{12}^{(4)}Z(\beta)Q(\Omega) + \rho_{12}^{(4)}F_3\} \quad \text{(A14)}$$

$$F_2 = r^{(4)}\{[1-(\rho_{12}^{(4)})^2]^{1/2}\rho_{12}^{(4)}Z(\beta)Z(\Omega) - \beta[1+(\rho_{12}^{(4)})^2]Z(\beta)Q(\Omega) + F_3\} \quad \text{(A15)}$$

$$F_3 = L(-\beta, -\beta, \rho_{12}^{(4)}) \quad \text{(A16)}$$

(see Abramowitz and Stegun, 1968). Here, the argument τ in $\rho_{12}^{(4)}(\tau)$ is omitted,

$$\Omega = -\beta\left[\frac{1-\rho_{12}^{(4)}}{1+\rho_{12}^{(4)}}\right]^{1/2} \quad \text{(A17)}$$

and β, $Z(\cdot)$ and $Q(\cdot)$ are defined in (18), (15) and (16), respectively.

The expected value in (A1) is a nonlinear function of $r_{12}(\tau)$, $r_{12}^{(2)}(\tau)$ and $r_{12}^{(4)}(\tau)$ and may be expanded by the Taylor series. By retaining only the zero and first-order terms, it is given approximately by

$$E[\zeta_1\zeta_2 H''_{1+}H''_{2+}] = a_1{}^2 r^{(4)} Z^2(\beta) + a_1{}^2 r^{(4)}\rho_{12}^{(4)} \times[-\beta Z(\beta) + Q(-\beta)]^2 + 2a_1 b_1 r^{(4)} Q(-\beta) \times[-\beta Z(\beta) + Q(-\beta)] + \rho_{12}\sigma_1\sigma_2 Q^2(-\beta) \quad \text{(A18)}$$

where

$$a_1 \approx (r^{(2)}/r^{(4)})^{1/2} \quad \text{(A19)}$$

$$b_1 \approx [r_{12}^{(2)} r^{(4)} - r_{12}^{(4)} r^{(2)}]/(r^{(4)})^2 \quad \text{(A20)}$$

$$\rho_{12}\sigma_1\sigma_2 \approx r_{12} - 2r_{12}^{(2)}\frac{r^{(2)}}{r^{(4)}} + r_{12}^{(4)}\left(\frac{r^{(2)}}{r^{(4)}}\right)^2. \quad \text{(A21)}$$

The integration in (A12) may also be facilitated by employing the Hermite polynomial series representation (Erdely et al., 1953) as follows:

$$\frac{1}{(1-p^2)^{1/2}}\exp\left[-\frac{1}{2(1-p^2)}(x^2+y^2-2pxy)\right] = 2\pi\sum_{n=0}^{\infty} p^n h_n(x)h_n(y)Z(x)Z(y) \quad \text{(A22)}$$

where

$$h_n(x) = \frac{(-1)^n}{(n!)^{1/2}}\frac{d^n Z(x)/dx^n}{Z(x)} \quad \text{(A23)}$$

is the Hermite polynomial function. Upon expanding the joint Gaussian probability density function $f_{\ddot{\zeta}_1,\ddot{\zeta}_2}(\ddot{\zeta}_1, \ddot{\zeta}_2)$ into the Hermite series representation, it is seen that the integrals in (A12) may be carried out easily. By retaining only the terms involving $n = 0$ and 1 in the series, (A18) may be obtained.

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