On Hopf algebra deformation approach to renormalization

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Abstract. The relation between Connes-Kreimer Hopf algebra approach to renormalization and deformation quantization is investigated. Both approaches rely on factorization, the correspondence being established at the level of Wiener-Hopf algebras, and double Lie algebras/Lie bialgebras, via r-matrices. It is conjectured that the *-product obtained by Lie bialgebra deformation quantization and the *-product corresponding in the sense of Kontsevich-Cattaneo to the QFT obtained via renormalization, correspond.

1 Introduction

Connes-Kreimer's Hopf algebra approach to renormalization [3, 8, 10] allows to identify an "algebraic interface" to renormalization, opening an avenue for connections with deformation quantization.

In this research note, we point out to some connections between Lie bialgebra quantization and Feynman path integral quantization methods.

Recall that renormalization is a factorization at the level of the convolution algebra of "commutative points" of the Hopf algebra of Feynman graphs, corresponding to a given renormalization scheme. We claim that this Birkhoff decomposition / Wiener-Hopf factorization is an algebraic geometry analog of the infinitesimal factorization [15] leading to Lie bialgebra quantization. The infinitesimal factorization is essentially equivalent to a Lie bialgebra structure, and hence, via a "standard procedure", to quantization ([5, 12]; [2]).

At a more specific level, a prescription scheme satisfying the multiplicativity condition is nothing else but a Baxter operator [13], leading to a solution of the modified classical Yang-Baxter equation, and therefore to a coboundary Lie bialgebra structure and deformation quantization [2], p.187.

Although conjectural at this point, this correspondence is supported by a preliminary result, the Factorization Theorem 3.1, which recognizes the factorization of

Poisson-Lie groups ([2], p.67) in the Birkhoff decomposition associated to a renormalization prescription scheme. The "deformed antipode", or rather deformed inverse of the bare Feynman value ([8], p.19), is interpreted as part of the "decomposition" of the convolution product. This prompts for an interpretation of the multiplicativity condition: a classical Yang-Baxter equation in disguise.

The results presented here, well-known individually, are assembled in a conjectural commutative diagram (section 4) which is expected to have an impact on the understanding of perturbative QFT, besides the obvious message that the details of the various receipts, computational schemes etc. of renormalization are essentially irrelevant to the overall physics scheme. It is expected that a unifying point of view including renormalization, deformation quantization and TQFTs will emerge.

The article is organized as follows. The algebraic interface to renormalization from [3, 8] is briefly recalled in §2. The factorization point of view is stated in §3, where Wiener-Hopf operators are defined as a special case of Baxter operators. An infinitesimal factorization occurs conform [15], determined by a certain operator satisfying a "Poincare-Bertrand" equation. The Lie algebra correspondent of the fore mentioned equation is a modified Yang-Baxter equation. This leads to a "Factorization Theorem" 3.1 on the convolution algebra, as an algebraic source for the Birkhoff decomposition in the analytic framework. The benefit is a prediction of a classical R-matrix, opening the path to Lie bialgebra quantization. We conclude in §4 with additional comments on the relation between deformation quantization and renormalization.

2   An algebraic interface to renormalization

We will start with a brief recall on renormalization as stated in [3, 8], using freely the concepts introduced in [6].

The main "characters in the renormalization play" are: a Lie algebra g generated by some Feynman graphs, with the corresponding Hopf algebra $H = U(g^*)$, a set of *states on Feynman graphs* (e.g. external structures, "colors" to distinguish various propagators etc.), allowing to define characters $\phi : H \to A$ into a commutative algebra as *bare values of the Feynman integral* corresponding to *Feynman rules* for a given regularization scheme, and a k-linear map $R : A \to A$ representing the *renormalization prescription* (e.g. minimal subtraction (MS) with zero subtraction point).

Then, Connes and Kreimer proved in [3] that the renormalized value of the Feynman integral ϕ_+ is a factor of the Birkhoff decomposition:

$$\phi = (\phi^-)^{-1} \star \phi^+$$

of the bare value of the dimensional regularization of the Feynman integral ϕ. It is obtained via the convolution with the counter term:

$$\phi^+ = S_R \star \phi.$$

Recall that, the Birkhoff decomposition of a meromorphic function (relative to a disk) is a factorization:

$$(2.1) \qquad\qquad\qquad \phi_+ = \phi \star \phi_-,$$

with $\phi_{\pm}$ holomorphic inside/outside the disk, agreeing on the common boundary. More generally it can be defined with respect to a decomposition of an algebra into subalgebras.

Definition 2.1. $\phi : \phi^- \to \phi^+$ *denotes the* Birkhoff decomposition *of the character* $\phi : H \to A$ *relative to the decomposition* $A = A_- \oplus A_+$ *corresponding to the projectors* (R^+, R^-) *iff* $\phi^{\pm}$ *are characters satisfying the following relation:*

$$(2.2) \qquad \phi_+ = \phi \star \phi_-, \quad R^{\pm}\phi_{\pm} = \phi_{\pm}.$$

The renormalization procedure based on the minimal subtraction prescription is essentially a Birkhoff decomposition, with the "finite part" of the Feynman amplitude (without poles in the analytic renormalization picture) provided by ϕ_+.

The above counter term is a deformation of the inverse $\phi^{-1} = \phi \circ S$ (with respect to convolution), and can be obtained inductively by the formula:

$$(2.3) \qquad S_R^{\phi}(\Gamma) = R(-\phi(\Gamma) - \sum_{\gamma \hookrightarrow \Gamma \to \gamma'} S_R^{\phi}(\gamma)\phi(\gamma')).$$

In the above formula, the sum runs over certain decompositions of the graph Γ. These possible decompositions can be encoded in a coproduct Δ ([8]; see also [6]).

Now the linear operator R has to satisfy a multiplicative constraint:

$$(2.4) \qquad R(x)R(y) + R(xy) = R(R(x)y) + R(xR(y)), \quad x, y \in A,$$

which ensure that S_R^{ϕ} is again a character ([9], p.14). This turns (A, R) into a Baxter algebra ([10], p.18). Moreover, R is usually idempotent (e.g. MS-prescription).

3 Relation with deformation quantization

Our point of view suggests to interpret the deformed inverse S_R^{ϕ} as part of the *factorization* enabling a Lie bialgebra deformation quantization, following [2], p.67. In order to do this, rewrite Equation 2.3 as follows:

$$(3.1) \qquad S_R^{\phi}(\Gamma) = R \circ m(-\phi(\Gamma) \otimes 1) + \sum_{\gamma \hookrightarrow \Gamma \to \gamma'} R \circ m(S_R^{\phi}(\gamma) \otimes \phi(\gamma')).$$

Now ϕ is a character of the convolution algebra $H^* = (Hom(H, A), \star, \eta)$, with:

$$< f \star g, X >_H = m_A \circ < f \otimes g, \Delta X >, \quad f, g \in Hom(H, A), X \in H,$$

or briefly $\star = m_A \circ (\cdot \otimes \cdot)\Delta$. Here H is regarded merely as a coalgebra and A as an algebra with multiplication denoted by m_A.

The multiplication m_A can be truncated using R, yielding a new convolution:

$$m_+ = R \circ m, \quad *_+ = m_+ \circ (\cdot \otimes \cdot)\Delta.$$

Although not associative when R is a (nontrivial) projector as above, the new multiplication m_+ is associative when R is the involution associated with a Wiener-Hopf operator (see (iii) of Lemma 3.1).

Proposition 3.1. *The deformed inverse S_R^ϕ is the inverse relative to the deformed convolution product $*_+$:*

$$S_R^\phi = \phi^{-1} \quad in \; (H_+^*, *_+).$$

In particular, in dimensional regularization, a natural implementation of the requirement that " ... it does not modify the pole terms in ϵ" [10], p.18, is to consider $R = R_-$ the projection on the pole part of the Taylor expansion.

The role of R and the relation with the deformation picture will be investigated next.

3.1   Wiener-Hopf operators

We will address the structure of Wiener-Hopf operators and the connection with Yang-Baxter equation.

Following [13], p.505, we introduce the following definition.

Definition 3.1. *An idempotent $R \in End(A)$ satisfying the Baxter identity is called a* Wiener-Hopf operator.

Note that, citing from [13], "In fact it was the algebraic study of Wiener-Hopf operators that motivated Glen Baxter to single out the Baxter identity." Operators of this form arise in the theory of Wiener-Hopf factorization ([14], p.483), which essentially corresponds to the Birkhoff decomposition via the Cayley transform.

Any Wiener-Hopf operator corresponds to a decomposition of the algebra ([14], p.181), yielding a decomposition of the associated Lie algebra.

Since to quantize a Lie bialgebra structure is needed (double Lie algebra [15]), this Wiener-Hopf structure seems to play in renormalization the role the double construction plays in quantization. For details we send the reader to [15], p.263 (Infinitesimal Form of the Factorization Theorem, r-matrices, Lie bialgebra structure and double Lie algebras), including the relation with Lax equations, Riemann problem etc. (p.262).

The "missing link", to be studied next, is the connection between the Baxter operator R and the classical r-matrix in the sense of [15], p.260, satisfying the modified Yang-Baxter equation (loc. cit. p.262):

(3.2) $[Rx, Ry] - R([Rx, y] + [x, Ry]) = -[x, y]$ $MYBE.$

Any solution of MYBE determines a *modified bracket*:

(3.3) $[x, y]_R = [Rx, y] + [x, Ry], \quad x, y \in A.$

satisfying the Jacoby identity (see [15], p.262).

3.2   Associative algebra level

The structure of a Wiener-Hopf algebra is equivalent to having a decomposition of the algebra as a sum of two subalgebras.

If R is an idempotent of $End(A)$, consider the associated decomposition $A = A_- \oplus A_+$. The symmetry about the "positive part" of the algebra (A, m) under the decomposition along the projector $R_+ = R$, will be called the *associated conjugation*:

(3.4) $C = R_+ - R_-.$

Here $R_- = I - R$ is the complementary projector.

An example of such an idempotent R_+ is the projection which cuts the pole part of a meromorphic function. Some well-known algebraic properties are summed up in the following lemma.

Lemma 3.1. *Let R be an idempotent of A, determining the above direct sum decomposition as k-modules. Then the following statements are equivalent:*

(i) R is a Wiener-Hopf operator, i.e. it satisfies the Baxter identity;

(ii) $A_\pm$ are subalgebras in A;

(iii) The Hilbert transform satisfies the Poincare-Bertrand identity (PBI - following [15], p.263):

$$(3.5) \qquad Rx \cdot Ry + xy = R(Rx \cdot y + x \cdot Ry), \quad x, y \in A.$$

Moreover, in this case the following twisted multiplication *is associative:*

$$(3.6) \qquad m_C(x, y) = C(x) \cdot y + x \cdot C(y), \quad x, y \in A.$$

proof 1. *Since both equations, Baxter equation and PBI, are linear, we can restrict to the corresponding subspaces, and establish the statements using a case-by-case analysis. For instance, if $x \in A_+$ and $y \in A_\pm$, then a direct computation shows that the Baxter identity is equivalent to $R(xy) = xy$, etc.*

The case-by-case analysis ($x, y, z \in A_\pm$) to establish the last statement may be reduced using the symmetry of the associator, and noting that on A_+, $m_C = 2m$, on A_- $m_C = -2m$, while on $A_+ \otimes A_-$ m_C vanishes (see also [15], p.263, Proposition 6). This completes the proof.

Now we will turn our attention to the infinitesimal level.

3.3 Lie algebra level

The alternation functor $Alt = \sum_{\sigma \in \Sigma} \epsilon(\sigma)$ allows to pass from associative algebras to Lie algebras, yielding the corresponding (well-known) Lie algebra properties.

Proposition 3.2. *Let $R \in End(A)$ and $(g, [,]) = Alt(A, m)$, i.e. $[x, y] = m(x, y) - m(y, x), x, y \in A$. If R is a Wiener-Hopf operator, then:*

(i)R satisfies the modified Yang-Baxter equation:

$$(3.7) \qquad [Rx, Ry] + R[x, y] = R([,]_R), \quad x, y \in g,$$

and the modified bracket $[,]_U$ is a Lie bracket.

(ii) $(g, [,]_R)$ is a double Lie algebra corresponding to the decomposition

$$g = g_- \oplus g_+, \quad g_\pm = Alt(A_\pm).$$

proof 2. *(i) is plainly obtained by alternating Baxter equation.*

Regarding the "infinitesimal factorization" of part (ii), we refer the reader to Proposition 5, [15], p.262.

As a consequence, R is an r-matrix, providing an infinitesimal factorization $g = g_- \oplus g_+$ via the Lie-algebra homomorphisms $R \pm 1$ ([15], 263). Moreover, note that an r-matrix exists iff the analogous Lie algebra factorization is uniquely solvable (loc. cit. p.264).

As proved in Proposition 3, [15], p.261, double Lie algebras and Lie bialgebras structures are equivalent in the case of skew symmetric solutions of MYBE, in the presence of an invariant inner product. This opens the avenue of deformation quantization, towards a QFT "a la Moyal".

3.4   The factorization property

We return to the Birkhoff decomposition, as interpreted in Proposition 3.1.

The truncated multiplication m_+ is not associative, since R is just a projector satisfying the Baxter identity. It is "half" of the factorization picture, and for reasons of symmetry, one has to consider the complementary projector $R_- = Id - R_+$ with the associated truncated multiplication $m_- = R_- \circ m$.

Define the two inversion maps relative to the corresponding convolutions:

$$S_\pm : H^* \to H^*, \quad S_\pm(\phi) = \phi^{-1} \ in \ H^*_\pm.$$

Note that since $R_+ + R_- = id$ (trivial overlapping of the two "charts") the "transition map" $T = (S_-)^{-1} S_+$ is the identity map on H^*, establishing the following "Factorization Theorem".

Theorem 3.1. *For any Wiener-Hopf algebra (A, m, R) and coalgebra (H, Δ), the convolution algebra $(H^*, \star)$ is factorizable, i.e. there is a canonical Birkhoff decomposition:*

$$\phi = \phi_-^{-1} \star \phi_+, \quad \phi \in Hom(H, A),$$

where $\phi_\pm = S_\pm(\phi)$ are the inverses relative to the truncated convolutions $*_\pm$.

A comparison with the Poisson-Lie group factorization [2], p.67) which is crucial for the Lie bialgebra quantization method prompts for the conjecture from the introduction, which will be reconsidered next.

4   Conclusions

Among the notable consequences of the above factorization, one expects the existence of a canonical classical R-matrix similar to the one present in the context of Poisson-Lie groups [2], p.67. This would allow to explore the deformation quantization path on the following diagram.

Star-products representing a quantization "a la Moyal", can be obtained via two quantization methods: the Lie bialgebra quantization and Feynman path integral quantization.

$$
\begin{array}{ccc}
Star-product & \xrightleftharpoons{\text{Moyal}} & QFT \\
& \overset{Kontsevich/Cattaneo-Felder}{\underset{Factorization}{\uparrow}} & \\
\overset{EK-R}{Quantization} & & \overset{Connes-Kreimer}{Renormalization} \\
Lie\ bialgebra & \xleftarrow{This\ article} & Wiener-Hopf\ operator.
\end{array}
$$
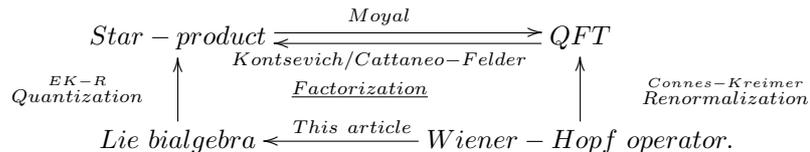
The star-product on a Poisson manifold was derived by Kontsevich [7] from a partition function of a QFT ([1]). For details on the left edge (Lie bialgebra quantization) see [2], Ch.6, p.170, and references therein ([5, 12] etc.).

The suggested "commutativity" of the above diagram needs a more detailed investigation, clarifying the role of the classical R-matrix expected above.

This will support the claim that renormalization is independent on the chosen prescription R, and contribute to the understanding of the renormalization group in the context of Weyl's program in quantum physics [11].

Moreover, it was suggested in [6] that the Feynman graphs (as a category of cobordism) should be thought of as a (quantum) "space-time". This interpretation is consistent with the duality "spectrum - convolution algebra" present in the deformation point of view of renormalization.

Our point of view, centered on the deformation theory via the r/R-matrix framework and integrable systems, was later investigated in [4]. A more detailed account, including these subsequent developments, will appear elsewhere.

5   Acknowledgments

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