An Axiomatic System for Peirce's Alpha Graphs

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Abstract. This paper presents a Hilbert-style system for Alpha graphs, the first part of Existential Graphs. A set of generalized Sheffer-strokes are the only connectives in the "symbol-based" formal system for Alpha graphs, and the most important advantage of the system is that both the decision procedure and the completeness's proof via countermodel are immediate.

1 Introduction

The present paper is an attempt to amalgamate two systems of logic that Peirce developed over his long career. In the paper titled "A Boolean Algebra with One Constant" of 1880, Peirce showed how all the elective functions of Boole could be expressed by use of a single primitive sign with the meaning of "neither ... nor ...". This is the first discovery of the truth-functional completeness of the 2-ary connective Sheffer's stroke function (henceforth, sheffer-stroke), "|" in notation. This fact is rediscovered by H.M. Sheffer in 1913 in his "A Set of Five Independent Postulates for Boolean Algebras, with application to logical constants". Then, the most important development is the presentation of the calculus in a strictly axiomatized form. Based on Peirce and Sheffer's taking $A|B$ as undefined, Nicod showed in his 1917 article "A Reduction in the Number of the Primitive Propositions of Logic" that the whole calculus could be based on the single axiom

$$[A|(B|C)]|\ ([D|(D|D)]|\ \{(E|B)|[(A|E)|(A|E)]\})$$

with

$$\alpha \qquad \alpha|(\beta|\gamma) \qquad /\therefore \gamma$$

as a rule of inference in place of the traditional *modus ponens*. But it can scarcely be said that the reduction achieved by Nicod is a simplification which makes the theory easier to grasp. Peirce himself also says: "Of course, it is not maintained that this notation is convenient." ([Pe33] 4.20)

The sheffer-stroke has two dual interpretations "neither ... nor ..." and "either not ... or not ...". It is worth noticing that introducing the sheffer-stroke into Existential Graphs should avoid this kind of troublesomeness resulted from Peirce's "notation in which the number of signs should be reduced to a minimum" ([Pe33] 4.12) and simultaneously save the elegance because of the least primitive connective. Throughout his scientific life Peirce explored, with seemingly endless creativity and stamina, one notational device after another. Peirce's chapters on Existential Graphs in volume 4 of his *Collected Papers* contain a wealth of ideas. We know that sheffer-stroke "either not ... or not ..." could be defined by two classical connectives negation and conjunction as "it is not the case that A and B". Interpreted as negation and conjunction, the only two primitive operations "Cut" and "Juxtaposition" underlie Existential Graphs. From this point of view, then, introducing the sheffer-stroke "either not ... or not ..." into Existential Graphs is very much apropos although the most outstanding characteristic of Existential Graphs is the "Iconicity". Of course according to Peirce a graph is in the main an "Icon" of the forms of relations in the constitution of its "Object," but nevertheless, it will ordinarily have symbolic features:

> Now since a diagram, though it will ordinarily have Symbolic Features, as well as features approaching the nature of Indices, is nevertheless in the main an Icon of the forms of relations in the constitution of its Object, the appropriateness of it for the representation of necessary inference is easily seen. ([Pe33] 4.531).

In accordance with this idea, certainly Existential Graphs can be regarded as an iconic system as well as a symbolic system. Based on the sheffer-stroke interpreted as "either not ... or not ..." and two rules of inference in place of the original ones, this paper develops an axiomatic system for the first part of the Existential Graphs, Alpha. The most outstanding advantage of this system is that the proof of every provable graph is effective. In the original occasion it is allowable to draw n graphs on the same area simultaneously, so in the system to be developed we adopt a generalized version instead of 2-ary sheffer-stroke, which occurs implicitly like many situations of Peirce himself.

We proceed as follows: Section 2 is a short introduction to the Peirce's Alpha Graphs, the readers who familiar with Existential Graphs should jump over this section. An axiomatics for the Alpha graphs defined by this paper and an informal definition of a deduction-tree of this axiomatics are given in section 3, moreover, this section describes a backward chaining procedure along with the inference rules for every graphs. Based on the semantics presented in section 4, section 5 establishes the soundness and completeness theorems for our formal system. In the concluding section we present some expectations for further research and defaults of this approach.

2 Alpha Graphs

Peirce sets up Existential Graphs with the intention of providing a logical analysis of mathematical reasoning. We start with by considering the grammar of the Alpha system. In fact, the language of the Alpha system can be described as a propositional language augmented with a propositional constant and based on a small complete set of connectives, namely negation and conjunction.

In the Alpha part there are just three primitive types of symbol:

1. A piece of paper or a blackboard upon which it is practicable to scribe the graphs, termed the *Sheet of Assertion*. Every part of the surface is called the *blank*.
2. Propositional signs (e.g., A, B, ...), probably any symbols, words or natural language sentences.
3. A self-returning finely drawn line known as a *Cut* or *Sep*.

Peirce occasionally employs a linear (or, bracket) notation for his graphs, a notation that is convenient for typesetting and space considerations though not as visually perspicuous as his two-dimensional notation [1]. In the situation without confusion, it is convenient to adopt this linear notation (more precisely, square bracket notation). Semantically, the sheet of assertion represents the universe of discourse. Writing the

[1] See, e.g., [Pe33], 4.378–389.

propositional signs on the sheet of assertion amounts to asserting their truth. For example, as Peirce states ([Pe33] 4.433), by writing

The pulp of some orange is red.

we assert that in our domain of discourse it is true that the pulp of some orange is red. Negation and conjunction are the principal logical connectives of the Alpha system. The sign for negation is the cut. By encircling the above assertion we get

[*The pulp of some orange is red*].

which asserts that it is false that the pulp of some orange is red. The conjunction of two or more assertions is obtained by juxtaposing the assertions together on the sheet. For example,

The pulp of some orange is red.
To express oneself naturally is the last perfection of a writer's art.

asserts that the pulp of some orange is red and to express oneself naturally is the last perfection of a writer's art. The other propositional connectives can now be defined in terms of cut and conjunction. Truth is represented by the empty sheet of assertion and falsity by an empty cut. In the Alpha system Implication is presented as $[A[B]]$.

One of the advantages of graphic notation is the ease of reading the graphs in many different ways, for example, the disjunction "A or B" may be read in at least five more ways [2]. In the same time, on the other hand, the syntactic history of Peirce's graphs is fundamentally different from the way a formula is composed out of the basic vocabulary of its system, that is to say, Peirce's graphs are pressed for the property of unique readability.

The rules are the following:

1. *Deletion* and *Insertion*: "Within an even finite number (including none) of seps, any graph may be erased; within an odd number [of seps] any graph may be inserted." ([Pe33] 4.492)
2. *Copying Rule*: "Any graph may be iterated within the same or additional seps, or if iterated, a replica may be erased, if the erasure leaves another outside the same or additional seps." ([Pe33] 4.492)
3. *Double Negation Rule*: "Anything can have double enclosures added or taken away, provided there be nothing within one enclosure but outside the other." ([Pe33] 4.379)

3 Axiomatics

In the system to be developed for Alpha the primitive symbols include sentence letters $p_0, p_1, \ldots,$ (p, q, r, s for metavariables) and the cut [ ] (or ( ), { } if necessary) in brackets-notation. And the number of the primitive connectives is infinite and each is a *generalized n-ary sheffer-stroke* (henceforth *nand*) which occurs implicitly, for $n \geq 0$.

Now define a set, *Ag*, of *Alpha graphs* in the following way:

Definition 1. *1. Every sentence letter is a graph.*

[2] According to a manuscript of Shin's.

2. *For n≥0, if* $\alpha_0, \ldots, \alpha_{n-1}$ *are all graphs, then the single cut of the juxtaposition* $[\alpha_0 \ldots \alpha_{n-1}]$ *of the n graphs* $\alpha_0, \ldots, \alpha_{n-1}$ *is a graph* [3].
3. *Nothing else is a graph.*

In the definition of "graph" there is no restriction on the order of the n graphs $\alpha_0, \ldots, \alpha_{n-1}$. This applies also to linear transcriptions of graphs. In other words, the order of the n graphs $\alpha_0, \ldots, \alpha_{n-1}$ in graph $[\alpha_0 \ldots \alpha_{n-1}]$ has no logical significance.

By the above definition, if $n=0$, then the empty cut (called *Enclosure*), [ ] in bracket notation, is a graph; and if $n=1$ and α_i is a sentence letter say p, then a single cut of α_i is a graph, i.e., [p]; similarly, $[[\alpha_i]]$ is a graph. Additionally, p, the scrolls [p[q]], [[p][q]], the *double cut* (namely the empty scroll) [[ ]], etc., are all graphs. Evidently, the empty space and the juxtaposition of the n graphs $\alpha_0, \ldots, \alpha_{n-1}$ such as pqs, p[q], are all "well-formed" graphs by Peirce's definition but no more by ours. So, it is not the case that all of the original Alpha graphs are captured by our definition. But the graphs at present do have the property of unique readability because the two operations Cut and Juxtaposition in fact have been handled unitarily as one connective though.

Definition 2. *The set of subgraphs of a graph* α *is the set* $Sub(\alpha)$ *such that:*

1. $Sub(p)=\{p\}$;
2. $Sub([\alpha_0 \ldots \alpha_{n-1}])=\{[\alpha_0 \ldots \alpha_{n-1}]\}\cup\bigcup_{i=0}^{n-1} Sub(\alpha_i)$

Sentence letters have no *immediate subgraphs*, and the collection of *immediate subgraphs* of $[\alpha_0 \ldots \alpha_{n-1}]$, $I\text{-}Sub([\alpha_0 \ldots \alpha_{n-1}])$ in notation, is the collection of graphs $\alpha_0, \ldots, \alpha_{n-1}$; similarly, for $\alpha_i \in \Gamma$, $I\text{-}Sub(\Gamma)=\bigcup_{i=0}^{n-1} I\text{-}Sub(\alpha_i)$.

By the definition of graphs every well-formed graph is a sentence letter or a single cut of a graph α_j, then every immediate subgraph of the graph $[\alpha_0 \ldots \alpha_{n-1}]$, i.e., $\alpha_0, \ldots, \alpha_{n-1}$, is a sentence letter or a single cut of a graph. For example, let α be

[p q (r[[ ]][s]) [ ]]

Then $I\text{-}Sub(\alpha)$ is exactly the set

{p, q, (r[[ ]][s]), [ ]}

which includes four members.

Let α, β be grphs and $\beta \in I\text{-}Sub(\alpha)$. Then we have the following notions:

Definition 3. 1. *α is a simple grapha if and only if α=[ ] or α=p or α=[p] for a propositional variable p;* [4]
2. *β is an atomic nand if each immediate subgraph α in I-Sub(α) is a simple graph.*

That is to say, a graph is an atomic *nand* if and only if its immediate subgraphs are all simple graphs.

Now we state the axioms and diagrammatic transformation rules for our graphs. There are two axiom schemas and two rules of inference. The two axiomatic schemas are:

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[3] Shin develops a definition for Peircean Alpha graphs. In her definition the implicit sheffer-stroke seems ready to come out at one's call. Our definition could be regarded as a refined version of hers. See [Sh02], p.65.

[4] This definition can be viewed as a contraction of the definition 4.3 of Shin's. See [Sh02], p.65.

Definition 4 (Axiom).

1. *Every atomic nand whose set of immediate subgraphs contains at least one empty cut is an axiom.*

2. *Every atomic nand whose set of immediate subgraphs contains some sentence letter and its negation is an axiom.*

Rules are schemas too. Let $\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}$ be graphs for m, $n \geq 0$, α and β denote these two sequences of graphs respectively. The two diagrammatic transformation rules, called *rule of double cut insertion* (rule 1) and *rule of unification of n graphs* (rule 2) respectively, are as follows:

Definition 5 (Rule).

1. *From* $[\alpha\beta]$ *infer* $[\alpha[[\beta]]]$;

2. *From n graphs* $[\alpha\ [\beta_0]]$, $\ldots$, $[\alpha[\beta_{n-1}]]$ *infer* $[\alpha[\beta]]$.

The definition of a graph being provable from a set of graphs is then defined as follows: Let $\Gamma\cup\{\alpha\}$ be a set of graphs, then α is a *provable* from Γ (written $\Gamma\vdash\alpha$) if and only if there is a finite nonempty sequence of graphs $\langle\alpha_0, \ldots, \alpha_{n-1}, \alpha\rangle$ such that each member of this sequence is either a member of Γ, an axiom, or follows from previous graphs by one of the two diagrammatic transformation rules. The sequence $\langle\alpha_0, \ldots, \alpha_{n-1}, \alpha\rangle$ is called a *deduction* of graph α and the number of $\langle\alpha_0, \ldots, \alpha_{n-1}, \alpha\rangle$ is called the length of a deduction. If Γ is empty (i.e., $\vdash\alpha$), then α is a *theorem.*

Generally speaking, Hilbert-style systems can be useful as formal representations of what is provable, but the actual finding of proofs in Hilbert-style systems is next to impossible. This is not the case in the system just stated. In this system, deductions will be presented as *trees*, called *deduction-trees*; the *nodes* will be labeled with graphs; the labels at the immediate successors of a node v are the premises of a rule application, the label at v the conclusion. At the *root* of the tree we find the conclusion of the whole deduction.

In accordance with the from-bottom-to-top direction in the two diagrammatic transformation rules, every deduction-tree grows upwards from its root, i.e., the graph needed to be proved. The procedure is as follows:

1. If a label at the node v is a graph which set of the immediate subgraphs has a graph with one double cut of a graph (or many graphs) as its member, then, in accordance with the rule of double cut insertion, we erase the double cut and take the result as the immediate predecessor of v.
2. If a label at the node v is a graph that the set of its immediate subgraphs has a graph with one cut of many graphs as its member, then, in accordance with the rule of unification of n graphs, we take the cut of many graphs apart and take the result as the immediate successors of v.

In general, the step (1) is considered prior to the step (2). Repeatedly use these two steps the deduction-tree grows from bottom to top gradually. Whenever all the labels at the tops of every branch are atomic *nand*s, the growth of the tree ends. Finally, write down on the paper all of the graphs from top to bottom line by line (and delete the repetitions, if any) which complete the proof. By constructing a deduction-tree with these two steps, the provability of a graph is reduced to the provability of a finite collection of atomic *nand*s, and when all of these atomic *nand*s are axioms then the original graph is provable.

4 Semantics

Having stated the axioms and rules, we can now turn to the task of showing that they are complete. To formulate the completeness issue precisely it is necessary to provide a semantics for the graphs. The semantics for the system is similar to that for propositional logic. Let Γ be a set of graphs, then we have

Definition 6 (truth assignment). *a truth assignment is a function* * *from* Γ *onto* $\{1, 0\}$ *such that* $([\alpha_0 \ldots \alpha_{n-1}])^* = 1$ *if and only if* $(\alpha_i)^* = 0$ *for some* $i < n$.

By this definition, the graph $[\alpha]$ is called the *negation* of the graph α, [5] and now the 2-ary sheffer stroke | is expressed by $[pq]$ in bracket-notation. Moreover, the definitions of all the seven signs in a Boolian Algebra are restated in bracket-notation as follows:

1. $\neg p =_{df}$ [p]
2. p$\wedge q =_{df}$ [[pq]]
3. p$\vee q =_{df}$ [[p][q]]
4. p$\rightarrow q =_{df}$ [p[q]]
5. $1 =_{df}$ [[ ]]
6. $0 =_{df}$ [ ]
7. p$\leftrightarrow q =_{df}$ [[pq] [[p][q]]]

Let * be a truth assignment function, Γ a set of graphs and α a graph. We say that α is *satisfiable* provided that $(\alpha)^*$=1; in this case * *satisfies* α, otherwise $(\alpha)^*$=0. Similarly, we say that Γ is *satisfiable* provided that for each $\alpha \in \Gamma$, $(\alpha)^*$=1; in this case * *satisfies* Γ (written $(\Gamma)^*$=1; otherwise $(\Gamma)^*$=0). We say that α is a *tautology* (written $\models \alpha$) provided that for each function *, $(\alpha)^*$=1. Finally, if $\Gamma \cup \{\alpha\}$ is a set of graphs we say that α is a logical consequence of Γ (written $\Gamma \models \alpha$) if every function * that satisfies Γ satisfies α.[6]

5 Soundness and Completeness

The soundness and completeness theorems together assert the equivalence of provability ($\vdash$) with tautologicality ($\models$).

We state three propositions here before our proof of the soundness and completeness theorems:

Lemma 1. *All axioms are tautologies.*

Proof. Without loss of generality, let α be a sequence of simple graphs, then according to our semantics defined above it is easy to prove that [[ ]α] is a tautology. Similarity for the second axiom. □

[5] This definition of negation differs from the others. Generally, negation is defined as $\neg A =_{df} A|A$ including Peirce's rubbing out the |. See the note of [Pe33] 4.20.

[6] This semantics is similar to which provided in [Ha95] in the case of Peirce diagrams.

Lemma 2. *The two rules preserve tautologicality.*

Proof. Leave it to the readers. □

Lemma 3. *Let* * *be a truth assignment. If* * *doesn't satisfy the premise(s) of one rule then* * *doesn't satisfy the conclusion of it.*

Proof. Let * be a truth assignment, α and β denote the sequence of graphs $\alpha_0 \ldots \alpha_{m-1}$ and $\beta_0 \ldots \beta_{n-1}$ respectively, and Γ is a set of graphs:

$$\{[\alpha[\beta_0]], \ldots, [\alpha[\beta_{n-1}]]\}$$

By hypothesis we have $([\alpha\beta])^*{=}0$ (the first case) and $(\Gamma)^*{=}0$ (the second case).

First case:

$([\alpha\beta])^*{=}0$

$\Rightarrow$For all $j{<}m$, $i{<}n$, $(\alpha_j)^* = (\beta_i)^* = 1$

$\Rightarrow ([\alpha[[\beta]]])^* = 0$.
that is to say, * doesn't satisfy $[\alpha[[\beta]]]$.

Second case:

$(\Gamma)^*{=}0$

$\Rightarrow$ For some $i{<}n$, $([\alpha[\beta_i]])^* = 0$

$\Rightarrow$ For all $j{<}m$, some $i < n$, $(\alpha_j) = 1$ and $(\beta_i) = 0$

$\Rightarrow ([\alpha[\beta]])^* = 0$.
that is to say, * doesn't satisfy $[\alpha[\beta]]$. □

Let α be a graph, now from the first two propositions by induction on the length of deductions we have:

Theorem 1 (Soundness). *If* $\vdash \alpha$ *then* $\models \alpha$.

Now we turn to the task of showing that the rules are complete.

Theorem 2 (Completeness). *If* $\models \alpha$ *then* $\vdash \alpha$.

Proof. We show the contrapositive. Assume that $\vdash \alpha$ does not hold. We construct a tree for α, thus each branch must be ended with a set of atomic *nands*, in which there must be at least one member that is not an axiom. Without loss of generality, let $\langle \alpha_0, \ldots, \alpha_{n-1}, \alpha \rangle$ be a branch of α, Γ a set of graphs at the top node α_0, $\{p_0, \ldots, p_{n-1}\}$ a complete set of letters occurring in Γ. Then Γ is not satisfiable. Now we have a function * such that for $i < n$,

1. If $p_i{\in}\mathit{I\text{-}Sub}(\Gamma)$, then $(p)^*{=}1$;
2. If $[p_i]{\in}\mathit{I\text{-}Sub}(\Gamma)$, then $(p)^*{=}0$.

By the third proposition proved above, α is not a tautology. □

6 Conclusion

This paper provides a new way to reason with Alpha graphs in a way to check easily whether an Alpha graph is tautologous or not, and the procedure presented in this paper can also be extended to the other two parts of Peirce's excellent logic system, Existential Graphs. On the other hand, Peirce always emphasized the experimental character of his rules and this character is missing in our approach. So, it should be understood as an additional approach for this area.

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References

[Ha95] E. Hammer: Logic and Visual Information, CSLI Publications, 1995.

[Kn62] W. Kneale and M. Kneale: The Development of Logic, Clarendon Press, 1962.

[Pe33] C. S. Peirce: Collected Papers of C. S. Peirce, volume 4, Charles Hartshorne and Paul Weiss, editors, Harvard University Press, 1933.

[Ro64] D. D. Roberts: "The Existential Graphs and Natural Deduction," In Edward Moore and Richard Robin, editors, Studies in the Philosophy of Charles Sanders Peirce. University of Massachusetts Press, 1964.

[Ro73] D. D. Roberts: The Existential Graphs of Charles S. Peirce, Mouton and Co., 1973.

[Sh13] H. Sheffer: "A Set of Five Independent Postulates for Boolean Algebras, with application to logical constants," Transactions of the American Mathematical Society, 1913.

[Sh02] S. J. Shin: The Iconic Logic of Peirce's Graphs, The MIT Press, 2002.

[So84] J. F. Sowa: Conceptual Structures: Information Processing in Mind and Machine, Addison-Wesley, Reading, 1984.

[Wh84] R. White: "Peirce's Alpha Graphs: The Completeness of Propositional Logic and the Fast Simplification of Truth Functions," Transactions of the Charles S. Peirce Society 20, 1984.

[Ze67] J. Zeman: "A System of Implicit Quantification," Journal of Symbolic Logic 32, 1967.

[Zh97] Q. Y. Zhang: "A Axiomatic System for Propositional Logic," Philosophical Researches (Chinese) 8, 1997.