第26卷第1期 数 学 研 究 与 评 论 Vol.26, No.1
2006年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2006

Article ID: 1000-341X(2006)01-0077-12 **Document code**: A

Fuzzifying Topological Linear Spaces Based on Continuous-Valued logic

ZHANG Guang-ji, ZOU Kai-qi, ZHANG Cheng

(University Key Lab. of Information Science and Engineering of Dalian University.
College of Information Engineering, Dalian University, Liaoning 116622, China)
(E-mail: zh_gj@sina.com)

Abstract: In this paper, we introduce the concept of fuzzifying topological linear space and discuss the structures and properties of the balanced neighborhood system of zero element. We also give the algebraic properties and the topological properties of fuzzifying convex set in the fuzzifying topological linear space.

Key words: continuous-valued logic; fuzzifying topological linear spaces; balanced set; zero element neighborhood system; convex set.
MSC(2000): 54A40
CLC number: O189.1

1. Introduction

The notion of fuzzy topological linear space was first introduced by Katsaras and Liu[3]. But, since there exist certain difficulties in their work, they have not been able to achieve much in this respect. Consulting the strengthened definition of fuzzy topology by Lowen[4], Wu Congxin[5] proposed a new definition of fuzzy topological linear space and discussed the following problems: the criteria of the fuzzy topological linear space by F-open neighborhood of θ_λ, fuzzy topological linear space of type (L), the algebraic and topological properties of the convex fuzzy set. Wu Congxin used fuzzy points introduced by Pu Baoming and Liu Yingming[6] as his tool in his work.

M.S.Ying[2] introduced fuzzifying topology and elementarily developed fuzzy topology from a new direction with the semantic method of continuous valued logic. On the basis of M.S.Ying' theory, we use semantic method of continuous-valued logic to develop fuzzy topological linear space from a completely different direction in this paper and thereby establish elementary fuzzifying topological linear space which is dual to the existing fuzzy topological linear space.

The reader is assumed to be familiar with Ying's paper[2].

First, we display the fuzzy logical and corresponding set-theoretical notations used in this paper:

(1) For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval[0,1]. A formula φ is valid, we write $\models \varphi$, if and only if $[\varphi] = 1$ for

Received date: 2003-12-18
Foundation item: the National Natural Science Foundation of China (60274016), the Project of Scientific Research in Hight Education Bureau Liaoning Province (2023901018).

every interpretation.

(2) $[\neg\alpha] := 1-[\alpha]$; $[\alpha\wedge\beta] := \min([\alpha],[\beta])$; $[\alpha\dot\wedge\beta] := [\alpha]\otimes[\beta] = \max(0,[\alpha]+[\beta]-1)$;

$$[\alpha\longrightarrow\beta] := [\alpha]\propto[\beta] = \min(1,1-[\alpha]+[\beta]); \quad [\forall x\alpha(x)] := \inf_{x\in X}[\alpha(x)]; \quad [x\in A] := A(x),$$

where X is the universe of discourse.

(3) $\alpha\vee\beta := \neg(\neg\alpha\wedge\neg\beta)$; $\alpha\longleftrightarrow\beta := (\alpha\longrightarrow\beta)\wedge(\beta\longrightarrow\alpha)$; $\exists x\alpha(x) := \neg\forall x\neg\alpha(x)$;
$A\subseteq B := (\forall x)(x\in A\longrightarrow x\in B)$; $A\equiv B := (A\subseteq B)\wedge(B\subseteq A)$.

Throughout this paper N_a will denote a neighborhood system of a over K where K is the number field and $a\in K$. That is $M\in N_a := (\exists\delta>0)(\forall b)(b\in M\longrightarrow|b-a|<\delta)$.

2. Fuzzifying topological linear space

Definition 2.1[2] *Let X be a universe of discourse, $\tau\in\mathcal{F}(\mathcal{P}(X))$ satisfy the following conditions:*

(T_1) $\models X\in\tau$;

(T_2) For any $A_1,A_2,\models(A_1\in\tau)\wedge(A_2\in\tau)\longrightarrow A_1\wedge A_2\in\tau$

(T_3) For any $\{A_\lambda|\lambda\in\Lambda\}\subseteq P(X)$,

$$\models\forall\lambda(\lambda\in\Lambda\longrightarrow A_\lambda\in\tau)\longrightarrow\bigcup_{\lambda\in\Lambda}A_\lambda\in\tau.$$

Then τ is called a fuzzifying topology and (X,τ) a fuzzifying topological space.

Definition2.2[2] *Let (X,τ) be a fuzzifying topological space and $x\in X$. The neighborhood system of x is denoted by $N_x\in\mathcal{F}(\mathcal{P}(\mathcal{X}))$ and defined as $A\in N_x := \exists B((B\in\tau)\wedge(x\in B\subseteq A))$, i.e.,*

$$N_x = \int_{P(X)}\sup_{x\in B\subseteq A}\tau(X)/A.$$

Proposition 2.1[2] *Let N_x be a fuzzifying neighborhood system of x, then*

(1) $\forall A,\models A\in N_x\longrightarrow x\in A$;

(2) $\forall A,B,\models(A\in N_x)\wedge(B\in N_x)\longrightarrow(A\cap B\in N_x)$;

(3) $\forall A,B,\models(A\subseteq B)\longrightarrow(A\in N_x\longrightarrow B\in N_x)$;

(4) $\forall A,\models(A\in\tau)\longleftrightarrow\forall x(x\in A\longrightarrow(\exists B)((B\in N_x)\wedge(B\subseteq A)))\longleftrightarrow\forall x(x\in A\longrightarrow A\in N_x)$.

Definition 2.3 *Let X be a linear space over K. A fuzzifying topological space (X,τ) is called a fuzzifying topological linear space if it has the following properties:*

(L_1) $\forall x,y\in X,A\in P(X)$,

$$\models(A\in N_{x+y})\longrightarrow(\exists B)(\exists C)((B\in N_x)\wedge(C\in N_y)\wedge(B+C\subseteq A)).$$

(L_2) $\forall x\in X,a\in K,A\in P(X)$,

$$\models(A\in N_{ax})\longrightarrow(\exists B)(\exists M)((B\in N_x)\wedge(M\in N_a)\wedge(\forall b)(b\in M\longrightarrow bB\subseteq A)).$$

Theorem 2.1 *Let (X,τ) be a fuzzifying topological linear space and let N_x be a fuzzifying neighborhood system of x. For any x,y,A we have*

(1) $\models A \in N_x \longrightarrow y + A \in N_{x+y}$;
(2) $\models A \in N_x \longrightarrow aA \in N_{ax}$ $(a \in K, a \neq 0)$;
(3) $\models A \in N_{x-y} \longrightarrow (\exists B)(\exists c)((B \in N_x) \wedge (C \in N_y) \wedge (B - C \subseteq A))$;
(4) $\models A \in \tau \longrightarrow x + A \in \tau$;
(5) $\models A \in \tau \longrightarrow aA \in \tau$ $(a \in K, a \neq 0)$;

Proof (1) According to (L_1),

$$N_x(A) = N_{(x+y)+(-y)}(A) \leq \sup_{B+C\subseteq A} \min(N_{x+y}(B), N_{(-y)}(C))$$
$$\leq \sup_{B+(-y)\subseteq A} \min(N_{x+y}(B), N_{(-y)}(C)) \leq \sup_{B\subseteq A+y} N_{x+y}(B) = N_{x+y}(y + A).$$

(2) According to (L_2), for any $a \in K, a \neq 0$,

$$N_x(A) = N_{\frac{1}{a}ax}(A) \leq \sup_{M\subseteq K} \inf_{b\in M} \sup_{bB\subseteq A} \min(N_{ax}(B), N_{\frac{1}{a}}(M)$$
$$\leq \sup_{M\subseteq K} \sup_{\frac{1}{a}B\subseteq A} \min(N_{ax}(B), N_{\frac{1}{a}}(M)) \leq \sup_{B\subseteq aA} N_{ax}(B) = N_{ax}(aA).$$

(3)

$$[A \in N_{x-y}] = N_{x+(-y)}(A) \leq \sup_{B+(-C)\subseteq A} \min(N_x(B), N_(- y))$$
$$\leq \sup_{B-C\subseteq A} \min(N_x(B), N_y(C))$$
$$= [(\exists B)(\exists C)((B \in N_x) \wedge (C \in N_y) \wedge (B - c \subseteq A))].$$

(4)

$$\tau(x + A) = \inf_{y\in x+A} N_y(x + A) = \inf_{y-x\in A} N_{x+(y-x)}(x + A) = \inf_{z\in A} N_{x+z}(x + A)$$
$$\geq \inf_{z\in A} N_z(A) = \tau(A).$$

(5) $\tau(aA) = \inf_{ax\in aA} N_{ax}(aA) = \inf_{x\in A} N_{ax}(aA) \geq \inf_{x\in A} N_x(A) = \tau(A)(a \in K.a \neq 0)$.

Corollary 2.1 *Let (X, τ) be a fuzzifying topological linear space and let θ be the zero element of X, then*

(1) $\models A \in N_\theta \longleftrightarrow x + A \in N_x$; (2) $\models A \in N_\theta \longleftrightarrow aA \in N_\theta$ $(a \neq 0)$.

Corollary 2.2 *Let (X, τ) be a fuzzifying topological linear space and let θ be the zero element of X, then*

(1) $\models x \in \bar{A} \longleftrightarrow \forall B(B \in N_\theta \longrightarrow x \in A + B)$;
(2) $\forall B \models A \in \tau \longrightarrow B + A \in \tau$;
(3) $\models x \in \mathring{A} \longleftrightarrow \exists B(B \in N_\theta \wedge x + B \subseteq A)$.

Definition 2.4 *Let (X, τ) be a fuzzifying topological linear space. The neighborhood system N_θ of zero element θ in X is called fuzzifying zero element neighborhood system of (X, τ).*

Theorem 2.2 *If N_θ is the zero neighborhood system of (X, τ),then*

(E_1) $\forall A, \models A \in N_\theta \longrightarrow \theta \in A$;

(E_2) $\forall A, B, \models (A \in N_\theta) \wedge (B \in N_\theta) \longrightarrow (A \cap B \in N_\theta)$;

(E_3) $\forall A, B, \models (A \subseteq B) \longrightarrow ((A \in N_\theta) \longrightarrow (B \in N_\theta))$;

(E_4) $\forall A, \models (A \in N_\theta) \longrightarrow (\exists B)(\exists C)((B \in N_\theta) \wedge (C \in N_\theta) \wedge (B + C \subseteq A))$;

(E_5) $\forall A, \forall a \neq 0, \models (A \in N_\theta) \longrightarrow (\exists B)((B \in N_\theta) \wedge (aB \subseteq A))$;

(E_6) $\forall x, A, \models (A \in N_\theta) \longrightarrow (\exists M)((M \in N_0) \wedge (\forall a \in M)(ax \in A))$;

(E_7) $\forall x, A, \models (A \in N_\theta) \longrightarrow (\exists B)((B \in N_\theta) \wedge (x + B \subseteq A))$.

Proof We prove only (E_6) and (E_7) since (E_1), (E_2) and (E_3) have a clear meaning from (1),(2) and (3) in Proposition 2.1, and (E_4), (E_5) are well-known from (L_1) and (L_2), respectively.

$$\begin{aligned}(E_6):\quad [A \in N_\theta] =& N_\theta(A) = N_{0x}(A) \leq \sup_{B} \sup_{M \subseteq K} \inf_{a \in M} \min(N_x(B), N_0(M), [aB \subseteq A]) \\ \leq& \sup_{B} \sup_{M \subseteq K} \inf_{a \in M} \min(B(x), N_0(M), [aB \subseteq A]) \leq \sup_{M \subseteq K} \inf_{a \in M} \min(N_0(M), \\ & [aB \subseteq A]) = [(\exists M)((M \in N_0) \wedge (\forall a \in M)(ax \in A))];\end{aligned}$$

$$\begin{aligned}(E_7):\quad & [(\exists B)((B \in N_\theta) \wedge (x + B \subseteq A))] = \sup_{C \subseteq A} N_\theta(C - x) \quad (x + B = C) \\ & \geq \sup_{C \subseteq A} \min(N_\theta(C), N_\theta(C - x)) \geq \sup_{C \subseteq A} \min(N_\theta(C), \inf_{x \in C} N_\theta(C - x)) \\ & = \sup_{C \subseteq A} \min(N_\theta(C), \inf_{x \in C} N_x(C)) = \sup_{C \subseteq A} \min(N_\theta(C), \tau(C)) = \sup_{C \subseteq A} \tau(C) \\ & \geq \sup_{\theta \in C \subseteq A} \tau(C) = N_\theta(A) = [A \in N_\theta].\end{aligned}$$

Definition 2.5 *Let X be a universe of discourse. A unary fuzzy predicate $\sigma \in \mathcal{F}(\mathcal{P}(\mathcal{X}))$, called fuzzy balance, is given as follows: $A \in \sigma := aA \subseteq A$ ($\forall a \in K, |a| \leq 1$).*

Theorem 2.3 *Let X be a linear space and $N \subseteq \sigma$ be a normal fuzzy subset on $\mathcal{P}(\mathcal{X})$. If N satisfies (E_1), (E_2), (E_3) and (E_5), then τ is a fuzzifying topology on X , which is defined as $A \in \tau := (\forall x)((x \in A) \longrightarrow (\exists B)((B \in N) \wedge (x + B \subseteq A)))$. Specially, if N satisfies (E_4), (E_6) and (E_7) also, then (X, τ) is a fuzzifying topological linear space, and N is the balanced neighborhood system of the zero element with respect to τ.*

Proof First,

$$\tau(A) = \inf_{x \in A} \sup_{x + B \subseteq A} N(B)$$

and from (E_1), for any $\neg(\theta \in A), N(A) = 0$.

(1) Because N is normal, there exists $B \in P(X)$ such that $N(B) = 1$. Hence,

$$\tau(X) = \inf_{x \in X} \sup_{x + B \subseteq X} N(B) = 1.$$

(2) From (E_2) and (E_5), we have $\models (A \in N) \wedge (B \in N) \longrightarrow (\exists C)((C \in N) \wedge (C \subseteq A \cap B))$.

Therefore,

$$\begin{aligned}\min(\tau(A),\tau(B)) &= \min(\inf_{x\in A}\sup_{x+C\subseteq A}N(C),\inf_{x\in B}\sup_{x+D\subseteq B}N(D))\\ &\le \min(\inf_{x\in A\cap B}\sup_{x+C\subseteq A}N(C),\inf_{x\in A\cap B}\sup_{x+D\subseteq B}N(D))\\ &= \inf_{x\in A\cap B}\min(\sup_{x+C\subseteq A}N(C),\sup_{x+D\subseteq B}N(D))\\ &\le \inf_{x\in A\cap B}\min(\sup_{(x+C)\cap(x+D)\subseteq A\cap B}N(C),\sup_{(x+C)\cap(x+D)\subseteq A\cap B}N(D))\\ &= \inf_{x\in A\cap B}\min(\sup_{x+(C\cap D)\subseteq A\cap B}N(C),\sup_{x+(C\cap D)\subseteq A\cap B}N(D))\\ &= \inf_{x\in A\cap B}\sup_{x+(C\cap D)\subseteq A\cap B}\min(N(C),N(D))\le \inf_{x\in A\cap B}\sup_{x+(C\cap D)\subseteq A\cap B}\sup_{U\subseteq(C\cap D)}N(U))\\ &\le \inf_{x\in A\cap B}\sup_{(x+U)\subseteq(A\cap B)}N(U)=\tau(A\cap B).\end{aligned}$$

(3)

$$\begin{aligned}\tau(\bigcup_{\lambda\in\Lambda}A_\lambda) &= \inf_{x\in\bigcup_{\lambda\in\Lambda}A_\lambda}\sup_{x+B\subseteq\bigcup_{\lambda\in\Lambda}A_\lambda}N(B)=\inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda}\sup_{x+B\subseteq\bigcup_{\lambda\in\Lambda}A_\lambda}N(B)\\ &\ge \inf_{\lambda\in\Lambda}\inf_{x\in A_\lambda}\sup_{x+B\subseteq A_\lambda}N(B)=\inf_{\lambda\in\Lambda}\tau(A_\lambda).\end{aligned}$$

(4) If N satisfies $(E_4),(E_6),(E_7)$ also, then for any $x\in X, N_x$ is the fuzzifying neighborhood system of (X,τ), where $N_x\in F(P(X))$ is defined as follows: $A\in N_x := A-x\in N$.

In fact,

$$\begin{aligned}\sup_{x\in B\subseteq A}\tau(B) &= \sup_{x\in B\subseteq A}\inf_{x\in B}\sup_{x+C\subseteq B}N(C)\le \sup_{x\in B\subseteq A}\sup_{x+C\subseteq B}N(C)=\sup_{\theta\in B-x\subseteq A-x}\sup_{C\subseteq B-x}N(C)\\ &= \sup_{\theta\in B-x\subseteq A-x}N(B-x)=N(A-x)=N_x(A).\end{aligned}$$

Conversely,

$$\begin{aligned}N_x(A)=N(A-x) &\le \inf_{y\in A-x}\sup_{y+B\subseteq A-x}N(B)=\inf_{z-x\in A-x}\sup_{z-x+B\subseteq A-x}N(B)\\ &= \inf_{z\in A}\sup_{z+B\subseteq A}N(B)=\inf_{z\in A}N(A-z)=\sup_{z\in D\subseteq A}\inf_{z\in D}N(D-z)\\ &= \sup_{z\in D\subseteq A}\inf_{z\in D}\sup_{z+C\subseteq D}N(C)=\sup_{z\in D\subseteq A}\tau(D).\end{aligned}$$

Therefore, $N_\theta(A)=N(A-\theta)=N(A)$.

Then the hypothesis of $N\subseteq\sigma$ implies that N is the balanced neighborhood system of the zero element with respect to τ.

(5) Finally, we will show that (X,τ) is the fuzzifying topological linear space, that is that $N_x(x\in X)$ satisfies (L_1) and (L_2).

$$N_{x+y}(A)=N(A-x-y)\le \sup_{E+F\subseteq A-x-y}\min(N(E),N(F))\le \sup_{E+F\subseteq A-x-y}\min(\sup_{x+B-x\subseteq E}N(B),$$

$$\sup_{y+C-y\subseteq F}N(C))=\sup_{E+F\subseteq A-x-y}\sup_{x+B-x\subseteq E}\sup_{y+C-y\subseteq F}\min(N(B),N(C))\le \sup_{(x+B)+(y+C)\subseteq A}\min(N(B),$$

$$N(C)) = \sup_{B+C\subseteq A} \min(N(B-x), N(C-y)) = \sup_{B+C\subseteq A} \min(N_x(B), N_y(C));$$

$$\begin{aligned}
N_{a_0x}(A) &= N(A-a_0x) \le [(\exists B)((B\in N)\wedge(B\subseteq A-a_0x))] \\
&\le [(\exists B)(\exists C)(\exists D)((C\in N)\wedge(D\in N)\wedge(C+(1+|a_0|)D\subseteq B)\wedge(B+a_0x\subseteq A)))] \\
&\le [(\exists B)(\exists C)(\exists D)(\exists M)((D\in N)\wedge(M\in N_0)\wedge(\forall(a-a_0)\in M)((a-a_0)x\in C)\wedge \\
&\quad (\frac{a}{1+|a_0|}D\subseteq D)\wedge(C+(1+|a_0|)D\subseteq B)\wedge(B+a_0x\subseteq A))] \\
&\le [(\exists B)(\exists C)(\exists D)(\exists M)((D\in N)\wedge(M\in N_0)\wedge(\forall(a-a_0)\in M)((a-a_0)x\in C)\wedge \\
&\quad (aD\subseteq(1+|a_0|)D)\wedge(C+(1+|a_0|)D\subseteq B)\wedge(B+a_0x\subseteq A))] \\
&\le (\exists C)(\exists D)(\exists M)((D\in N)\wedge(M\in N_0)\wedge(\forall(a-a_0)\in M)((a-a_0)x\in C)\wedge \\
&\quad (aD\subseteq(1+|a_0|)D)\wedge(C+(1+|a_0|)D+a_0x\subseteq A))] \\
&\le [(\exists C)(\exists D)(\exists M)((D\in N)\wedge(M\in N_0)\wedge \\
&\quad (\forall(a-a_0)\in M)((a-a_0)x\in C)\wedge(C+aD+a_0x\subseteq A))] \\
&\le [(\exists D)(\exists M)((D\in N)\wedge(M\in N_0)\wedge(\forall(a-a_0)\in M)((a-a_0)x+aD+a_0x\subseteq A))] \\
&\le [(\exists D)(\exists M)((D\in N)\wedge(M\in N_{a_0})\wedge(\forall a\in M)(a(x+D)\subseteq A))] \\
&\le [(\exists D)(\exists M)((D\in N_x)\wedge(M\in N_{a_0})\wedge(\forall a\in M)(aD\subseteq A))].
\end{aligned}$$

In the following examples, K can be any non-discrete field; for instance, the field p-adic numbers, or the field of quaternions with their usual absolute values, or any subfield of these such as the rational, real or complex number field (with the respective induced absolute).

Example 2.1 Let A be any non-empty set, K^A the set of all mappings $\alpha \longrightarrow \xi_\alpha$ of A into K; we write $x=(\xi_\alpha), y=(\eta_\alpha)$ to denote elements x,y of K^A. Defining addition by $x+y=(\xi_\alpha+\eta_\alpha)$ and scalar multiplication by $\lambda x=(\lambda\xi_\alpha)$, it is immediate that K^A becomes a linear space over K . For any finite subset $H\subseteq A$ and any real number $\varepsilon>0$, let $V_{H.\varepsilon}$ be the subset $\{x: |\xi_\alpha|\le\varepsilon \ \ if \ \ \alpha\in H\}$ of K^A. Consider the fuzzy subset N on $\mathcal{P}(\mathcal{K}^\mathcal{A})$ defined as follows:

$$N(V_{H.\varepsilon}) = \begin{cases} 1, & \text{if } \ \varepsilon\ge 1, \\ \varepsilon, & \text{if } \ \varepsilon<1, \end{cases}$$

then it is clear that N is a balance neighborhood system of the zero element for a unique fuzzifying topology under which K^A is a fuzzifying topological linear space.

Example 2.2 Let X be any non-empty topological space; the set of all continuous functions f on X into K such that $\sup_{x\in X}|f(t)|$ is finite is a subset of K^A, which is a linear space $C_K(X)$ under the operations addition and scalar multiplication induced by the linear space K^X (Example 2.1). Let U_n be the set $\{f: \sup_{t\in X}|f(t)|\le n^{-1}\}(n\in N)$. It is clear that the fuzzy set N defined as $N(U_n)=n^{-1}(n=1,2,3,\ldots\ldots)$ is a balance neighborhood system of the zero element for a unique fuzzifying topology under which $C_K(X)$ is a fuzzifying topological linear space.

Example 2.3 Let $K[t]$ be the ring of polynomials $f[t]=\Sigma_n\alpha_nt^n$ over K in one indeterminate t. With multiplication to left multiplication by polynomials of degree 0, $K[t]$ becomes a linear space over K. Let r be a fixed real number such that $0 < r \leq 1$ and denote by V_ε the set of polynomials for which $\Sigma_n|\alpha_n|^r \leq \varepsilon(\varepsilon > 0)$. Consider the fuzzy subset N defined as follows :

$$N(V_\varepsilon) = \begin{cases} 1, & \text{if } \ \varepsilon \geq 1, \\ \varepsilon, & \text{if } \ \varepsilon < 1, \end{cases}$$

then it is clear that N is a balance neighborhood system of the zero element for a unique fuzzifying topology under which $K[t]$ is a fuzzifying topological linear space.

Definition 2.6[9] *Let Σ be a class of fuzzifying topological spaces. A unary predicate $T_3 \in \mathcal{F}(\Sigma)$, called fuzzy regular, is given as follows:*

$$T_3(X,\tau) := (\forall x)(\forall A)((x \in A) \wedge (A \in \tau) \longrightarrow (\exists B)((B \in N_x) \wedge (\bar{B} \subseteq A))),$$

Theorem 2.4 *Let (X,τ) be a fuzzifying topological linear space, then $\models T_3(X,\tau)$.*

Proof For any x, A and $x \in A$,

$$\tau(A) \leq N_x(A) = N_{x-\theta}(A) \leq \sup_{B-C\subseteq A} \min(N_x(B), N_\theta(C)) = \sup_B \sup_{B-C\subseteq A} \min(N_x(B),$$

$$N_\theta(C)) = \sup_B \min(N_x(B), \sup_{B-C\subseteq A} N_y(C+y)).$$

Since $\forall y \in A^C, (A^C + C) \cap B = \Phi$ (if $(A^C + C) \cap B \neq \Phi$, there exist $z \in A^C, c \in C$ and $b \in B$ such that $z = b - c \in B - C \subseteq A$. This contradicts the fact that $z \in A^C$), which means that $(A^C + C) \subseteq B^C$).

Hence,

$$\forall y \in A^C, N_y(y+C) \leq N_y(A^C + C) \leq N_y(B^C), \sup_{B-C\subseteq A} N_y(y+C) \leq \inf_{y\in A^C} N_y(B^C),$$

which implies that

$$\tau(A) \leq \sup_B \min(N_x(B), \inf_{y\in A^C} N_y(B^C)), \qquad [A \in \tau] \leq [(\exists B)((B \in N_x) \wedge (\bar{B} \subseteq A)].$$

This completes the proof.

3. Fuzzifying convex sets

Definition 3.1 *Let X be a linear space. A unary fuzzy predicate $\vartheta \in \mathcal{F}(\mathcal{F}(X))$, called fuzzy convex, is given as follows:*

$$A \in \vartheta := (\forall x)(\forall y)((x \in A) \wedge (y \in A) \longrightarrow (\lambda x + (1-\lambda)y) \in A), \qquad \forall \lambda \in [0,1].$$

Example 3.1 Let $X = R^2, A \in \mathcal{F}(R^2)$ and

$$A((x,y)) = \begin{cases} 1, & \text{if } \ x = y = 0 \\ \alpha - \frac{1}{2}, & \text{if } \ x > 0, y = \alpha x, \frac{1}{2} \leq \alpha \leq 1, \\ 1 - \frac{\alpha}{2}, & \text{if } \ x > 0, y = \alpha x, 1 \leq \alpha \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

By routine calculations it follows that $[A\in\vartheta]=1$.

Theorem 3.1 *Let X be a linear space, $A\in\mathcal{F}(X)$, and $x_0\in X$, then*

$$\models A\in\vartheta\longleftrightarrow x_0+A\in\vartheta.$$

Proof

$$\begin{aligned}[A\in\vartheta]&=\inf_{x,y}\min(1,1-\min(A(x),A(y))+A(\lambda x+(1-\lambda)y))\\&=\inf_{x,y}\min(1,1-\min((A+x_0)(x+x_0),(A+x_0)(y+x_0))+\\&\quad(A+x_0)(\lambda(x+x_0)+(1-\lambda)(y+x_0)))\\&=\inf_{x_1,y_1}\min(1,1-\min((A+x_0)(x_1),(A+x_0)(y_1))+(A+x_0)(\lambda(x_1)+(1-\lambda)(y_1)))\\&=[x_0+A\in\vartheta].\end{aligned}$$

Theorem 3.2 *Let X be a linear space. For any $a_1,a_2\in R_+$, we have*

$$\models A\in\vartheta\longleftrightarrow(a_1+a_2)A\equiv a_1A+a_2A.$$

Proof

$$\begin{aligned}[(a_1+a_2)A\equiv a_1A+a_2A]&=\inf_{z\in X}(1-|(a_1+a_2)A(z)-(a_1A+a_2A)(z)|)\\&=\inf_{z\in X}(1-\sup_{a_1x+a_2y=z}|A(\frac{a_1x+a_2y}{a_1+a_2})-\min(A(x),A(y))|)\\&=\inf_{x,y\in A}(1-|\min(A(x),A(y))-A(\lambda x+(1-\lambda)y)|)\\&=[A\in\vartheta].\ \ (\lambda=\frac{a_1}{a_1+a_2}).\end{aligned}$$

Theorem 3.3 *Let X be a linear space, $A,B,A_\lambda\in\mathcal{F}(\mathcal{X})(\lambda\in *)$, then*

(1) $\models A\in\vartheta\longrightarrow aA\in\vartheta\quad\forall a\in R_+$;

(2) $\models(A\in\vartheta)\wedge(B\in\vartheta)\longrightarrow((A+B)\in\vartheta)$;

(3) $\models(\forall\kappa)((\kappa\in\Lambda)\longrightarrow\vartheta(A_\kappa))\longrightarrow\vartheta(\bigcap_{\kappa\in\Lambda}A_\kappa)$.

Proof (1) The proof is easy.

(2)

$$\begin{aligned}[(A+B)\in\vartheta]&=\inf_{x,y}\min(1,1-\min((A+B)(x),(A+B)(y))+(A+B)(\lambda x+(1-\lambda)y))\\&\geq\inf_{x,y}\min(1,1-\min(\sup_{x_1+x_2=x}\min(A(x_1),B(x_2)),\sup_{y_1+y_2=y}\min(A(y_1),B(y_2)))+\\&\quad\sup_{x_1+x_2=x,y_1+y_2=y}(A+B)((\lambda x_1+(1-\lambda)y_1)+(\lambda x_2+(1-\lambda)y_2)))\\&\geq\min(1,1-\sup_{x_1,x_2,y_1,y_2}\min(\min(A(x_1),A(y_1)),\min(B(x_2),B(y_2)))+\\&\quad\sup_{x_1,x_2,y_1,y_2}\min(A(\lambda x_1+(1-\lambda)y_1),B(\lambda x_2+(1-\lambda)y_2)))\\&\geq\min(\inf_{x_1,y_1}\min(1,1-\min(A(x_1),A(y_1))+\\&\quad A(\lambda x_1+(1-\lambda)y_1)),\inf_{x_2,y_2}\min(1,1-\min(B(x_2),B(y_2))+\\&\quad B(\lambda x_2+(1-\lambda)y_2)))=[(A\in\vartheta)\wedge(B\in\vartheta)]\ \ (\lambda\in[0,1]).\end{aligned}$$

(3)

$$\begin{aligned}
[\vartheta(\bigcap_{\kappa\in\Lambda} A_\kappa)] &= \inf_{x,y}\min(1, 1-\min((\bigcap_{\kappa\in\Lambda} A_\kappa)(x), (\bigcap_{\kappa\in\Lambda} A_\kappa)(y)) + (\bigcap_{\kappa\in\Lambda} A_\kappa)(\lambda x+(1-\lambda)y))\\
&= \inf_{x,y}\min(1, 1-\min(\inf_{\kappa\in\Lambda} A_\kappa(x), \inf_{\kappa\in\Lambda} A_\kappa(y)) + \inf_{\kappa\in\Lambda} A_\kappa(\lambda x+(1-\lambda)y))\\
&\geq \inf_{x,y}\min(1, 1-\inf_{\kappa\in\Lambda}\min(A_\kappa(x), A_\kappa(y)) + \inf_{\kappa\in\Lambda} A_\kappa(\lambda x+(1-\lambda)y))\\
&\geq \inf_{\kappa\in\Lambda}\inf_{x,y}\min(1, 1-\min(A_\kappa(x), A_\kappa(y)) + A_\kappa(\lambda x+(1-\lambda)y))\\
&= \inf_{\kappa\in\Lambda}[\vartheta(A_\kappa)] = [(\forall\kappa)((\kappa\in\Lambda)\longrightarrow(\vartheta(A_\kappa)))].
\end{aligned}$$

Theorem 3.4 *Let X and Y be two linear spaces, $A, B \in \mathcal{F}(X\times Y)$, and $C = \{(y,z)|y\in X, z\in Y, (y,z_1)\in A, (y,z_2)\in B, z_1+z_2=z\}$, then $\models (A\in\vartheta)\wedge(B\in\vartheta)\longrightarrow C\in\vartheta$.*

Proof

$$\begin{aligned}
[C\in\vartheta] &= \inf_{y_1,y_2,z_1,z_2}\min(1, 1-\min(C(y_1,z_1), C(y_2,z_2)) + C(\lambda(y_1,z_1)+(1-\lambda)(y_2,z_2)))\\
&= \inf_{y_1,y_2,z_1,z_2}\min(1, 1-\min(C(y_1,z_1^1+z_1^2), C(y_2,z_2^1+z_2^2))+\\
&\quad C(\lambda(y_1,z_1^1+z_1^2)+(1-\lambda)(y_2,z_2^1+z_2^2)))\\
&= \inf_{y_1,y_2,z_1,z_2}\min(1, 1-\min(\min(A(y_1,z_1^1), B(y_1,z_1^2)), \min(A(y_2,z_2^1), B(y_2,z_2^2)))+\\
&\quad \min(A(\lambda y_1+(1-\lambda)y_2, \lambda z_1^1+(1-\lambda)z_2^1), B(\lambda y_1+(1-\lambda)y_2, \lambda z_1^2+(1-\lambda)z_2^2)))\\
&\geq \min(\inf_{y_1,y_2,z_1,z_2}\min(1, 1-\min(A(y_1,z_1^1), A(y_2,z_2^1))+\\
&\quad A(\lambda y_1+(1-\lambda)y_2, \lambda z_1^1+(1-\lambda)z_2^1)),\\
&\quad \inf_{y_1,y_2,z_1,z_2}\min(1, 1-\min(B(y_1,z_1^2), B(y_2,z_2^2)) + B(\lambda y_1+(1-\lambda)y_2, \lambda z_1^2+(1-\lambda)z_2^2)))\\
&= \min(\inf_{y_1,y_2,z_1,z_2}\min(1, 1-\min(A(y_1,z_1^1), A(y_2,z_2^1)) + A(\lambda(y_1,z_1^1)+(1-\lambda)(y_2,z_2^1))),\\
&\quad \inf_{y_1,y_2,z_1,z_2}\min(1, 1-\min(B(y_1,z_1^2), B(y_2,z_2^2)) + B(\lambda(y_1,z_1^2)+(1-\lambda)(y_2,z_2^2))))\\
&= [(A\in\vartheta)\wedge(B\in\vartheta)].
\end{aligned}$$

Theorem 3.5 *Let X and Y be two linear spaces and $f: X\longrightarrow Y$ a linear map. If $A\in\mathcal{F}(X)$ and $B\in\mathcal{F}(Y)$, then*

(1) $\models \vartheta(A)\longrightarrow\vartheta(f(A))$; *(2)* $\models \vartheta(B)\longleftrightarrow\vartheta(f^{-1}(B))$.

Proof (1)

$$\begin{aligned}
[\vartheta(f(A))] &= \inf_{x_1,x_2\in Y}\min(1, 1-\min(f(A)(y_1), f(A)(y_2)) + f(A)(\lambda y_1+(1-\lambda)y_2))\\
&= \inf_{y_1,y_2\in Y}\min(1, 1-\min(\sup_{x_1\in f^{-1}(y_1)} A(x_1), \sup_{x_2\in f^{-1}(y_2)} A(x_2))+\\
&\quad \sup_{x_1\in f^{-1}(y_1), x_2\in f^{-1}(y_2)} A(\lambda x_1+(1-\lambda)x_2))\\
&\geq \inf_{y_1,y_2\in Y}\inf_{x_1\in f^{-1}(y_1), x_2\in f^{-1}(y_2)}\min(1, 1-\min(A(x_1), A(x_2)) + A(\lambda x_1+(1-\lambda)x_2))\\
&= \inf_{x_1,x_2\in X}\min(1, 1-\min(A(x_1), A(x_2)) + A(\lambda x_1+(1-\lambda)x_2)) = [\vartheta(A)].
\end{aligned}$$

(2)

$$\begin{aligned}[\vartheta(f^{-1}(B))] &= \inf_{x_1,x_2\in X} \min(1, 1-\min(f^{-1}(B)(x_1), f^{-1}(B)(x_2)) + f^{-1}(B)(\lambda x_1 + (1-\lambda)x_2)) \\ &= \inf_{x_1,x_2\in X} \min(1, 1-\min(B(f(x_1)), B(f(x_2))) + B(f(\lambda x_1 + (1-\lambda)x_2))) \\ &= \inf_{x_1,x_2\in X} \min(1, 1-\min(B(f(x_1)), B(f(x_2))) + B(\lambda f(x_1) + (1-\lambda)f(x_2))) \\ &= \inf_{y_1,y_2\in Y} \min(1, 1-\min(B(y_1), B(y_2)) + B(\lambda y_1 + (1-\lambda)y_2)) = [\vartheta(B)].\end{aligned}$$

Theorem 3.6 *Let (X,τ)be a fuzzifying topological linear spaces, then* $\models A\in\vartheta \longrightarrow \bar{A}\in\vartheta$.

Proof We only need to prove that, for any $x, y \in X, \bar{A}(\lambda x + (1-\lambda)y) \geq \min(\bar{A}(x), \bar{B}(y))$.

In fact,from $N_x^A(\Phi) = \sup_{B\cap A=\Phi} N_x(B) = \sup_{B\subseteq A^C} N_x(B) = N_x(A^C)$ we have

$$\begin{aligned}\bar{A}(\lambda x + (1-\lambda)y) &= 1 - N_{\lambda x+(1-\lambda)y}(A^C) = 1 - N^A_{\lambda x+(1-\lambda)y}(\Phi) \\ &\geq 1 - \sup_{\lambda B+(1-\lambda)C\subseteq\Phi} \min(N^A_{\lambda x}(\lambda B), N^A_{(1-\lambda)y}((1-\lambda)C)) \\ &= 1 - \sup_{\lambda B+(1-\lambda)C\subseteq\Phi} \min(N^A_x(B), N^A_y(C)) \geq 1 - \max(N^A_x(\Phi), N^A_y(\Phi)) \\ &= 1 - \max(N_x(A^C), N_y(A^C)) = \min(1-N_x(A^C), 1-N_y(A^C)) \\ &= \min(\bar{A}(x), \bar{A}(y)).\end{aligned}$$

Theorem 3.7 *Let (X,τ) be a fuzzifying topological linear spaces, then* $\models A\in\vartheta \longrightarrow \mathring{A}\in\vartheta$.

Proof Notice that, for any $x, y \in X$,

$$\begin{aligned}&\min(\mathring{A}(x), \mathring{A}(y)) = \min(N_x(A), N_y(A)) = \min(N_\theta(A-x), N_\theta(A-y)) \\ &= \min(\sup_{\theta\in aB\subseteq A-x} \tau(aB), \sup_{\theta\in bB\subseteq A-y} \tau(bB)) \leq \min(\sup_{x\in x+aB\subseteq A} \tau(B), \sup_{y\in y+bB\subseteq A} \tau(B)) \\ &\leq \sup_{x\in x+aB\subseteq A, y\in y+bB\subseteq A} \tau(B) \leq \sup_{\lambda x+(1-\lambda)y\in\lambda x+(1-\lambda)y+(\lambda a+(1-\lambda)b)B\subseteq A} \tau(B) \\ &\leq \sup_{\lambda x+(1-\lambda)y\in\lambda x+(1-\lambda)y+(\lambda a+(1-\lambda)b)B\subseteq A} \tau(\lambda x + (1-\lambda)y + (\lambda a + (1-\lambda)b)B) \\ &= N_{\lambda x+(1-\lambda)y}(A) = \mathring{A}(x + (1-\lambda)y).\end{aligned}$$

Therefore, $\min(1, 1-\min(\mathring{A}(x), \mathring{A}(y)) + \mathring{A}(\lambda x + (1-\lambda)y)) = 1$.

Theorem 3.8 *Let (X,τ)be a fuzzifying topological linear spaces, then* $\models A\in\vartheta \longrightarrow (y\in \bar{A}\wedge x\in\mathring{A} \longrightarrow (1-\lambda)x + \lambda y \in \mathring{A})$.

Proof Applying Corollary 2.2(1) and (3) we have

$$[y\in\bar{A}] = \inf_B \min(1, 1-N_\theta(B) + [y\in A+B]), [x\in\mathring{A}] = \sup_B \min(N_\theta(B), [x+B\subseteq A]).$$

If $[B\in N_\theta] = 0$, the result holds. Now, we suppose $[B\in N_\theta] = 1$. Then for all $0 < \varepsilon < 1, [y\in\bar{A}] = [y\in A+\varepsilon B] \leq [(1-\lambda)x + \lambda y + \varepsilon B \subseteq (1-\lambda)x + \lambda(A+\varepsilon B) + \varepsilon B] = [(1-\lambda)x + \lambda y + \varepsilon B \subseteq (1-\lambda)(x + \varepsilon(1+\lambda)(1-\lambda)^{-1}B) + \lambda A]$ and for some sufficiently small $\varepsilon > 0, [x\in\mathring{A}] = [x + \varepsilon(1+\lambda)(1-\lambda)^{-1}B \subseteq A]$.

Therefore,for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}&[(1-\lambda)x+\lambda y+\varepsilon B\subseteq A]\\ &\quad\geq[(1-\lambda)x+\lambda y+\varepsilon B\subseteq(1-\lambda)(x+\varepsilon(1+\lambda)(1-\lambda)^{-1})B+\lambda A]+\\ &\qquad[(1-\lambda)(x+\varepsilon(1+\lambda)(1-\lambda)^{-1}B+\lambda A\subseteq(1-\lambda)A+\lambda A]-1\geq[(y\in\bar{A})\owedge(x\in\mathring{A})],\end{aligned}$$

i,e.,

$$[(1-\lambda)x+\lambda y\in\mathring{A})]=\min(N_\theta(B),[(1-\lambda)x+\lambda y+\varepsilon B\subseteq A])\geq[(y\in\bar{A})\owedge(x\in\mathring{A})].$$

Theorem 3.9 *Let (X,τ) be a fuzzifying topological linear spaces, $z\in X$, then*

$$\models A\in\vartheta\longrightarrow(\forall x(x\in\mathring{A}\longrightarrow((1-\mu)x+\mu z\in A\longrightarrow z\in\mathring{A})))\quad(\mu>1).$$

Proof By Theorem 3.8, we have

$$\begin{aligned}&[\forall x((x\in\mathring{A})\otimes((1-\mu)x+\mu z\in A))]=\inf_x\max(0,\mathring{A}(x)+A((1-\mu)x+\mu z)-1)\\ &\quad\leq\inf_x\max(0,\mathring{A}(x)+\bar{A}((1-\mu)x+\mu z)-1)=\inf_x\max(0,\mathring{A}(x)+\bar{A}(y)-1)\\ &\quad\leq\mathring{A}((1-\lambda)x+\lambda y)=\mathring{A}(z),\end{aligned}$$

where $y=(1-\mu)x+\mu z,\lambda=\mu^{-1}$.

Theorem 3.10 *Let (X,τ)be a fuzzifying topological linear spaces and $z\in X$, then $\models A\in\vartheta\longrightarrow(\bar{A}\equiv\bar{\mathring{A}})$.*

Proof Firstly, we know that $[\bar{\mathring{A}}\subseteq\bar{A}]=1$ for any $A\in\mathcal{F}(\mathcal{X})$. Secondly, we will show that $\models A\in\vartheta\longrightarrow(\bar{A}\subseteq\bar{\mathring{A}})$. If $[x\in\bar{A}]>t$, from Theorem 6.1(2)[2(1)], there exists a net S^* such that $[(S^*\subseteq A)\wedge(S^*\rhd x)]>t$. Now, we take x_{S^*} from S^*, and set S^{**} as follows:

$$S^{**}=\{(1-\lambda)x_{S^*}+\lambda y|\quad x_{S^*}\in S^*,y\in\mathring{A}\}.$$

It is clear that S^{**} is also a net and $[(1-\lambda)x_{S^*}+\lambda y\in\mathring{A}]\geq[x_{S^*}\in A]$ from Theorem 3.8. In other words, $[S^{**}\subseteq\mathring{A}]\geq[S^*\subseteq A]$.

Since $S^{**}\not\mathrel{\widetilde{\subseteq}}A$ implies $S^*\not\mathrel{\widetilde{\subseteq}}A$, where $\widetilde{\subseteq}$ is the binary crisp predicates "almost in", we have

$$[S^{**}\rhd x]=1-\sup_{S^{**}\not\widetilde{\subseteq}B}N_x(B)\geq1-\sup_{S^{*}\not\widetilde{\subseteq}B}N_x(B)=[S^*\rhd x].$$

Therefore,

$$[x\in\bar{\mathring{A}}]=[\exists S((S\subseteq\mathring{A})\wedge(S\rhd x))]\geq[(S^{**}\subseteq\mathring{A})\wedge(S^{**}\rhd x)]\geq[(S^*\subseteq\mathring{A})\wedge(S^*\rhd x)]>t.$$

Since t is arbitrary, it holds that $[A\in\vartheta\longrightarrow(\bar{A}\subseteq\bar{\mathring{A}})]=1$.

References:

[1] ROSSER J B, TURQUETTE A R. *Many-Valued Logic* [M]. Amsterdam: North-Holland, 1952.

[2] YING Ming-sheng. *A new approach for fuzzy topology (I), (II), (III)* [J]. Fuzzy Sets and Systems, 1991, **39**: 303–321; 1992, **47**: 221–232; 1993, **55**: 193–207.

[3] KATSARAS A K, LIU D B. *Fuzzy vector spaces and fuzzy topological vector spaces* [J]. J. Math. Anal. Appl., 1977, **58**: 135–146.

[4] LOWEN R. *Fuzzy topological spaces and compactness* [J]. J. Math. Anal. Appl., 1976, **56**: 621–633.

[5] WU Cong-xin. *On the fuzzy topological linear space (I)* [J]. Fuzzy Math., 1981, **1**: 1–14. (in Chinese)

[6] PU Pao-ming, LIU Ying-ming. *Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence* [J]. J. Math. Anal. Appl., 1980, **76**: 571–599.

[7] YING Ming-sheng. *On the method of neighborhood systems in fuzzy topology* [J]. Fuzzy Sets and Systems, 1994, **68**: 227–238.

[8] YING Ming-sheng. *Fuzzifying topology based on complete residuated lattice-valued logic (I)* [J]. Fuzzy Sets and Systems, 1993, **56**: 337–373.

[9] SHEN Ji-zhong. *Separation axiom in fuzzifying topology* [J]. Fuzzy Sets and Systems, 1993, **57**: 111–123.

[10] SHEN Ji-zhong. *Fuzzifying groups based on complete residuated lattice-valued logic* [J]. Inform. Sci., 1993, **75**: 165–186.

[11] PAVELKA J. *On fuzzy logic 1,2,3* [J]. Z. fur Mathematik Logic u Grundlagen d Mathematic, 1979, **25**: 45–52, 119–134, 447–464.

[12] CHANG C L. *Fuzzy topological spaces* [J]. J. Math. Anal. Appl., 1968, **24**: 182–190.

基于连续值逻辑上的不分明化拓扑线性空间

张广济， 邹开其， 张 成

(大连大学信息工程学院，大连大学信息科学与工程辽宁省高校重点实验室，辽宁 大连 116622)

摘要: 在连续逻辑的语义框架下，我们给出了不分明化拓扑线性空间的概念，讨论了零元平衡邻域系的结构和性质，并在不分明化拓扑线性空间中给出不分明化凸集的代数与拓扑性质.

关键词: 连续值逻辑；不分明化拓扑线性空间；平衡集；零元邻域系；凸集.