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Derivations of Certain Lie Algebras of Upper Triangular Matrices over Commutative Rings

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Abstract: Let R be an arbitrary commutative ring with identity. Denote by $\mathbf{t}$ the Lie algebra over R consisting of all upper triangular n by n matrices and let $\mathbf{b}$ be the Lie subalgebra of $\mathbf{t}$ consisting of all matrices of trace 0. The aim of this paper is to give an explicit description of the derivation algebras of the Lie algebras $\mathbf{t}$ and $\mathbf{b}$, respectively.

Key words: derivations of Lie algebras, commutative rings.
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1. Introduction

Let R be a commutative ring with identity and R^* the group of invertible elements of R. Let $M_n(R)$ be the R-algebra of n by n matrices over R that has a structure of a Lie algebra over R with the bracket $[x,y]=xy-yx$. We denote by $\mathbf{t}$ (resp., $\mathbf{n}$) the subset of $M_n(R)$ consisting of all upper triangular (resp., strictly upper triangular) matrices. When $n>1$, let $\mathbf{b}$ be the subset of $\mathbf{t}$ consisting of all matrices of trace 0. Cao[1−3] described the automorphism groups of $\mathbf{t},\mathbf{n}$ and $\mathbf{b}$ respectively, when they are viewed as Lie algebras. Jφndrup[4] gave a complete description of the derivations of $\mathbf{t}$, when $\mathbf{t}$ is viewed as a ring.

2. Preliminaries

Following the notations in [1] mainly, we denote by E the identity matrix in $M_n(R)$ and by E_{ij} the matrix in $M_n(R)$ whose sole nonzero entry is 1 in the (i,j) position. Let L denote $\mathbf{t}$ or $\mathbf{b}$ and let Der$L$ be the derivation Lie algebra of the Lie algebra L. Let

$$\mathbf{n_1}=\mathbf{n},\quad \mathbf{n_2}=[\mathbf{n},\mathbf{n_1}],\quad \mathbf{n_3}=[\mathbf{n},\mathbf{n_2}],\ldots$$

be the lower central series of $\mathbf{n}$. Every $\mathbf{n_k}$ is a characteristic ideal of $\mathbf{t}$, which is stable under the action of any derivation of $\mathbf{t}$. Let RE be the set $\{rE\mid r\in R\}$ of scarlar matrices in $\mathbf{t}$. We denote by $Z(L)$ the center of the Lie algebra L. The following three lemmas are trivial.

Lemma 2.1 *If $n>1$, and $n\in R^*$, then both $\mathbf{b}$ and RE are ideals of $\mathbf{t}$ and $\mathbf{t}=\mathbf{b}\oplus RE$*[1].

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Lemma 2.2 *Let* $\mathbf{d}$ *be the subset of* $\mathbf{t}$ *consisting of all diagonal matrices in* $\mathbf{t}$. *Then* $\mathbf{d}$ *is a Lie subalgebra of* $\mathbf{t}$ *and* $\mathbf{t} = \mathbf{d} \oplus \mathbf{n}$.

We denote by $\mathrm{Hom}_R(\mathbf{d}, R)$ the set consisting of all homomorphisms $\sigma : \mathbf{d} \to R$ of R-modules. It forms a new R-module. If $1 \leq i \leq n$, then $\chi_i : \mathbf{d} \to R$, defined by $\chi_i(\mathrm{diag}(d_1, d_2, \ldots, d_n)) = d_i$, is a standard homomorphism in $\mathrm{Hom}_R(\mathbf{d}, R)$.

Lemma 2.3 $\mathrm{Hom}_R(\mathbf{d}, R)$ *is a free* R*-module of rank* n *with a basis:* $\chi_1, \chi_2, \ldots, \chi_n$. *In other words,*

$$\mathrm{Hom}_R(\mathbf{d}, R) = R\chi_1 \oplus R\chi_2 \oplus \cdots \oplus R\chi_n.$$

3. The Standard Derivations of t

We now construct some standard derivations as follows.

(A) Central derivations

Any homomorphism $\sigma : \mathbf{d} \to R$ of R-modules may be extended to a derivation σ' of the Lie algebra $\mathbf{t}$ by:

$$\sigma'(D + x) = \sigma(D)E,$$

for all $D \in \mathbf{d}, x \in \mathbf{n}$. σ' is called a central derivation of $\mathbf{t}$ induced by σ. Let Φ denote the set consisting of all central derivations of $\mathbf{t}$. Then Φ forms an R-submodule of $\mathrm{Der}\mathbf{t}$.

Lemma 3.1 Φ *is a Lie subalgebra of* $\mathrm{Der}\mathbf{t}$.

Proof Let

$$\sigma_1 = r_1\chi_1 + r_2\chi_2 + \cdots + r_n\chi_n \in \mathrm{Hom}_R(\mathbf{d}, R),$$

and

$$\sigma_2 = s_1\chi_1 + s_2\chi_2 + \cdots + s_n\chi_n \in \mathrm{Hom}_R(\mathbf{d}, R).$$

Denote $\sum_{i=1}^{n} r_i$ by r, $\sum_{i=1}^{n} s_i$ by s, and $rs_i - sr_i$ by p_i. Let σ denote $\sum_{i=1}^{n} p_i\chi_i$. Then we have that $[\sigma_1', \sigma_2'] = \sigma' \in \Phi$. We are done.

(B) Inner derivations

Let $T \in \mathbf{t}$, then $ad\ T : x \to [T, x], x \in \mathbf{t}$, is a derivation of $\mathbf{t}$, called the inner derivation of $\mathbf{t}$ induced by T. Let $\mathrm{ad}\mathbf{t}$ denote the set consisting of all $\mathrm{ad}T$, with $T \in \mathbf{t}$, which forms an ideal of $\mathrm{Der}\mathbf{t}$. We see that $\mathrm{ad}\mathbf{t}$ is isomorphic to the quotient Lie algebra of $\mathbf{t}$ to $Z(\mathbf{t})$.

Lemma 3.2 *(1)* $\mathrm{ad}\mathbf{t} \subseteq \mathbf{\Phi}$, *when* $n = 1$; *(2)* $\Phi \cap \mathrm{ad}\mathbf{t} = \mathbf{0}$, *when* $n \geq 2$.

Proof If $n = 1$, we see that $\mathbf{t} = RE$. Then $\mathrm{ad}\mathbf{t} = \mathbf{0} \subseteq \mathbf{\Phi}$. If $n \geq 2$, let $\sigma' = \mathrm{ad}T \in \Phi \cap \mathrm{ad}\mathbf{t}$, for some $T \in \mathbf{t}$, where $\sigma : \mathbf{d} \to R$ is a homomorphism of R-modules. Then $\sigma'(\mathbf{n}) = (\mathrm{ad}T)(\mathbf{n}) = [T, \mathbf{n}] = 0$, forcing $T \in Z(\mathbf{t})$. Thus we have that $\sigma' = \mathrm{ad}T = 0$.

4. The Derivation Algebra of t

If $n > 1$, for $1 \leq k \leq n-1$, we assume that $n = kq + r$ with q and r two non-negative integer numbers and $r \leq k-1$. Let $D_k = \mathrm{diag}(E_k, 2E_k, \ldots, qE_k, (q+1)E_r) \in \mathbf{d}$, $k = 1, 2, \ldots, n-1$ where E_k denotes the $n \times n$ identity matrix.

Theorem 4.1 *Let R be an arbitrary commutative ring with identity. Then*

(1) $\mathrm{Der}\mathbf{t} = \Phi$, *when* $n = 1$.

(2) $\mathrm{Der}\mathbf{t} = \Phi \oplus \mathrm{ad}\mathbf{t}$, *when* $n \geq 2$.

Proof If $n = 1$, it is obvious that $\mathrm{Der}\mathbf{t} = \Phi$. From now on, we assume that $n > 1$. Let π denote the set $\Phi \oplus \mathrm{ad}\mathbf{t}$. For any $\varphi \in \mathrm{Der}\mathbf{t}$, we will show that $\varphi \in \pi$.

Firstly, we will prove that there exists some $T \in \mathbf{t}$ such that

$$(\mathrm{ad}T + \varphi)(\mathbf{d}) \subseteq \mathbf{d}.$$

For any $H \in \mathbf{d}$, suppose that

$$\varphi(H) \equiv (\sum_{1 \leq i < j \leq n} a_{ij}(H) E_{ij}) (\mathrm{mod}\ \mathbf{d}),$$

where $a_{ij}(H) \in R$ is relative to H. By $[D_1, H] = 0$, we have that

$$[H, \varphi(D_1)] = [D_1, \varphi(H)],$$

which follows that

$$\sum_{1 \leq i < j \leq n} (\chi_i(H) - \chi_j(H)) a_{ij}(D_1) E_{ij} = \sum_{1 \leq i < j \leq n} (\chi_i(D_1) - \chi_j(D_1)) a_{ij}(H) E_{ij}.$$

This yields that $(\chi_i(H) - \chi_j(H)) a_{ij}(D_1) = (\chi_i(D_1) - \chi_j(D_1)) a_{ij}(H)$, for any $1 \leq i < j \leq n-1$. In particular, we have that

$$a_{i,i+1}(H) = (\chi_{i+1}(H) - \chi_i(H)) a_{i,i+1}(D_1), \ i = 1, 2, \ldots, n-1.$$

Let $T_1 = \sum_{i=1}^{n-1} a_{i,i+1}(D_1) E_{i,i+1}$. Then $(\varphi - \mathrm{ad}T_1)(\mathbf{d}) \subseteq \mathbf{d} + \mathbf{n_2}$. By replacing φ with $\varphi - \mathrm{ad}T_1$, then we may assume that $\varphi(\mathbf{d}) \subseteq \mathbf{d} + \mathbf{n_2}$. If $n = 2$, this step is completed. If $n > 2$, for any $H \in \mathbf{d}$, we now suppose that

$$\varphi(H) \equiv (\sum_{1 \leq i < j \leq n-1} b_{i,j+1}(H) E_{i,j+1}) (\mathrm{mod}\ \mathbf{d}),$$

where $b_{i,j+1}(H) \in R$ is relative to H. By $[D_2, H] = 0$, we have that $[H, \varphi(D_2)] = [D_2, \varphi(H)]$ which follows that

$$\sum_{1 \leq i < j \leq n-1} (\chi_i(H) - \chi_{j+1}(H)) b_{i,j+1}(D_2) E_{i,j+1} = \sum_{1 \leq i < j \leq n-1} (\chi_i(D_2) - \chi_{j+1}(D_2)) b_{i,j+1}(H) E_{i,j+1}.$$

This yields that

$$(\chi_i(H) - \chi_{j+1}(H)) b_{i,j+1}(D_2) = (\chi_i(D_2) - \chi_{j+1}(D_2)) b_{i,j+1}(H),$$

for any $1 \leq i < j \leq n-1$. In particular, we have that

$$b_{i,i+2}(H) = (\chi_{i+2}(H) - \chi_i(H))b_{i,i+2}(D_2), \quad i = 1, 2, \ldots, n-2.$$

Let $T_2 = \sum_{i=1}^{n-2} b_{i,i+2}(D_2)E_{i,i+2}$. Then $(\varphi - \mathrm{ad}T_2)(\mathbf{d}) \subseteq \mathbf{d} + \mathbf{n_3}$. By replacing φ with $\varphi - \mathrm{ad}T_2$, then we may assume that $\varphi(\mathbf{d}) \subseteq \mathbf{d} + \mathbf{n_3}$. If $n = 3$, this step is completed. If $n > 3$, we repeat above replacement. After $n-2$ steps, we may assume that $\varphi(\mathbf{d}) \subseteq \mathbf{d} + \mathbf{n}_{n-1}$. For any $H \in \mathbf{d}$, suppose that $\varphi(H) \equiv c_{1,n}(H)E_{1,n}(\mathrm{mod}\mathbf{d})$, where $c_{1,n}(H) \in R$ is relative to H. By $[D_{n-1}, H] = 0$, we have that $[H, \varphi(D_{n-1})] = [D_{n-1}, \varphi(H)]$, which follows that

$$(\chi_1(H) - \chi_n(H))c_{1,n}(D_{n-1}) = (\chi_1(D_{n-1}) - \chi_n(D_{n-1}))c_{1,n}(H).$$

So we have that

$$c_{1,n}(H) = (\chi_n(H) - \chi_1(H))c_{1,n}(D_{n-1}).$$

Let $T_n = c_{1,n}(D_{n-1})E_{1,n}$. Then $(\varphi - \mathrm{ad}T_n)(\mathbf{d}) \subseteq \mathbf{d}$. By replacing φ with $\varphi + \mathrm{ad}T_n$, then we may assume that $\varphi(\mathbf{d}) \subseteq \mathbf{d}$.

Secondly, we will prove that there exists some $D \in \mathbf{d}$ such that $(\varphi + \mathrm{ad}D)(E_{i,i+1}) = 0$, for $i = 1, 2, \ldots, n-1$, under the assumption that $\varphi(\mathbf{d}) \subseteq \mathbf{d}$.

For $1 \leq i \leq n-1$, suppose that

$$\varphi(E_{i,i+1}) = \sum_{1 \leq k < l \leq n} x_{kl}^{(i)} E_{kl}$$

with $x_{kl}^{(i)} \in R$. Let $D \in \mathbf{d}$. By applying φ on the the two sides of

$$[D, E_{i,i+1}] = (\chi_i(D) - \chi_{i+1}(D))E_{i,i+1},$$

we have that

$$[\varphi(D), E_{i,i+1}] + [D, \varphi(E_{i,i+1})] = (\chi_i(D) - \chi_{i+1}(D))\varphi(E_{i,i+1}).$$

It follows that

$$(\chi_i - \chi_{i+1})(\varphi(D))E_{i,i+1} + \sum_{1 \leq k < l \leq n} x_{kl}^{(i)}(\chi_k(D) - \chi_l(D))E_{kl} = (\chi_i(D) - \chi_{i+1}(D)) \sum_{1 \leq k < l \leq n} x_{kl}^{(i)} E_{kl}.$$

So

$$\chi_i(\varphi(D)) = \chi_{i+1}(\varphi(D)),$$

and

$$x_{kl}^{(i)}(\chi_k(D) - \chi_l(D)) = x_{kl}^{(i)}(\chi_i(D) - \chi_{i+1}(D)),$$

for any $1 \leq k < l \leq n$. If $(k, l) \neq (i, i+1)$, we may choose $D \in \mathbf{d}$ such that $\chi_i(D) = \chi_{i+1}(D)$ and $\chi_k(D) = \chi_l(D) + 1$, then we see that $x_{kl}^{(i)} = 0$. This implies that $\varphi(E_{i,i+1}) = r_i E_{i,i+1}$ for some $r_i \in R$. Let $D = \mathrm{diag}(0, r_1, r_1 + r_2, \ldots, \sum_{i=1}^{n-1} r_i)$. Then we have that $(\varphi + \mathrm{ad}D)(E_{i,i+1}) = 0$, $i = 1, 2, \ldots, n-1$. The fact that $\mathbf{n}$ is generated by all $E_{i,i+1}, i = 1, 2, \ldots, n-1$ forces that $(\varphi + \mathrm{ad}D)(\mathbf{n}) = \mathbf{0}$. By replacing φ with $\varphi + \mathrm{ad}D$, we may assume that $\varphi(\mathbf{n}) = \mathbf{0}$ and $\varphi(\mathbf{d}) \subseteq \mathbf{d}$.

Now we intend to prove that φ is a central derivation of $\mathbf{t}$. Let $D \in \mathbf{d}, 1 \le i \le n-1$. By applying φ on the two sides of

$$[D, E_{i,i+1}] = (\chi_i(D) - \chi_{i+1}(D))E_{i,i+1},$$

we have that $(\chi_i - \chi_{i+1})(\varphi(D)) = 0$. This means that $\varphi(D) = r_D E$ for a unique $r_D \in R$. Thus we get a homomorphism $\sigma : \mathbf{d} \to R$ of R-modules, defined by $\sigma(D) = r_D$. It is obvious that $\varphi(D + x) = \sigma(D)E$ for $x \in \mathbf{n}, \mathbf{D} \in \mathbf{d}$. Hence φ is the central derivation σ' of $\mathbf{t}$ induced by σ. □

5. The derivation algebra of b

We now use the result on the derivation algebra of $\mathbf{t}$ to discuss the derivations of the Lie subalgebra $\mathbf{b}$ of $\mathbf{t}$. In this section, we assume that $n > 1$ and $n \in R^*$. It is obvious that the restriction of an inner derivation of $\mathbf{t}$ to $\mathbf{b}$ is a derivation of $\mathbf{b}$, which is also called an inner derivation of $\mathbf{b}$. For a derivation φ of $\mathbf{t}$, we denote by $\varphi_{\mathbf{b}}$ the map $\mathbf{b} \to \mathbf{t}$ defined by $\varphi_{\mathbf{b}}(x) = \varphi(x)$, for all $x \in \mathbf{b}$.

Theorem 5.1 *Let $n > 1$ and let R be a commutative ring with identity in which $n \in R^*$. Then* $\mathrm{Der}\mathbf{b} = (\mathrm{ad}\mathbf{t})_{\mathbf{b}}$.

Proof For any $\psi \in Der\ \mathbf{b}$, ψ can be lifted to a derivation of $\mathbf{t}$, by acting trivially on RE. The lift of $\psi \in \mathrm{Der}\mathbf{b}$ to $\mathbf{t}$ is denoted by $\psi_{\mathbf{t}}$. By 4.1, we may assume that $\psi_{\mathbf{t}} = \mathrm{ad}T + \sigma'$, for suitable $T \in \mathbf{t}$ and $\sigma \in \mathrm{Hom}_R(\mathbf{d}, R)$. Then $\psi = (\psi_{\mathbf{t}})_{\mathbf{b}} = (\mathrm{ad}T + \sigma')_{\mathbf{b}} = (\mathrm{ad}T)_{\mathbf{b}} + (\sigma')_{\mathbf{b}}$. This means that $(\sigma')_{\mathbf{b}}$ is a derivation of $\mathbf{b}$, forcing $(\sigma')_{\mathbf{b}} = 0$. Hence $\psi = (\mathrm{ad}T)_{\mathbf{b}}$. □

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可换环上一些上三角矩阵李代数的导子

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摘要: 设 R 是任意含单位元的可换环， $\mathbf{t}$ 是 R 上 $n \times n$ 上三角矩阵组成的李代数， $\mathbf{b}$ 是 R 上迹为零的 $n \times n$ 上三角矩阵组成的李代数. 本文明确给出了 $\mathbf{t}$ 和 $\mathbf{b}$ 的导子代数.

关键词: 李代数的导子; 可换环.