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KKM Type Theorems and Coincidence Theorems in Topological Spaces

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Abstract A class of finitely continuous topological spaces (in short, FC-spaces) is introduced. Some new KKM type theorems and coincidence theorems involving admissible set-valued mappings and the set-valued mapping with compactly local intersection property are proved in FC-spaces. As applications, some new fixed point theorems are obtained in FC-spaces. These theorems improve and generalize many known results in recent literature.

Keywords FC-space; KKM type theorem; coincidence theorem; admissible set-valued map; compactly local intersection property.

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1. Introduction

In 1987, Horvath[1], replacing convex hulls by contractible sets, gave a purely topological version of the KKM theorem. Since then, Park and Kim[2] introduced the concepts of admissible set-valued mappings and generalized convex (or G-convex) spaces. Verma[3] introduced the concepts of G-H-convex spaces. Ben-El-Mechaiekh et.al[4] introduced the concepts of L-convex spaces. They established some KKM type theorems in these spaces respectively. L-convex space includes all the above abstract convex spaces as special cases. Ding[5] proved some new KKM type theorems and coincidence theorems involving admissible set-valued mappings and the set-valued mappings with compactly local intersection property in L-convex spaces. Ding [6] established some new generalized KKM type theorems for generalized G-KKM and S-KKM type mappings from a nonempty set into a G-convex space. Deng and Xia[7] generalized the corresponding results in [6] to general topological spaces without any convexity assumptions.

Inspired by the above research works, in this paper, we first introduce a class of finite continuous topological spaces (in short, FC-spaces) without any convexity structure. Then some new KKM type theorems involving admissible set-valued mappings and the set-valued mapping with compactly local intersection property are proved in FC-spaces. As applications, some new

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coincidence theorems and fixed point theorems are obtained in FC-spaces. Our results unify and generalize many known results in recent literature.

2. Preliminaries

Let X and Y be two nonempty sets. We will denote by 2^Y and $\langle X\rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of X, respectively. For any $A\in\langle X\rangle$, we denote by $|A|$ the cardinality of A. Let Δ_n be the standard n-dimensional simplex with vertices $e_0,e_1,\dots,e_n$. If J is a nonempty subset of $\{0,1,\dots,n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j:j\in J\}$.

The following notions were introduced by Ding[8].

Let A be a subset of a topological space X. A is called to be compactly open (resp., compactly closed) in X if for any nonempty compact subset K of X, $A\bigcap K$ is open (resp., closed) in K. For any given subset A of X, define the compact closure and the compact interior of A, denoted by $\mathrm{ccl}(A)$ and $\mathrm{cint}(A)$, as

$$\mathrm{ccl}(A)=\bigcap\{B\subset X:A\subset B\text{ and }B\text{ is compactly closed in }X\},$$

$$\mathrm{cint}(A)=\bigcup\{B\subset X:B\subset A\text{ and }B\text{ is compactly open in }X\}.$$

It is easy to see that $\mathrm{cint}(A)$ (resp., $\mathrm{ccl}(A)$) is compactly open (resp., compactly closed) in X and for each nonempty compact subset K of X, since $\mathrm{ccl}(A)\bigcap K=K\bigcap(\bigcap\{B\subset X:A\subset B$ and B is compactly closed in $X\})=\bigcap\{B\bigcap K\subset K:A\bigcap K\subset B\bigcap K$ and $B\bigcap K$ is closed in $K\}$, we have $\mathrm{ccl}(A)\bigcap K=\mathrm{cl}_K(A\bigcap K)$. Since $\mathrm{cint}(A)\bigcap K=K\bigcap(\bigcup\{B\subset X:B\subset A$ and $B\subset X$ is compactly open$\})=\bigcup\{K\bigcap B\subset K:K\bigcap B\subset K\bigcap A$ and $K\bigcap B\subset K$ is open$\}$,we have $\mathrm{cint}(A)\bigcap K=\mathrm{int}_K(A\bigcap K)$, where $\mathrm{cl}_K(A\bigcap K)$ and $\mathrm{int}_K(A\bigcap K)$ denote the closure and the interior of $A\bigcap K$ in K, respectively. It is clear that a subset A of X is compactly open (resp., compactly closed) in X if and only if $\mathrm{cint}(A)=A$ (resp., $\mathrm{ccl}(A)=A$).

Definition 2.1 *Let X be a set and Y be a topological space. A mapping $G:X\to 2^Y$ is said to be transfer compactly open-valued (resp., transfer compactly closed-valued) on X if for $x\in X$ and for each nonempty compact subset K of Y, $y\in G(x)\bigcap K$ (resp., $y\notin G(x)\bigcap K$) implies that there exists a point $x'\in X$ such that $y\in\mathrm{int}_K(G(x')\bigcap K)$ (resp., $y\notin\mathrm{cl}_K(G(x')\bigcap K)$). Clearly, each open-valued (resp., closed-valued) mapping is transfer open-valued (resp., transfer closed-valued)[9]) and is also compactly open-valued (resp., compactly closed-valued). Each transfer open-valued (resp., transfer closed-valued) mapping is transfer compactly open-valued (resp., transfer compactly closed-valued) and the inverse is not true in general.*

Definition 2.2 *Let X and Y be two topological spaces. $G:X\to 2^Y$ is a set-valued mapping.*

(1) G is said to be compact if $G(X)$ is included in a compact subset of Y;

(2) G is said to have the local intersection property on X if for each $x\in X$ with $G(x)\neq\emptyset$, there exists an open neighborhood $\mathcal{N}(x)$ of x in X such that $\bigcap_{z\in\mathcal{N}(x)}G(z)\neq\emptyset$[10];

(3) G is said to have the compactly local intersection property on X if for each nonempty

compact subset K of X and for each $x \in K$ with $G(x) \neq \emptyset$, there exists an open neighborhood $\mathcal{N}(x)$ of x in X such that $\bigcap_{z\in\mathcal{N}(x)\bigcap K} G(z) \neq \emptyset$[11].

Clearly, if G has the compactly local intersection property, then for any compact subset K of X, the restriction $G\mid_K$: $K \to 2^Y$ of G on K has the local intersection property. It is also clear that each set-valued mapping with local intersection property has the compactly local intersection property and the inverse is not true in general.

The following notion was introduced by Ding [12].

Definition 2.3 *$(X, \{\varphi_N\})$ is said to be a finitely continuous space (in short, FC-space) if X is a topological space and for each $N = \{x_0, \dots, x_n\} \in \langle X\rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. A subset M of an FC-space X is said to be an FC-subspace of X if for each $N = \{x_0, \dots, x_n\} \in \langle X\rangle$ and for any $\{x_{i_0}, \dots, x_{i_k}\} \subset N \bigcap M$, $\varphi_N(\Delta_k) \subset M$ where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \dots, k\})$.*

Remark 2.1 It is clear that FC-space is a new class of topological spaces without any linear and convexity structure. FC-space includes H-space [1], G-convex space[2], G-H space [3], L-space[4], and many topological spaces with abstract convexity structure as special cases, see [1–6] and the references therein.

The following notions were introduced by Park[2].

Let X and Y be two topological spaces. For a given class U of set-valued mappings, $U(X, Y)$ denotes the set of set-valued mappings $T : X \to Y$ belonging to U, and U_c the set of finite composites of set-valued mappings in U.

Let $\mathcal{U}$ denote the class of set-valued mappings satisfying the following properties:

(1) $\mathcal{U}$ contains the class C of (single-valued) continuous mappings;

(2) Each $F \in \mathcal{U}_c(X, Y)$ is upper semicontinuous (in short, u.s.c.) on X with nonempty compact values;

(3) For any standard n-dimensional simplex Δ_n, each $F \in \mathcal{U}_c(\Delta_n, \Delta_n)$ has a fixed point.

A class $\mathcal{U}_c^k(X, Y)$ is defined as follows: $F \in \mathcal{U}_c^k(X, Y)$ if and only if for any compact subset K of X there exists an $F^* \in \mathcal{U}_c(K, Y)$ such that $F^*(x) \subset F(x)$, $\forall\ x \in K$. Clearly, $\mathcal{U} \subset \mathcal{U}_c \subset \mathcal{U}_c^k$.

Lemma 2.1[11] *Let X and Y be topological spaces and $G : X \to 2^Y$ be a set-valued mapping with nonempty values. Then the following conditions are equivalent:*

(I) G has the compactly local intersection property;

(II) For each compact subset K of X and for each $y \in Y$, there exists an open subset O_y of X (which may be empty) such that $O_y \bigcap K \subset G^{-1}(y)$ and $K = \bigcup_{y\in Y}(O_y \bigcap K)$;

(III) For each compact subset K of X, there exists a set-valued mapping $F : X \to 2^Y$ such that for any $y \in Y$, $F^{-1}(y)$ is open or empty in X; $F^{-1}(y) \bigcap K \subset G^{-1}(y)$, $\forall\ y \in Y$, and $K = \bigcup_{y\in Y}(F^{-1}(y) \bigcap K)$;

(IV) For each compact subset K of X and for each $x \in K$, there exists $y \in Y$ such that $x \in \mathrm{cint}G^{-1}(y) \bigcap K$ and $K = \bigcup_{y\in Y}(\mathrm{cint}G^{-1}(y) \bigcap K) = \bigcup_{y\in Y}(G^{-1}(y) \bigcap K)$;

(V) $G^{-1} : Y \to 2^X$ is transfer compactly open-valued on Y.

3. KKM type theorem and coincidence theorems

Theorem 3.1 *Let Y be a topological space, $(X, \{\varphi_N\})$ be an FC-space, $F \in \mathcal{U}_c^k(X, Y)$ and $G : X \to 2^Y$ such that*

(i) For each $x \in X$, $G(x)$ is compactly open in Y;

(ii) For each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ and for each $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$, $F(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^k (Y \setminus G(x_{i_j}))$ where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \dots, k\})$.

Then we have

(a) For any $N = \{x_0, \dots, x_n\} \in \langle X \rangle$, $F(\varphi_N(\Delta_n)) \bigcap (\bigcap_{i=0}^n (Y \setminus G(x))) \neq \emptyset$.

(b) For any $N = \{x_0, \dots, x_n\} \in \langle X \rangle$, there exists a $y \in F(\phi_N(\Delta_n))$ such that $G^{-1}(y) \bigcap N = \emptyset$.

Proof We first prove the conclusions (a) and (b) are equivalent.

(a)⇒(b). By (a), for each $N \in \langle X \rangle$ and for any $x \in N$, $F(\varphi_N(\Delta_n)) \bigcap (\bigcap_{x \in N} (Y \setminus G(x))) \neq \emptyset$, it follows that there exists a $y \in F(\varphi_N(\Delta_n))$ such that $y \in \bigcap_{x \in N} (Y \setminus G(x))$, that is $y \notin \bigcup_{x \in N} G(x)$, i.e., $N \bigcap G^{-1}(y) = \emptyset$ and so the conclusion (b) holds.

(b)⇒(a) is easy, we omit its proof here.

Hence, it is enough to show that the conclusion (a) holds. Suppose the conclusion (a) is not true. Then there exists a set $N = \{x_0, ..., x_n\} \in \langle X \rangle$ such that $F(\varphi_N(\Delta_n)) \bigcap (\bigcap_{x \in N} (Y \setminus G(x))) = \emptyset$. It follows that

$$F(\varphi_N(\Delta_n)) \subset \bigcup_{x \in N} G(x). \tag{3.1}$$

Since $\varphi_N(\Delta_n)$ is compact in X and $F \in \mathcal{U}_c^k(X, Y)$, there exists an $\tilde{F} \in \mathcal{U}_c(\varphi_N(\Delta_n), Y)$ such that

$$\tilde{F}(x) \subset F(x), \quad \forall \ \ x \in \varphi_N(\Delta_n). \tag{3.2}$$

Since $\tilde{F}$ is u.s.c with compact values and $\varphi_N(\Delta_n)$ is compact, $\tilde{F}(\varphi_N(\Delta_n))$ is compact in Y. By (3.1) and (3.2), we have

$$\tilde{F}(\varphi_N(\Delta_n)) = \bigcup_{i=0}^n (G(x_i) \bigcap \tilde{F}(\varphi_N(\Delta_n))).$$

By (i), $\{G(x_i) \bigcap \tilde{F}(\varphi_N(\Delta_n))\}_{i=0}^n$ is an open cover of $\tilde{F}(\varphi_N(\Delta_n))$. Let $\{\psi_i\}_{i=0}^n$ be the continuous partition of unity subordinated to the open cover, i.e., for each $i \in \{0, 1, ..., n\}$, $\psi_i : \tilde{F}(\varphi_N(\Delta_n)) \to [0, 1]$ is continuous;

$$\{y \in \tilde{F}(\varphi_N(\Delta_n)) : \psi_i(y) \neq 0\} \subset G(x_i) \bigcap \tilde{F}(\varphi_N(\Delta_n)) \subset G(x_i); \tag{3.3}$$

and $\sum_{i=0}^n \psi_i(y) = 1$, $\forall \, y \in \tilde{F}(\varphi_N(\Delta_n))$. Define a mapping $\psi : \tilde{F}(\varphi_N(\Delta_n)) \to \Delta_n$ by

$$\psi(y) = \sum_{i=0}^n \psi_i(y) e_i, \ \forall \, y \in \tilde{F}(\varphi_N(\Delta_n)).$$

Then ψ is continuous and hence $\psi \tilde{F} \varphi_N \in \mathcal{U}_C(\Delta_n, \Delta_n)$. Therefore, $\psi \tilde{F} \varphi_N$ has a fixed point

$z_0 \in \Delta_n$, that is $z_0 \in \psi\tilde{F}\varphi_N(z_0)$. Hence there exists a $y_0 \in \tilde{F}(\varphi_N(z_0))$ such that

$$z_0 = \psi(y_0) = \sum_{j\in J(y_0)} \psi_j(y_0)e_j \ \in \Delta_{J(y_0)},$$

where $J(y_0) = \{j \in \{0, 1, ..., n\} : \psi_j(y_0) \neq 0\}$. It follows from (ii) that

$$y_0 \in \tilde{F}(\varphi_N(z_0)) \subset \tilde{F}(\varphi_N(\Delta_{J(y_0)})) \subset F(\varphi_N(\Delta_{J(y_0)})) \subset \bigcup_{j\,\in J(y_0)} (Y \setminus G(x_j)).$$

Therefore, there exists a $j_0 \in J(y_0)$ such that $y_0 \notin G(x_{j_0})$. On the other hand, by the definition of $J(y_0)$, we have $\psi_{j_0}(y_0) \neq 0$ and it follows from (3.3) that $y_0 \in G(x_{j_0})$,which is a contradiction. This completes the proof. □

Remark 3.1 Theorem 3.1 does not require the space X to possess any convexity structure. Theorem 3.1 generalizes Theorem 3.1 of Ding [5] from L-convex space to FC-space.

Here, we give a concrete example of Theorem 3.1.

Let $X = (0, 1)\bigcup(2, 3)$ and $Y = [0, 4)$ with the usual topology. For each $N = \{x_0, x_1, ..., x_n\} \in \langle X\rangle$, define a mapping $\varphi_N : \Delta_n \to X$ by $\varphi_N(\alpha) = \frac{1}{3}\sum_{i=0}^{n} \alpha_i x_i, \forall\alpha = \{\alpha_0, ..., \alpha_n\} \in \Delta_n$. Then φ_N is continuous. Hence $(X, \{\varphi_N\})$ is an FC-space. Define $G : X \to 2^Y$ by $G(x) = (x+1, 4), x \in X$ and $F : X \to 2^Y$ by $F(x) = [0, x], x \in X$. It is easy to see that F and G satisfy all the conditions of Theorem 3.1, hence we have the results of Theorem 3.1. However X only has topological structure and not any abstract convexity structure.

Corollary 3.1 *Let Y be a topological space, $(X, \{\varphi_N\})$ be an FC-space, $F \in \mathcal{U}_c^k(X, Y)$ and $T : X \to 2^Y$ such that for each $x \in X$, $T(x)$ is compactly closed in Y and for each $N = \{x_0, x_1, ..., x_n\} \in \langle X\rangle$ and for each $\{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\}$, $F(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} T(x_{i_j})$ where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \ldots, k\})$. Then*

$$F(\phi_N(\Delta_n))\bigcap\ (\bigcap_{x\in N} T(x)) \neq \emptyset, \forall\, N \in \langle X\rangle.$$

Proof Let $G(x) = Y \setminus T(x)$ for each $x \in X$. The conclusion of Corollary 3.1 follows from Theorem 3.1.

Remark 3.2 Corollary 3.1 generalizes the Corollary of Park and Kim[2] from G-convex space to FC-space, and the Corollary 3.1 of Ding[5] from L-convex space to FC-space.

If $X = Y$ and F is the identity mapping on X, then the Corollary 3.1 reduces to the following Corollary 3.2.

Corollary 3.2 *Let $(X, \{\varphi_N\})$ be an FC-space and $T : X \to 2^X$ be a set-valued mapping such that for each for each $x \in X$, $T(x)$ is compactly closed in X and for each $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ and for each nonempty $\{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\}$, $\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} T(x_{i_j})$.*
Then $\varphi_N(\Delta_n)\bigcap(\bigcap_{x\,\in N} T(x)) \neq \emptyset, \forall\, N \in \langle X\rangle$.

Remark 3.3 Corollary 3.2 is different from Theorem 3.1 of Deng and Xia[7].

Theorem 3.2 *Let Y be a topological space and $(X, \{\phi_N\})$ be an FC-space, $F \in \mathcal{U}_c^k(X, Y)$ and*

$T : X \to 2^Y$ such that

(i) For each $x \in X$, $T(x)$ is compactly closed in Y;

(ii) For each $N = \{x_0, \dots, x_n\} \in \langle X\rangle$ and each nonempty $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$, $F(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} T(x_{i_j})$;

(iii) There exists a nonempty compact subset K of Y such that either

(a) For some $M \in \langle X\rangle$, $\mathrm{cl}F(X) \setminus K \subset \bigcup_{x\in M}(Y \setminus T(x))$; or

(b) For each $N \in \langle X\rangle$, there exists a compact FC-subspace L_N of X containing N such that $F(L_N) \setminus K \subset \bigcup_{x\in L_N}(Y \setminus T(x))$.

Then $\mathrm{cl}F(X) \bigcap K \bigcap (\bigcap_{x\in X} T(x)) \neq \emptyset$.

Proof Define a mapping $G : X \to 2^Y$ by $G(x) = Y \setminus T(x)$. By (i), $G(x)$ is compactly open in Y for each $x \in X$. Suppose the conclusion is false. Then we have

$$\mathrm{cl}F(X)\bigcap K \subset \bigcup_{x\in X}((Y \setminus T(x))\bigcap K) = \bigcup_{x\in X}(G(x)\bigcap K). \tag{3.4}$$

Since $\mathrm{cl}F(X)\bigcap K$ is compact in K, by (3.4), there exists an $N = \{x_0, \dots, x_n\} \in \langle X\rangle$ such that

$$\mathrm{cl}F(X)\bigcap K \subset \bigcup_{x\in N} G(x). \tag{3.5}$$

Case (iii)(a). By the condition (iii)(a) and (3.5), there exists a finite set $N_1 = N\bigcup M = \{x_0, \dots, x_n, x_{n+1}, \dots, x_{n+m}\} \in \langle X\rangle$, such that

$$\mathrm{cl}F(X) \subset \bigcup_{i=0}^{n+m} G(x_i). \tag{3.6}$$

By the conditions (i), (ii) and Theorem 3.1, we have

$$F(\varphi_{N_1}(\Delta_{n+m}))\bigcap(\bigcap_{x\in N_1}(Y \setminus G(x))) \neq \emptyset,$$

that is $F(\varphi_{N_1}(\Delta_{n+m})) \not\subset \bigcup_{i=0}^{n+m} G(x_i)$ which is a contradiction with (3.6).

Case (iii)(b). Let L_N be the compact FC-subspace of X in the condition (iii)(b). Since $F \in \mathcal{U}_c^k(X, Y)$, there exists an $\tilde{F} \in \mathcal{U}_c(L_N, Y)$ such that $\tilde{F}(x) \subset F(x)$ for all $x \in L_N$. By (ii), we have that for each $A = \{x_0, \dots, x_n\} \in \langle L_N\rangle$ and for each $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$,

$$\tilde{F}(\varphi_A(\Delta_k)) \subset F(\varphi_A(\Delta_k)) \subset \bigcup_{j=0}^{k}(Y \setminus G(x_{i_j})),$$

where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \dots, k\})$. By $\mathcal{U}_c \subset \mathcal{U}_c^k$, we have $\tilde{F} \in \mathcal{U}_c^k(L_N, Y)$, By Theorem 3.1, there exists a $y \in \tilde{F}(\varphi_A(\Delta_n))$ such that $G^{-1}(y)\bigcap A = \emptyset$, It follows that $y \notin G(x)$ for all $x \in A$. Hence we have

$$y \in \tilde{F}(\varphi_A(\Delta_n))\bigcap\ (\bigcap_{x\in A}(Y \setminus G(x))) = \bigcap_{x\in A}(\tilde{F}(\varphi_A(\Delta_n))\bigcap T(x)).$$

Since L_N is FC-subspace of X, we have $\varphi_A(\Delta_n) \subset L_N$ and $y \in \bigcap_{x\in A}(\tilde{F}(L_N)\bigcap T(x))$. Since $\tilde{F}$ is u.s.c. with compact valued and L_N is compact, $\tilde{F}(L_N)$ is compact in Y. By (i), $T(x)$

is compactly closed. Hence the family $\{\tilde{F}(L_N)\bigcap T(x) : x \in L_N\}$ has the finite intersection property. It follows that

$$\tilde{F}(L_N)\bigcap(\bigcap_{x\in L_N} T(x)) \neq \emptyset.$$

Take any $z \in \tilde{F}(L_N)\bigcap(\bigcap_{x\in L_N} T(x)) = \tilde{F}(L_N)\setminus\bigcup_{x\in L_N}(Y\setminus T(x))$, i.e.,

$$z \in \tilde{F}(L_N)\,\text{and}\, z \notin \bigcup_{x\in L_N}(Y\setminus T(x)) = \bigcup_{x\in L_N} G(x). \tag{3.7}$$

By (iii)(b), we have $z \in K$, it follows from (3.5) that

$$z \in \text{cl}F(X)\bigcap K \subset \bigcup_{x\in N} G(x).$$

Hence there exists an $x^* \in N \subset L_N$ such that $z \in G(x^*)$ which contradicts (3.7). This completes the proof. □

Remark 3.4 Theorem 3.2 generalizes the Theorem 3.2 of Ding[13] and the Theorem 3.2 of Ding[5] from G-convex spaces to FC-spaces and from L-convex spaces to FC-spaces respectively, and the conditions are better than that in Theorem 3.2 of Ding[5].

Theorem 3.3 *Let Y be a topological space and $(X,\{\phi_N\})$ be an FC-space. Let $F \in \mathcal{U}_c^k(X,Y)$ and $T : X \to 2^Y$ be such that*

(i) T is transfer compactly closed-valued on X;

(ii) For each $N = \{x_0,\dots,x_n\} \in \langle X\rangle$ and any nonempty $\{e_{i_0},\dots,e_{i_k}\} \subset \{e_0,\dots,e_n\}$, $F(\varphi_N(\Delta_k)) \subset \bigcup_{j=0}^k \text{ccl}T(x_{i_j})$;

(iii) There exists a nonempty compact subset K of Y such that either

(a) For some $M \in \langle X\rangle$, $\text{cl}F(X)\setminus K \subset \bigcup_{x\in M}(Y\setminus \text{ccl}T(x))$; or

(b) For each $N \in \langle X\rangle$, there exists a compact FC-subspace L_N of X containing N such that $F(L_N)\bigcap(\bigcap_{x\in L_N}\text{ccl}T(x)) \subset K$.

Then we have

$$\text{cl}F(X)\bigcap K\bigcap(\bigcap_{x\in X} T(x)) \neq \emptyset.$$

Proof It is easy to show that $\text{ccl}T : X \to 2^Y$ satisfies all the conditions of Theorem 3.2. Hence we have

$$\text{cl}F(X)\bigcap K\bigcap(\bigcap_{x\in X}\text{ccl}T(x)) \neq \emptyset.$$

Since T is transfer compactly closed-valued and $\text{cl}F(X)\bigcap K$ is compact. we must have

$$\text{cl}F(X)\bigcap K\bigcap(\bigcap_{x\in X} T(x)) = \text{cl}F(X)\bigcap K\bigcap(\bigcap_{x\in X}\text{ccl}T(x)) \neq \emptyset.$$

This completes the proof. □

Remark 3.5 Theorem 3.3 generalizes Theorem 3.3 of Ding[5], Theorems 3.2 and 3.3 of Ding[8] and Theorems 3 and 4 of Park and Kim[2] from L-convex space and G-convex spaces to FC-space.

By applying Theorem 3.3 we can obtain the following coincidence theorem.

Theorem 3.4 *Let $(X, \{\varphi_N\})$ be an FC-space and K be a nonempty compact subset of a topological space Y. Let $F \in \mathcal{U}_c^k(X,Y)$ and $G, P : Y \to 2^X$ be such that*

(i) G satisfies one of the conditions (I)–(V) in Lemma 2.1;

(ii) For each $y \in F(X)$, $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ and

$$\{x_{i_j} : j = 0, \ldots, k\} \subset \{x_0, \ldots, x_n\} \bigcap (\mathrm{cint} G^{-1})^{-1}(y)$$

imply $\varphi_N(\Delta_k) \subset P(y)$, where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \ldots, k\})$;

(iii) $\mathrm{cl}F(X) \bigcap K \subset \bigcup_{x\in X} G^{-1}(x)$;

(iv) One of the following conditions holds:

(a) For some $M \in \langle X\rangle$, $\mathrm{cl}F(X) \setminus K \subset \bigcup_{x\in M} \mathrm{cint}G^{-1}(x)$;

(b) For each $N \in \langle X\rangle$, there exists a compact FC-subspace L_N of X containing N such that $F(L_N) \setminus K \subset \bigcup_{x\in L_N} \mathrm{cint}(G^{-1}(x))$.

Then there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in P(y_0)$ and $y_0 \in F(x_0)$, i.e., $(x_0, y_0)$is a coincidence point of P and F.

Proof Define a mapping $T : X \to 2^Y$ by

$$T(x) = Y \setminus G^{-1}(x) = \{y \in Y : x \notin G(y)\}, \forall\, x \in X.$$

Then, by (i) and Lemma 2.1, T is transfer compactly closed-valued on X.

Case (iv)(a). By (iv)(a) and $Y \setminus \mathrm{ccl}T(x) = \mathrm{cint}G^{-1}(x)$ for each $x \in X$, it is easy to see that the condition (iii)(a) of Theorem 3.3 holds.

Case(iv)(b). By (iv)(b), for each $N \in \langle X\rangle$, there exists a compact FC-subspace L_N of X containing N such that

$$F(L_N) \setminus K \subset \bigcup_{x\in L_N} \mathrm{cint}(G^{-1}(x)) = \bigcup_{x\in L_N} (Y \setminus \mathrm{ccl}T(x)),$$

that is, $F(L_N) \bigcap \left(\bigcap_{x\in L_N} \mathrm{ccl}T(x)\right) \subset K$. Hence the condition (iii)(b) of Theorem 3.3 holds. By the condition (iii), we have $clF(X) \bigcap K \subset \bigcup_{x\in X} G^{-1}(x)$, that is

$$\mathrm{cl}F(X) \bigcap K \bigcap \Big(\bigcap_{x\in X} (Y \setminus G^{-1}(x))\Big) = \emptyset,$$

i.e.,

$$\mathrm{cl}F(X) \bigcap K \bigcap \Big(\bigcap_{x\in X} T(x)\Big) = \emptyset.$$

Hence the conclusion of Theorem 3.3 does not hold. By Theorem 3.3, there exists an $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$, and a nonempty set $\{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\}$ such that

$$F(\varphi_N(\Delta_k)) \not\subset \bigcup_{j=0}^{k} \mathrm{ccl}(T(x_{i_j})) = \bigcup_{j=0}^{k} (Y \setminus \mathrm{cint}(G^{-1}(x_{i_j}))),$$

where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \ldots, k\}$. So there exist $y_0 \in F(\varphi_N(\Delta_k))$ and $x_0 \in \varphi_N(\Delta_k)$ such that

$$y_0 \in F(x_0) \text{ and } y_0 \notin \bigcup_{j=0}^{k} (Y \setminus \mathrm{cint}G^{-1}(x_{i_j})).$$

It follows that $y_0 \in \mathrm{cint}G^{-1}(x_{i_j}),\ \ \forall\ \ j=0,\dots,k$. Hence we have

$$\{x_{i_j}: j=0,\dots,k\} \in \langle (\mathrm{cint}G^{-1})^{-1}(y_0)\rangle.$$

By the condition (ii), we obtain $\varphi_N(\Delta_k) \subset P(y_0)$. Therefore, $x_0 \in P(y_0)$ and $y_0 \in F(x_0)$, i.e., (x_0, y_0) is a coincidence point of F and P. This completes the proof. □

Remark 3.6 Theorem 3.4 generalizes Theorem 3.4 of Ding[5] from L-convex space to FC-space and the conditions are better. Theorem 3.4 also generalizes Theorems 4.2 and 4.3 of Ding [8] and Theorem 1 of Park and Kim[2] in several aspects.

Theorem 3.5 *Let $(X, \{\varphi_N\})$ be an FC-space and K be a nonempty compact subset of X. Let $G, P : X \to 2^X$ be such that*

(i) G satisfies one of the conditions (I)–(V) in Lemma 2.1;

(ii) For each $x \in X$, $N = \{x_0,\dots,x_n\} \in \langle X\rangle$ and

$$\{x_{i_j}: j=0,\dots,k\} \subset \{x_0,\dots,x_n\} \bigcap (\mathrm{cint}G^{-1})^{-1}(x)$$

imply $\varphi_N(\Delta_k) \subset P(x)$;

(iii) For each $x \in K$, $G(x) \neq \emptyset$;

(iv) For each $N \in \langle X\rangle$, there exists a compact FC-subspace L_N of X containing N such that $L_N \setminus K \subset \bigcup_{x\in L_N} \mathrm{cint}(G^{-1}(x))$.

Then P has fixed point in X.

Proof Let $Y = X$ and $F(x) = \{x\}$ be the identity mapping. Then $F \in \mathcal{U}_c(X,X) \subset \mathcal{U}_c^k(X,X)$. It is easy to check that all conditions of Theorem 3.4 are satisfied. The conclusion follows from Theorem 3.4.

Corollary 3.3 *Let $(X, \{\varphi_N\})$ be an FC-space and K be a compact subset of X. And let $G : X \to 2^X$ satisfy the conditions (i), (iii) and (iv) of Theorem 3.5 and the condition (ii) be replaced by the following condition:*

(ii)' For each $x \in X$, $G(x)$ is an FC-subspace of X.

Then G has fixed point in X.

Remark 3.7 Theorem 3.5 and Corollary 3.3 generalize Theorem 3.5 and Corollary 3.1 of Ding[5] to FC-spaces.

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