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The Convergence of Hermite Interpolation Operators on the Real Line

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Abstract The present paper investigates the convergence of Hermite interpolation operators on the real line. The main result is: Given $0<\delta_0<1/2$, $0<\varepsilon_0<1$. Let $f\in C_{(-\infty,\infty)}$ satisfy $|y_k|=O(e^{(1/2-\delta_0)x_k^2})$ and $|f(x)|=O(e^{(1-\varepsilon_0)x^2})$. Then for any given point $x\in\mathbb{R}$, we have $\lim_{n\to\infty}H_n(f,x)=f(x)$.

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1. Introduction

Let $W(x)=e^{-x^2/2}$, and $p_n(x)=\gamma_n x^n+\cdots$, $\gamma_n>0$ denote the nth Hermite polynomial for the weight function $W(x)$ so that

$$\int_{-\infty}^{\infty}p_n(x)p_m(x)W^2(x)\mathrm{d}x=\begin{cases}0, & n\neq m,\\ 1, & n=m.\end{cases}$$

As usual, the zeros of $p_n(x)$ are supposed to be

$$-\infty<x_n<x_{n-1}<\cdots<x_1<\infty.$$

Take the zeros of nth Hermite polynomial as the interpolation nodes. For $f\in C_{(-\infty,\infty)}$, we define Hermite interpolation operators as follows:

$$H_n(f,x)=\sum_{k=1}^{n}\left(\frac{p_n(x)}{p_n'(x_k)(x-x_k)}\right)^2\left[f(x_k)\left(1-\frac{p_n''(x_k)}{p_n'(x_k)}(x-x_k)\right)+y_k(x-x_k)\right].$$

It is easy to verify that

$$H_n(f,x_k)=f(x_k),\quad H_n'(f,x_k)=y_k.$$

The main result of this paper is

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Theorem *Given $0 < \delta_0 < 1/2$, $0 < \varepsilon_0 < 1$. Let $f \in C_{(-\infty,\infty)}$ satisfy*

$$|y_k| = O(e^{(1/2-\delta_0)x_k^2}), \ \ |f(x)| = O(e^{(1-\varepsilon_0)x^2}). \tag{1}$$

Then for any given point $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} H_n(f,x) = f(x).$$

2. Lemmas

Lemma 1 *For the nth Hermite polynomial $p_n(x)$, the following inequalities hold:*

$$p_n'(x) = \sqrt{2n} \cdot p_{n-1}(x), \tag{2}$$

$$\left|\frac{p_n''(x_k)}{p_n'(x_k)}\right| \le C(1+|x_k|), \tag{3}$$

$$\sup_{x\in\mathbf{R}} |p_n(x)| \cdot \omega(x) \sim a_n^{-\frac{1}{2}} n^{\frac{1}{6}}, \tag{4}$$

$$\frac{a_n}{n}\left|p_n'(x_k)\right| \cdot \omega(x_k) \sim a_n^{-\frac{1}{2}} \left(\max\left\{n^{-\frac{2}{3}}, 1-\frac{|x_k|}{a_n}\right\}\right)^{\frac{1}{4}}, \tag{5}$$

where a_μ is the μth Mhskar-Rahmanov-Saff number, that is, the positive root of the equation

$$\mu = \frac{2}{\pi}\int_0^1 a_\mu t Q'(a_\mu t)(1-t^2)^{-\frac{1}{2}} \mathrm{d}t.$$

Proof Readers can find (2) in Ref. [4], (3) in Ref. [3] and (4), (5) in Ref. [1].

Lemma 2 *Let $x \in [x_{j+1}, x_j]$ for some j and $E = \{k : k \ne j, j+1\}$. Then for $k \in E$, we have*

$$\sum_{k\in E} \frac{1}{|x-x_k|} \le Cn^{\frac{1}{2}} \log n.$$

Proof By the definition of E, we have

$$\sum_{k\in E}\frac{1}{|x-x_k|} = \sum_{k=1}^{j-1}\frac{1}{|x-x_k|} + \sum_{k=j+2}^{n}\frac{1}{|x-x_k|} \le \sum_{k=1}^{j-1}\frac{1}{|x_j-x_k|} + \sum_{k=j+2}^{n}\frac{1}{|x_{j+1}-x_k|}.$$

From[1]

$$x_j - x_{j+1} \sim \frac{a_n}{n}\max\left\{n^{-\frac{2}{3}}, 1-\frac{|x_j|}{a_n}\right\}^{-\frac{1}{2}}, \tag{6}$$

and[2] $a_n = Cn^{1/2}$, it follows that

$$n^{-\frac{1}{2}} \le |x_j - x_{j+1}| \le n^{-\frac{1}{6}}.$$

Therefore,

$$\sum_{k\in E}\frac{1}{|x-x_k|} \le Cn^{\frac{1}{2}}\sum_{k=1}^{n}\frac{1}{k} \le Cn^{\frac{1}{2}}\log n.$$

Lemma 3 *Let*

$$l_k(x) = \frac{p_n(x)}{p_n'(x_k)(x-x_k)}.$$

Then for sufficiently large n, we have

$$\sum_{k=1}^{n} l_k^2(x)e^{(1/2-\delta_0)x_k^2}|x-x_k| \le C\frac{\log n}{n^{1/6}}e^{x^2}.$$

Proof In the sequel, we suppose $x \in [x_{j+1}, x_j]$ for some $1 \le j \le n-1$. The case $x \in (-\infty, x_n)$ or $x \in (x_1, +\infty)$ can be treated similarly. Let

$$E_1 = \{k : |x_k| \le n^{1/4}\}, \ \ E_2 = \{k : |x_k| > n^{1/4}\}.$$

By the assumption, we have

$$\begin{aligned}
\sum_{k=1}^{n} l_k^2(x)e^{(1/2-\delta_0)x_k^2}|x-x_k| \le & C\sum_{k=1}^{n}\frac{p_n^2(x)}{(p_n'(x_k))^2|x-x_k|}\exp\{(1/2-\delta_0)x_k^2\} \\
\le & C(\sum_{k\in E_1\cap E} + \sum_{k\in E_2\cap E})\frac{p_n^2(x)}{(p_n'(x_k))^2|x-x_k|}\exp\{(1/2-\delta_0)x_k^2\}+ \\
& C\frac{p_n^2(x)}{(p_n'(x_j))^2|x-x_j|}\exp\{(1/2-\delta_0)x_j^2\}+ \\
& C\frac{p_n^2(x)}{(p_n'(x_{j+1}))^2|x-x_{j+1}|}\exp\{(1/2-\delta_0)x_{j+1}^2\} \\
=: & C(A_1+A_2+A_3+A_4).
\end{aligned}$$

Applying (4) and (5) gives

$$\begin{aligned}
A_1 + A_2 \le & \sum_{k\in E}\frac{(p_n(x)\omega(x))^2\exp\{-(1/2+\delta_0)x_k^2\}}{(p_n'(x_k)\omega(x_k))^2|x-x_k|}e^{x^2} \\
\le & C\sum_{k\in E_1\cap E}\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2})^2|x-x_k|}\exp\{-(1/2+\delta_0)x_k^2\}e^{x^2}+ \\
& C\sum_{k\in E_2\cap E}\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2|x-x_k|}\exp\{-(1/2+\delta_0)x_k^2\}e^{x^2}.
\end{aligned}$$

With Lemma 2 and $a_n = Cn^{1/2}$, we deduce that

$$A_1 + A_2 \le C\frac{\log n}{n^{1/6}}e^{x^2} + Cn^{1/6}\log n\exp\{-(1/2+\delta_0)n^{-1/2}\}e^{x^2} \le C\frac{\log n}{n^{1/6}}e^{x^2}$$

holds for sufficiently large n. Next we will estimate A_3 and A_4. Write

$$\begin{aligned}
A_3 = & \frac{p_n^2(x)}{|p_n'(x_j)|^2|x-x_j|}\exp\{(1/2-\delta_0)x_j^2\} \\
= & \frac{|p_n(x)|\omega(x)|(p_n(x)-p_n(x_j))|\omega(x)\omega^2(x_j)}{|p_n'(x_j)|^2\omega^2(x_j)|x-x_j|}e^{x^2}\cdot\exp\{(1/2-\delta_0)x_j^2\}.
\end{aligned}$$

If $j \in E_1$, then (4),(5) and the mean value theorem together yield that

$$\begin{aligned}
A_3 \le & C\frac{a_n^{-1/2}n^{1/6}|p_n'(\xi_j)|\omega(\xi_j)}{(\frac{n}{a_n}a_n^{-1/2})^2}\omega^{-1}(\xi_j)\omega(x_j)\omega(x)e^{x^2}\cdot\exp\{-\delta_0 x_j^2\} \\
\le & Cn^{-7/12}|p_n'(\xi_j)|\omega(\xi_j)\omega^{-1}(\xi_j)\omega(x_j)\omega(x)e^{x^2}\cdot\exp\{-\delta_0 x_j^2\},
\end{aligned}$$

where $\xi_j \in (x, x_j)$. By (2), $p_n'(\xi_j) = \sqrt{2n}p_{n-1}(\xi_j)$. We obtain that

$$A_3 \leq Cn^{-7/12}\sqrt{2n}a_n^{-1/2}n^{1/6}\omega^{-1}(\xi_j)\omega(x_j)\omega(x)e^{x^2} \leq Cn^{-1/6}e^{x^2},$$

where we should note that $\omega^{-1}(\xi_j)\cdot\omega(x)\cdot\omega(x_j) \leq 1$ for $\xi_j \in (x, x_j)$.

For $j \in E_2$, by Lemma 1 and the mean value theorem again,

$$\begin{aligned} A_3 &\leq C\frac{a_n^{-1/2}n^{1/6}\sqrt{2n}a_n^{-1/2}n^{1/6}}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2}\cdot\exp\{-\delta_0 n^{1/2}\}e^{x^2} \\ &\leq Cn^{1/6}\exp\{-\delta_0 n^{1/2}\}e^{x^2} \leq Cn^{-1/6}e^{x^2} \end{aligned}$$

for n large enough. The estimate of A_4 can be processed in a similar way. Combining all the above estimates, we complete the proof of Lemma 3.

Lemma 4 *Let $f(x) \in C_{(-\infty,\infty)}$ satisfy (1). Then for any given point $x \in \mathbb{R}$, we have*

$$\sum_{k=1}^{n}\frac{p_n^2(x)}{(p_n'(x_k))^2|x-x_k|}\left|\frac{p_n''(x_k)}{p_n'(x_k)}\right||f(x)-f(x_k)| \to 0, \quad n\to\infty.$$

Proof Suppose $x \in [x_{j+1}, x_j)$ without loss of generality. By (3) we have

$$\begin{aligned} &\sum_{k=1}^{n}\frac{p_n^2(x)}{(p_n'(x_k))^2|x-x_k|}\left|\frac{p_n''(x_k)}{p_n'(x_k)}\right||f(x)-f(x_k)| \\ &\leq \sum_{k=1}^{n}\frac{p_n^2(x)}{(p_n'(x_k))^2|x-x_k|}(1+|x_k|)|f(x)-f(x_k)| \\ &\leq C\left(\sum_{|x_k|\leq 2|x|}+\sum_{|x_k|\geq 2|x|}\right)\frac{p_n^2(x)}{(p_n'(x_k))^2|x-x_k|}(1+|x_k|)|f(x)-f(x_k)| \\ &=: C(I_1+I_2). \end{aligned}$$

Let $\omega(f,t)_{[a,b]}$ be the modulus of continuity of $f(x)$ on the interval $[a,b]$. Then

$$\begin{aligned} I_1 \leq &\sum_{|x_k|\leq 2|x|}\frac{p_n^2(x)}{(p_n'(x_k))^2|x-x_k|}(1+|x_k|)\omega(f,1/\log n)_{[-4|x|,4|x|]}(1+|x-x_k|\log n) \\ \leq &\omega(f,1/\log n)_{[-4|x|,4|x|]} \cdot \\ &\sum_{|x_k|\leq 2|x|}\frac{(p_n(x)\omega(x))^2\omega^2(x_k)e^{x^2}}{(p_n'(x_k)\omega(x_k))^2|x-x_k|}(1+|x_k|)(1+|x-x_k|\log n) \\ \leq &\omega(f,1/\log n)_{[-4|x|,4|x|]}\left(\sum_{|x_k|\leq 2|x|}\frac{(p_n(x)\omega(x))^2\omega^2(x_k)}{(p_n'(x_k)\omega(x_k))^2|x-x_k|}(1+|x_k|)e^{x^2} + \right. \\ &\left.\sum_{|x_k|\leq 2|x|}\frac{(p_n(x)\omega(x))^2\omega^2(x_k)}{(p_n'(x_k)\omega(x_k))^2}(1+|x_k|)\log n\cdot e^{x^2}\right) \\ =:&C\omega(f,1/\log n)_{[-4|x|,4|x|]}(I_{11}+I_{12}). \end{aligned}$$

Applying Lemmas 1 and 2 yields that

$$I_{11} \le \sum_{\substack{|x_k|\le 2|x|\\ k\ne j,j+1}} \frac{(p_n(x)\omega(x))^2\omega^2(x_k)}{(p_n'(x_k)\omega(x_k))^2|x-x_k|}(1+4|x|)e^{x^2}+$$
$$\frac{p_n(x)\omega(x)p_n'(\xi_j)\omega(\xi_j)}{(p_n'(x_j)\omega(x_j))^2}\omega^{-1}(\xi_j)\omega^2(x_j)\omega(x)(1+4|x|)e^{x^2}+$$
$$\frac{p_n(x)\omega(x)p_n'(\xi_{j+1})\omega(\xi_{j+1})}{(p_n'(x_{j+1})\omega(x_{j+1}))^2}\omega^{-1}(\xi_{j+1})\omega^2(x_{j+1})\omega(x)(1+4|x|)e^{x^2}$$
$$\le C\left(\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2})^2}n^{1/2}\log n + \frac{a_n^{-1/2}n^{1/6}\sqrt{2n}a_n^{-1/2}n^{1/6}}{(\frac{n}{a_n}a_n^{-1/2})^2}\omega^{-1}(\xi_j)\omega^2(x_j)\omega(x)+\right.$$
$$\left.\frac{a_n^{-1/2}n^{1/6}\sqrt{2n}a_n^{-1/2}n^{1/6}}{(\frac{n}{a_n}a_n^{-1/2})^2}\omega^{-1}(\xi_{j+1})\omega^2(x_{j+1})\omega(x)\right)(1+4|x|)e^{x^2}$$
$$\le \frac{\log n}{n^{1/6}}(1+4|x|)e^{x^2},$$

where $\omega^{-1}(\xi_j)\omega^2(x_j)\omega(x)\le 1$ and $\omega^{-1}(\xi_{j+1})\omega^2(x_{j+1})\omega(x)\le 1$ are used, with ξ_j and ξ_{j+1} located between x, x_j and between x, x_{j+1} respectively. Similarly, by Lemma 1, with an easier argument, $I_{12}\to 0$, $n\to\infty$. Thus $I_1\to 0$, $n\to\infty$.

Now we estimate I_2. By (4), (5) and Lemma 2, and with the inequality $|x_k|\le Ca_n$([4]) in mind, we have

$$I_2 \le C\left(\sum_{2|x|\le|x_k|\le n^{1/12}} + \sum_{|x_k|>n^{1/12}}\right)\frac{(p_n(x)\omega(x))^2\omega^2(x_k)}{(p_n'(x_k)\omega(x_k))^2|x-x_k|}(1+|x_k|)e^{(1-\varepsilon_0)x_k^2}e^{x^2}$$
$$\le C\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2})^2}n^{1/12}n^{1/2}\log n\cdot e^{x^2}+$$
$$\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2}a_n n^{1/2}\log n\cdot e^{-\varepsilon_0 n^{1/6}}\cdot e^{x^2}$$
$$\le Ce^{x^2}\left(\frac{\log n}{n^{1/12}}+n^{2/3}\log n\cdot e^{-\varepsilon_0 n^{1/6}}\right).$$

The proof of Lemma 4 is completed.

Lemma 5 *Let $f(x)\in C_{(-\infty,\infty)}$ satisfy (1). Then for any given point $x\in\mathbb{R}$, we have*

$$\sum_{k=1}^{n}|f(x)-f(x_k)|\cdot l_k^2(x)\to 0,\quad n\to\infty.$$

Proof Write

$$\sum_{k=1}^{n}|f(x)-f(x_k)|\cdot l_k^2(x)$$
$$\le \sum_{|x_k|\le 2(|x|+1)}\omega(f,|x-x_k|)\cdot l_k^2(x)+\sum_{|x_k|>2(|x|+1)}|f(x)-f(x_k)|\cdot l_k^2(x)$$
$$\le \omega(f,1/\log n)_{[-4|x|-4,4|x|+4]}\sum_{|x_k|\le 2(|x|+1)}(1+\log n|x-x_k|)\cdot l_k^2(x)+$$

$$\sum_{|x_k|>2(|x|+1)} |f(x)-f(x_k)|\cdot l_k^2(x) =: S_1+S_2.$$

Then Lemma 3 implies $S_1 \to 0,\ n\to\infty$. We estimate S_2. Assume $n^{1/12} > 2(|x|+1)$ for sufficiently large n:

$$\begin{aligned} S_2 &= \sum_{|x_k|>2(|x|+1)} |f(x)-f(x_k)|\cdot l_k^2(x) \\ &\le C\left(\sum_{2(|x|+1)<|x_k|\le n^{1/12}} + \sum_{|x_k|>n^{1/12}}\right) e^{(1-\varepsilon_0)x_k^2}\frac{(p_n(x)\omega(x))^2\omega^2(x_k)}{(p_n'(x_k)\omega(x_k))^2(x-x_k)^2}e^{x^2} \\ &=: S_{21}+S_{22}. \end{aligned}$$

By (6) and $2(|x|+1)<|x_k|\le n^{1/12}$, for sufficiently large n we have

$$x_k - x_{k+1} \sim \frac{a_n}{n} \sim n^{-1/2}.$$

Then

$$\left(\sum_{2(|x|+1)<|x_k|\le n^{1/12}} \frac{1}{(x-x_k)^2}\right) \le C\frac{n^{1/12}}{n^{-1/2}} = Cn^{7/12}.$$

Therefore it follows from Lemma 1 that

$$S_{21} \le C\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2})^2}\cdot n^{7/12}e^{x^2} \le Cn^{-1/12}e^{x^2},$$

while by (6),

$$\begin{aligned} S_{22} &\le C\frac{(a_n^{-1/2}n^{1/6})^2}{(\frac{n}{a_n}a_n^{-1/2}n^{-1/6})^2}e^{x^2}\sum_{|x_k|>n^{1/12}}\frac{e^{-\varepsilon_0 x_k^2}}{(x-x_k)^2} \\ &\le Cn^{-1/3}e^{x^2}e^{-\varepsilon_0 n^{1/6}}\cdot n \le Cn^{2/3}e^{x^2}e^{-\varepsilon_0 n^{1/6}}, \end{aligned}$$

which completes the proof of Lemma 5.

3. Proof of the Theorem

By noting the definition of Hermite-Fejér operator $F_n(f,x)$, we see that

$$F_n(f,x) = \sum_{k=1}^n f(x_k)\left[1-(x-x_k)\frac{p_n''(x_k)}{p_n'(x_k)}\right]l_k^2(x). \tag{7}$$

Let $f(x)\equiv 1$, whence $F_n(f,x)\equiv 1$, we have

$$1 = \sum_{k=1}^n \left[1-(x-x_k)\frac{p_n''(x_k)}{p_n'(x_k)}\right]l_k^2(x).$$

We rewrite

$$|H_n(f,x)-f(x)| \le \sum_{k=1}^n \left(\frac{p_n(x)}{p_n'(x_k)(x-x_k)}\right)^2 |f(x_k)-f(x)|\left|1-\frac{p_n''(x_k)}{p_n'(x_k)}(x-x_k)\right| +$$

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\frac{p_n(x)}{p_n'(x_k)(x-x_k)}\right)^2 |y_k||x-x_k| \\
\leq &\sum_{k=1}^{n}\left(\frac{p_n(x)}{p_n'(x_k)(x-x_k)}\right)^2 |f(x_k)-f(x)|+ \\
&\sum_{k=1}^{n}\left(\frac{p_n(x)}{p_n'(x_k)(x-x_k)}\right)^2 |f(x_k)-f(x)|\left|\frac{p_n''(x_k)}{p_n'(x_k)}(x-x_k)\right|+ \\
&\sum_{k=1}^{n}\left(\frac{p_n(x)}{p_n'(x_k)(x-x_k)}\right)^2 |y_k||x-x_k| \\
=:&J_1+J_2+J_3.
\end{aligned}$$

By the assumption , $|y_k| \leq Ce^{(2/1-\delta_0)x_k^2}$. Applying Lemma 5 to J_1, Lemma 4 to J_2 and Lemma 3 to J_3, we obtain for any given point $x \in \mathbb{R}$ that

$$H_n(f,x)-f(x) \to 0, \quad n \to \infty.$$

For the Hermite-Fejer operator $F_n(x)$ (see (7)), we have the following corollary:

Corollary *Given* $0 < \varepsilon_0 < 1$. *Let* $f \in C_{(-\infty,\infty)}$ *satisfy* $|f(x)| = O(e^{(1-\varepsilon_0)x^2})$. *Then for any given point* $x \in \mathbb{R}$, *we have* $\lim_{n\to\infty} F_n(f,x) = f(x)$.

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