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A NUMERICAL METHOD IN TERMS OF THE THIRD DERIVATIVE FOR A DELAY INTEGRAL EQUATION FROM BIOMATHEMATICS

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ABSTRACT. This paper presents a numerical method for approximating the positive, bounded and smooth solution of a delay integral equation which occurs in the study of the spread of epidemics. We use the cubic spline interpolation and obtain an algorithm based on a perturbed trapezoidal quadrature rule.

Key words and phrases: Delay Integral Equation, Successive approximations, Perturbed trapezoidal quadrature rule.

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1. INTRODUCTION

Consider the delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds. \tag{1.1}$$

This equation is a mathematical model for the spread of certain infectious diseases with a contact rate that varies seasonally. Here $x(t)$ is the proportion of infectives in the population at time t, $\tau > 0$, is the lenght of time in which an individual remains infectious and $f(t, x(t))$ is the proportion of new infectives per unit time.

There are known results about the existence of a positive bounded solution (see [3], [8]), which is periodic in certain conditions ([9]), or about the existence and uniqueness of the positive periodic solution (in [10], [11]). In [8] the author obtains the existence and uniqueness of

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the continuous positive bounded solution, and in [6] a numerical method for the approximation of this solution is provided using the trapezoidal quadrature rule.

Here, we obtain the existence and uniqueness of the positive, bounded and smooth solution and use the cubic spline of interpolation from [5] to approximate this solution on the initial interval $[-\tau, 0]$. We suppose that on $[-\tau, 0]$, the solution Φ is known only in the discrete moments $t_i, i = \overline{0, n}$, and use the values $\Phi(t_i) = y_i$, $i = \overline{0, n}$ for the spline interpolation. Afterward, we outline a numerical method and an algorithm to approximate the solution on $[0, T]$, with $T > 0$ fixed, using the quadrature rule from [1] and [2].

2. EXISTENCE AND UNIQUENESS OF THE POSITIVE SMOOTH SOLUTION

Impose the initial condition $x(t) = \Phi(t), t \in [-\tau, 0]$ to equation (1.1) and consider $T > 0$, be fixed. We obtain the initial value problem:

$$x(t) = \begin{cases} \int_{t-\tau}^{t} f(s, x(s))ds, & \forall t \in [0, T] \\ \\ \Phi(t), & \forall t \in [-\tau, 0]. \end{cases} \tag{2.1}$$

Suppose that the following conditions are fulfilled:

(i) $\Phi \in C^1[-\tau, 0]$ and we have

$$b = \Phi(0) = \int_{-\tau}^{0} f(s, x(s))ds \text{ with } \Phi'(0) = f(0, b) - f(-\tau, \Phi(-\tau));$$

(ii) $b > 0$ and $\exists a, M, \beta \in \mathbb{R}, M > 0, 0 < a \leq \beta$ such that $a \leq \Phi(t) \leq \beta, \quad \forall t \in [-\tau, 0]$;

(iii) $f \in C([-\tau, T] \times [a, \beta]),\ f(t, x) \geq 0,\ f(t, y) \leq M,\ \forall t \in [-\tau, T],\ \forall x \geq 0,\ \forall y \in [a, \beta]$;

(iv) $M\tau \leq \beta$ and there is an integrable function g such that $f(t, x) \geq g(t),\ \forall t \in [-\tau, T]$, $\forall x \geq a$ and

$$\int_{t-\tau}^{t} g(s)ds \geq a, \quad \forall t \in [0, T];$$

(v) $\exists L > 0$ such that $|f(t, x) - f(t, y)| \leq L\,|x - y|\,,\ \forall t \in [-\tau, T],\ \forall x, y \in [a, \infty)$.

Then, we obtain the following result:

Theorem 2.1. *Suppose that assumptions (i)-(v) are satisfied. Then the equation (1.1) has a unique continuous solution* $x(t)$ *on* $[-\tau, T]$, *with* $a \leq x(t) \leq \beta$, $\forall t \in [-\tau, T]$ *such that* $x(t) = \Phi(t)$ *for* $t \in [-\tau, 0]$. *Also,*

$$\max\{|x_n(t) - x(t)| : t \in [0, T]\} \longrightarrow 0$$

as $n \to \infty$ *where* $x_n(t) = \Phi(t)$ *for* $t \in [-\tau, 0]$, $n \in \mathbb{N}$, $x_0(t) = b$ *and*

$$x_n(t) = \int_{t-\tau}^{t} f(s, x_{n-1}(s))ds$$

for $t \in [0, T]$, $n \in \mathbb{N}^*$. *Moreover, the solution* x *belongs to* $C^1[-\tau, T]$.

Proof. From [9] under the conditions (i), (ii), (iv), (v), it follows that the existence of an unique positive continuous on $[-\tau, T]$ solution for (1.1) such that $x(t) \geq a$, $\forall t \in [-\tau, T]$ and $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$. Using Theorem 2 from [7] we conclude that $\max\{|x_n(t) - x(t)| : t \in [0, T]\} \longrightarrow 0$ as $n \to \infty$. From (iv) we see that

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds \leq \int_{t-\tau}^{t} Mds = M\tau \leq \beta,\ \forall t \in [0, T].$$

Because $a \leq \Phi(t) \leq \beta$ for $t \in [-\tau, 0]$ and $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$ we deduce that $a \leq x(t) \leq \beta$, $\forall t \in [-\tau, T]$, and the solution is bounded. Since x is a solution for (1.1) we have $x(t) = \int_{t-\tau}^{t} f(s, x(s))ds$, for all $t \in [0, T]$, and because $f \in C([-\tau, T] \times [a, \beta])$ we can state that x is differentiable on $[0, T]$, and x' is continuous on $[0, T]$. From condition (iii) it follows that x is differentiable with x' continuous on $[-\tau, 0]$ (including the continuity in the point $t = 0$). Then $x \in C^1[-\tau, T]$ and the proof is complete. □

Corollary 2.2. *In the conditions of the Theorem2.1, if* $f \in C^1([-\tau, T] \times [a, \beta])$, $\Phi \in C^2[-\tau, 0]$ *and*

$$\Phi''(0) = \frac{\partial f}{\partial t}(0, b) + \frac{\partial f}{\partial x}(0, b)\left[f(0, b) - f(-\tau, , \Phi(-\tau))\right] - \frac{\partial f}{\partial t}(-\tau, \Phi(-\tau)) - \frac{\partial f}{\partial x}(-\tau, \Phi(-\tau))\, \Phi'(-\tau),$$

then $x \in C^2[-\tau, T]$.

Proof. Follows directly from the above theorem. □

3. APPROXIMATION OF THE SOLUTION ON THE INITIAL INTERVAL

Suppose that on the interval $[-\tau, 0]$ the function Φ is known only in the points, t_i, $i = \overline{0, n}$, which form the uniform partition

(3.1) $$\Delta_n : -\tau = t_0 < t_1 < \cdots < t_{n-1} < t_n = 0,$$

where we have the values $\Phi(t_i) = y_i$, $i = \overline{0, n}$ and $t_{i+1} - t_i = h = \frac{\tau}{n}$, $\forall i = \overline{0, n-1}$.

Let

$$m_0 = \frac{1}{h}\left(\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4}\right), \text{ and}$$
$$M_0 = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11\Delta^4 y_0}{12}\right),$$

where,

$$\Delta y_0 = y_1 - y_0, \qquad \Delta^2 y_0 = y_2 - 2y_1 + y_0,$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0, \qquad \Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

We build a cubic spline of interpolation which corresponds to the following conditions:

(3.2) $$\Phi(t_i) = y_i, i = \overline{0, n}, \qquad \Phi'(t_0) = m_0, \qquad \Phi''(t_0) = M_0.$$

This spline function is $s \in C^2[-\tau, 0]$ which, according to [5], have on the each subinterval $[t_{i-1}, t_i]$, $i = \overline{1, n}$, the expression:

(3.3) $$s(t) = \frac{M_i - M_{i-1}}{6h_i} \cdot (t - t_{i-1})^3 + \frac{M_{i-1}}{2} \cdot (t - t_{i-1})^2 + m_{i-1} \cdot (t - t_{i-1}) + y_{i-1},$$

for $t \in [t_{i-1}, t_i]$ where $y_i = s(t_i)$, $m_i = s'(t_i)$, $M_i = s''(t_i)$, $i = \overline{0, n}$. For these values, according to [5], there exists the recurence relations:

(3.4) $$\begin{cases} M_i = 6 \cdot \dfrac{y_i - y_{i-1}}{h^2} - \dfrac{6m_{i-1}}{h} - 2M_{i-1} \\ m_i = 3 \cdot \dfrac{y_i - y_{i-1}}{h} - 2m_{i-1} - \dfrac{1}{2}M_{i-1} \cdot h \end{cases}, \quad i = \overline{1, n}.$$

Then, from [5] Lemma 2.1, there exists a unique cubic spline function of interpolation, s, which satisfy the conditions:

$$\begin{cases} s(t_i) = y_i, \quad \forall i = \overline{0,n} \\ s'(t_0) = m_0 \\ s''(t_0) = M_0, \end{cases} \tag{3.5}$$

and on the each subinterval of Δ_n is defined by the relation (3.3).

Between the values of s' and s'' on the knots there exists the relations:

$$\begin{cases} M_i + 2M_{i-1} = 6 \cdot \dfrac{y_i - y_{i-1} - m_{i-1} \cdot h}{h^2} \\ \\ M_i + M_{i-1} = \dfrac{2\,(m_i - m_{i-1})}{h} \end{cases}, \quad i = \overline{1,n}. \tag{3.6}$$

This spline function s approximates the solution Φ on the interval $[-\tau, 0]$ and we have $s(t_i) = \Phi(t_i) = y_i,\ \forall i = \overline{0,n}$.

Remark 3.1. Since $\Phi \in C^2[-\tau, 0]$, we can estimate the error of this approximation, $\|\Phi - s\|$, where $\|\cdot\|$ is the Čebyšev norm on the set of continuous functions on an compact interval of the real axis: $\|u\| = \max\{|u(t)| : t$ lies in an compact interval$\}$, for any continuous function u on this interval. If we know the value $\|\Phi''\|_2 = \left(\int_{-\tau}^{0} [\Phi''(t)]^2 dt\right)^{\frac{1}{2}}$, then for each $t \in [-\tau, 0]$ we have

$$|\Phi(t) - s(t)| \leq \|\Phi - s\| \leq \|\Phi''\|_2 \cdot h^{\frac{3}{2}} \leq \sqrt[2]{\tau}\, \|\Phi''\| \cdot h^{\frac{3}{2}},$$

according to [4] page 127. Else, if we know only the values $y_i = \Phi(t_i), i = \overline{0,n}$ then $|\Phi(t) - s(t)| \leq \|\Phi - s\|$, $\forall t \in [-\tau, 0]$, and for $\|\Phi - s\|$ we use the error estimation from [7], since $\Phi \in C^2[-\tau, 0]$ and then Φ is a Lipschitzian function on $[-\tau, 0]$.

In [7], it has been shown that

$$\|\Phi - s\| \leq \max\{\|s - F_1\|, \|s - F_2\|\}$$

where

$$\|s - F_1\| = \max\{a_i : i = \overline{1,n}\}$$
$$\|s - F_2\| = \max\{b_i : i = \overline{1,n}\}$$

and

$$a_i = \left\|(s - F_1)\ |_{[t_{i-1}, t_i]}\right\|,$$
$$b_i = \left\|(s - F_2)\ |_{[t_{i-1}, t_i]}\right\|,$$
$$F_1(t) = \sup\{\Phi(t_k) - \|\Phi\|_L \cdot |t - t_k| : k = \overline{0,n}\},$$
$$F_2(t) = \inf\{\Phi(t_k) + \|\Phi\|_L \cdot |t - t_k| : k = \overline{0,n}\},$$

with

$$\|\Phi\ |_{\Delta_n}\|_L = \max\{|\ [t_{i-1}, t_i; \Phi]\ | : i = \overline{1,n}\}$$

if $[t_{i-1}, t_i; \Phi] = [\Phi(t_i) - \Phi(t_{i-1})]/(t_i - t_{i-1})$ is the divided difference of the function Φ on the knots t_{i-1}, t_i.

4. MAIN RESULTS

From Theorem 2.1 follows that the equation (1.1) has a unique positive, bounded and smooth solution on $[-\tau, T]$. Let φ be this solution, which, by virtue of Theorem 2.1, can be obtained by successive approximations method on $[0, T]$.

So, we have :

$$\text{(4.1)}\qquad \begin{cases} \varphi_m(t) = \Phi(t) \text{ , for } t \in [-\tau, 0],\ \forall m \in \mathbb{N} \text{ and} \\ \varphi_0(t) = \Phi(0) = b = \int_{-\tau}^{0} f(s, \Phi(s))ds, \\ \varphi_1(t) = \int_{t-\tau}^{t} f(s, \varphi_0(s))ds = \int_{t-\tau}^{t} f(s, b)ds, \\ \cdots \\ \varphi_m(t) = \int_{t-\tau}^{t} f(s, \varphi_{m-1}(s))ds, \\ \cdots \end{cases} , \quad \text{for } t \in [0, T].$$

To obtain the sequence of successive approximations (4.1) we compute the integrals using a quadrature rule.

We assume that there is $l \in \mathbb{N}^*$ such that $T = l\tau$. On each interval $[i\tau, (i+1)\tau],\ \ i = \overline{0, l-1}$ we establish an equidistant partition. Then on the interval $[-\tau, T]$ we have $q = l \cdot n + n + 1$ knots which realize the division:

$$\text{(4.2)}\qquad -\tau = t_0 < t_1 < \cdots < t_{n-1} < 0 = t_n < t_{n+1} < \cdots < t_{q-1} < t_q = T,$$

having $t_{i+1} - t_i = h,\ \forall i = \overline{n, q-1}$. We can see that $t_j - \tau = t_{j-n},\ \forall j = \overline{n, q}$.

In the aim to compute the integrals from (4.1) we use the following quadrature rule of N.S. Barnett and S.S. Dragomir in [1]:

$$\text{(4.3)}\qquad \int_a^b F(t)dt = \frac{(b-a)}{2n} \sum_{i=0}^{n-1} [F(t_i) + F(t_{i+1})] - \frac{(b-a)^2}{12n^2} [F'(a) - F'(b)] + R_n(F),$$

where

$$t_i = a + i \cdot \frac{b-a}{n}, \quad i = \overline{0, n},$$

and

$$|R_n(F)| \le \frac{(b-a)^4}{160n^3} \cdot \|F'''\| \,, \quad \text{ if } F \in C^3[a, b].$$

Here, we consider the function F defined by $F(t) = f(t, x(t))$, for $t \in [-\tau, T]$. Since the solution of (2.1) verify the relation,

$$x'(t) = f(t, x(t)) - f(t - \tau, x(t - \tau)), \quad \forall t \in [0, T],$$

we point out the following connection between the smoothness of x and f.

Remark 4.1. In the conditions of Corollary 2.2, if $\Phi \in C^3[-\tau, 0]$, $f \in C^2([-\tau, T] \times [a, \beta])$, and

$$\Phi'''(0) = \lim_{t>0, t\to 0} \frac{d}{dt} \left[\frac{\partial f}{\partial t}(t, \varphi(t)) + \frac{\partial f}{\partial x}(t, \varphi(t))\varphi'(t) \right. \\ \left. - \frac{\partial f}{\partial t}(t - \tau, \varphi(t - \tau)) - \frac{\partial f}{\partial x}(t - \tau, \varphi(t - \tau))\, \varphi'(t - \tau) \right],$$

then $x \in C^3[-\tau, T]$.

If $x \in C^3[-\tau, T]$ and $f \in C^3([-\tau, T] \times [a, \beta])$ then $F \in C^3[-\tau, T]$ and $F'''(t) = [f(t, x(t))]_t'''$, $\forall t \in [-\tau, T]$. For this function F we apply the quadrature rule (4.3) and obtain the approximative values of the solution φ at the points $t_k,\ k = \overline{n+1, q}$, as in the following formula:

$$
\begin{aligned}
\varphi_m(t_k) &= \int_{t_k-\tau}^{t_k} f(s, \varphi_{m-1}(s))ds \\
&= \frac{\tau}{2n} \sum_{i=0}^{n-1} [f(t_{k+i} - \tau, \varphi_{m-1}(t_{k+i} - \tau)) + f(t_{k+i+1} - \tau, \varphi_{m-1}(t_{k+i+1} - \tau))] \\
&\quad - \frac{\tau^2}{12n^2} \left[\frac{\partial f}{\partial t}(t_k, \varphi_{m-1}(t_k)) \right. \\
&\quad + \frac{\partial f}{\partial x}(t_k, \varphi_{m-1}(t_k)) \cdot \varphi'_{m-1}(t_k) - \frac{\partial f}{\partial t}(t_k - \tau, \varphi_{m-1}(t_k - \tau)) \\
&\quad \left. - \frac{\partial f}{\partial x}(t_k - \tau, \varphi_{m-1}(t_k - \tau)) \cdot \varphi'_{m-1}(t_k - \tau) \right] + r_{m,k}^{(n)}(f),
\end{aligned} \tag{4.4}
$$

$\forall\ m \in \mathbb{N}^*$ and $\forall k = \overline{n+1, q}$, where,

$$\varphi'_{m-1}(t) = f(t, \varphi_{m-2}(t)) - f(t - \tau, \varphi_{m-2}(t - \tau)), \quad \forall t \in [0, T],\ \forall m \in \mathbb{N},\ m \geq 2,$$

and $\varphi'_0(t) = 0,\ \varphi'_1(t) = f(t, b) - f(t - \tau, b),\ \forall t \in [0, T]$.

To estimate the remainder we need to obtain an upper bound for the third derivative $[f(t, x(t))]_t'''$. After elementary calculus we have for $t \in [-\tau, T]$:

$$
\begin{aligned}
[f(t, x(t))]_t''' &= \frac{\partial^3 f}{\partial t^3}(t, x(t)) + 3\frac{\partial^3 f}{\partial t^2 \partial x}(t, x(t)) \cdot x'(t) \\
&\quad + 3\frac{\partial^3 f}{\partial t \partial x^2}(t, x(t)) \cdot [x'(t)]^2 + \frac{\partial^3 f}{\partial x^3}(t, x(t)) [x'(t)]^3 \\
&\quad + 3\frac{\partial^2 f}{\partial t \partial x}(t, x(t)) x''(t) + 3\frac{\partial^2 f}{\partial x^2}(t, x(t)) x'(t) x''(t) + \frac{\partial f}{\partial x}(t, x(t)) x'''(t).
\end{aligned}
$$

We denote

$$
\begin{aligned}
M_0 = M &= \max \{|f(t, x)| : t \in [-\tau, T],\ x \in [a, \beta]\} \\
\left\| \frac{\partial^\alpha f}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \right\| &= \max \left\{ \left| \frac{\partial^{|\alpha|} f(t, x)}{\partial t^{\alpha_1} \partial x^{\alpha_2}} \right| : t \in [-\tau, T],\ x \in [a, \beta],\ \alpha_1 + \alpha_2 = |\alpha| \right\} \\
M_1 &= \max \left\{ \left\| \frac{\partial f}{\partial t} \right\|, \left\| \frac{\partial f}{\partial x} \right\| \right\}, \\
M_2 &= \max \left\{ \left\| \frac{\partial^2 f}{\partial t^2} \right\|, \left\| \frac{\partial^2 f}{\partial t \partial x} \right\|, \left\| \frac{\partial^2 f}{\partial x^2} \right\| \right\}, \\
M_3 &= \max \left\{ \left\| \frac{\partial^3 f}{\partial t^3} \right\|, \left\| \frac{\partial^3 f}{\partial t^2 \partial x} \right\|, \left\| \frac{\partial^3 f}{\partial t \partial x^2} \right\|, \left\| \frac{\partial^3 f}{\partial x^3} \right\| \right\}.
\end{aligned}
$$

Consequently, we obtain the estimations:

$$
\begin{aligned}
\left| [f(t, x(t))]_t''' \right| &\leq M_3(1 + 6M_0 + 12M_0^2 + 8M_0^3) \\
&\quad + M_2(8M_1 + 32M_0M_1 + 32M_0^2M_1) \\
&\quad + 4M_1^3 + 8M_0M_3 = M''', \qquad \forall t \in [-\tau, T],
\end{aligned}
$$

and

$$\left|r_{m,k}^{(n)}(f)\right| \le \frac{\tau^4 M'''}{160n^3}, \qquad \forall m \in \mathbb{N}^*, \ \forall k = \overline{n+1,q}.$$

Then, to compute the integrals (4.1), we can use the following algorithm:

$$\begin{aligned}
\varphi_1(t_k) &= \frac{\tau}{2n}\sum_{i=0}^{n-1}\left[f(t_{k+i}-\tau,\varphi_0(t_{k+i}-\tau)) + f(t_{k+i+1}-\tau,\varphi_0(t_{k+i+1}-\tau))\right] \\
&\quad - \frac{\tau^2}{12n^2}\left[\frac{\partial f}{\partial t}(t_k,\varphi_0(t_k)) + \frac{\partial f}{\partial x}(t_k,\varphi_0(t_k))\cdot\varphi_0'(t_k)\right. \\
&\quad \left. - \frac{\partial f}{\partial t}(t_k-\tau,\varphi_0(t_k-\tau)) - \frac{\partial f}{\partial x}(t_k-\tau,\varphi_0(t_k-\tau))\cdot\varphi_0'(t_k-\tau)\right] + r_{1,k}^{(n)}(f) \\
&=: \widetilde{\varphi_1}(t_k) + r_{1,k}^{(n)}(f),
\end{aligned}$$

(4.5) $\varphi_2(t_k)$

$$\begin{aligned}
&= \frac{\tau}{2n}\sum_{i=0}^{n-1}\left[f(t_{k+i}-\tau,\varphi_1(t_{k+i}-\tau)) + f(t_{k+i+1}-\tau,\varphi_1(t_{k+i+1}-\tau))\right] \\
&\quad - \frac{\tau^2}{12n^2}\left[\frac{\partial f}{\partial t}(t_k,\varphi_1(t_k)) + \frac{\partial f}{\partial x}(t_k,\varphi_1(t_k))\cdot\varphi_1'(t_k)\right. \\
&\quad \left. -\frac{\partial f}{\partial t}(t_k-\tau,\varphi_1(t_k-\tau)) - \frac{\partial f}{\partial x}(t_k-\tau,\varphi_1(t_k-\tau))\cdot\varphi_1'(t_k-\tau)\right] + r_{1,k}^{(n)}(f) \\
&= \frac{\tau}{2n}\sum_{i=0}^{n-1}\left[f(t_{k+i}-\tau,\widetilde{\varphi_1}(t_{k+i}-\tau) + r_{1,k+i-n}^{(n)}(f))\right. \\
&\quad \left. + f(t_{k+i+1}-\tau,\widetilde{\varphi_1}(t_{k+i+1}-\tau) + r_{1,k+i+1-n}^{(n)}(f))\right] \\
&\quad - \frac{\tau^2}{12n^2}\cdot\left[\frac{\partial f}{\partial t}(t_k,\widetilde{\varphi_1}(t_k) + r_{1,k}^{(n)}(f))\right. \\
&\quad + \frac{\partial f}{\partial x}(t_k,\widetilde{\varphi_1}(t_k) + r_{1,k}^{(n)}(f))\cdot(f(t_k,b) - f(t_k-\tau,b)) \\
&\quad - \frac{\partial f}{\partial t}(t_k-\tau,\widetilde{\varphi_1}(t_k-\tau) + r_{1,k-n}^{(n)}(f)) \\
&\quad \left. -\frac{\partial f}{\partial x}(t_k-\tau,\widetilde{\varphi_1}(t_k-\tau) + r_{1,k-n}^{(n)}(f))(f(t_k-\tau,b) - f(t_k-2\tau,b))\right] + r_{2,k}^{(n)}(f) \\
&= \frac{\tau}{2n}\sum_{i=0}^{n-1}\left[f(t_{k+i}--\tau,\widetilde{\varphi_1}(t_{k+i}-\tau)) + f(t_{k+i+1}-\tau,\widetilde{\varphi_1}(t_{k+i+1}-\tau))\right] \\
&\quad - \frac{\tau^2}{12n^2}\cdot\left[\frac{\partial f}{\partial t}(t_k,\widetilde{\varphi_1}(t_k)) + \frac{\partial f}{\partial x}(t_k,\widetilde{\varphi_1}(t_k))\cdot(f(t_k,b) - f(t_k-\tau,b))\right. \\
&\quad - \frac{\partial f}{\partial t}(t_k-\tau,\widetilde{\varphi_1}(t_k-\tau)) \\
&\quad \left. -\frac{\partial f}{\partial x}(t_k-\tau,\widetilde{\varphi_1}(t_k-\tau))\cdot(f(t_k-\tau,b) - f(t_k-2\tau,b))\right] + \widetilde{r_{2,k}^{(n)}(f)} \\
&=: \widetilde{\varphi_2(t_k)} + \widetilde{r_{2,k}^{(n)}(f)}, \qquad \forall k = \overline{n+1,q}.
\end{aligned}$$

We have the remainder estimation:

$$\left|\widetilde{r_{2,k}^{(n)}}(f)\right| \leq \frac{\tau^4 M'''}{160n^3}\left[1+\tau L+\frac{\tau^2 M_2(1+2M)}{6n^2}\right], \qquad \forall k=\overline{n+1,q}.$$

By induction, for $m \geq 3$ we obtain:

$$\begin{aligned}
(4.6)\quad \varphi_m(t_k) &= \frac{\tau}{2n}\sum_{i=0}^{n-1}\Big[f(t_{k+i}-\tau,\widetilde{\varphi_{m-1}}(t_{k+i}-\tau)+\widetilde{r_{m-1,k+i-n}^{(n)}(f)})\\
&\quad + f(t_{k+i+1}-\tau,\widetilde{\varphi_{m-1}}(t_{k+i+1}-\tau,\widetilde{\varphi_{m-1}}(t_{k+i+1}-\tau)+\widetilde{r_{m-1,k+i+1-n}}(f))\Big]\\
&\quad - \frac{\tau^2}{12n^2}\cdot\Big[\frac{\partial f}{\partial t}(t_k,\widetilde{\varphi_{m-1}}(t_k)+\widetilde{r_{m-1,k}^{(n)}}(f))+\frac{\partial f}{\partial x}(t_k,\widetilde{\varphi_{m-1}}(t_k)\\
&\quad + \widetilde{r_{m-1,k}^{(n)}}(f))\cdot\varphi_{m-1}'(t_k)-\frac{\partial f}{\partial t}(t_k-\tau,\widetilde{\varphi_{m-1}}(t_k-\tau)+\widetilde{r_{m-1,k-n}^{(n)}}(f))\\
&\quad -\frac{\partial f}{\partial x}(t_k-\tau,\widetilde{\varphi_{m-1}}(t_k-\tau)+\widetilde{r_{m-1,k-n}^{(n)}}(f))\cdot\varphi_{m-1}'(t_k-\tau)\Big]\\
&= \frac{\tau}{2n}\sum_{i=0}^{n-1}\Big[f(t_{k+i}-\tau,\widetilde{\varphi_{m-1}}(t_{k+i}-\tau)+\widetilde{r_{m-1,k+i-n}^{(n)}(f)})\\
&\quad +f(t_{k+i+1}-\tau,\widetilde{\varphi_{m-1}}(t_{k+i+1}-\tau)+\widetilde{r_{m-1,k+i+1-n}}(f))\Big]\\
&\quad - \frac{\tau^2}{12n^2}\cdot\Big[\frac{\partial f}{\partial t}(t_k,\widetilde{\varphi_{m-1}}(t_k)+\widetilde{r_{m-1,k}^{(n)}}(f))+\frac{\partial f}{\partial x}(t_k,\widetilde{\varphi_{m-1}}(t_k)\\
&\quad + \widetilde{r_{m-1,k}^{(n)}}(f))\cdot(f(t_k,\widetilde{\varphi_{m-2}}(t_k)+\widetilde{r_{m-2,k}^{(n)}}(f))\\
&\quad - f(t_k-\tau,\widetilde{\varphi_{m-2}}(t_k-\tau)+\widetilde{r_{m-2,k-n}^{(n)}}(f))-\frac{\partial f}{\partial t}(t_k-\tau,\widetilde{\varphi_{m-1}}(t_k-\tau)\\
&\quad + \widetilde{r_{m-1,k-n}^{(n)}}(f))-\frac{\partial f}{\partial x}(t_k-\tau,\widetilde{\varphi_{m-1}}(t_k-\tau)\\
&\quad + \widetilde{r_{m-1,k-n}^{(n)}}(f))\cdot(f(t_k-\tau,\widetilde{\varphi_{m-2}}(t_k-\tau)+\widetilde{r_{m-2,k-n}^{(n)}}(f))\\
&\quad - f(t_k-2\tau,\widetilde{\varphi_{m-2}}(t_k-2\tau)+\widetilde{r_{m-2,k-2n}^{(n)}}(f)))\Big]]+r_{m,k}^{(n)}(f)\\
&= \frac{\tau}{2n}\sum_{i=0}^{n-1}\left[f\left(t_{k+i}-\tau,\widetilde{\varphi_{m-1}}(t_{k+i}-\tau)\right)+f\left(t_{k+i+1}-\tau,\widetilde{\varphi_{m-1}}(t_{k+i+1}-\tau)\right)\right]\\
&- \frac{\tau^2}{12n^2}\cdot\Big[\frac{\partial f}{\partial t}\left(t_k,\widetilde{\varphi_{m-1}}(t_k)\right)+\frac{\partial f}{\partial x}\left(t_k,\widetilde{\varphi_{m-1}}(t_k)\right)\cdot(f\left(t_k,\widetilde{\varphi_{m-2}}(t_k)\right)\\
&\quad - f(t_k-\tau,\widetilde{\varphi_{m-2}}(t_k-\tau))-\frac{\partial f}{\partial t}\left(t_k-\tau,\widetilde{\varphi_{m-1}}(t_k-\tau)\right)\\
&\quad - \frac{\partial f}{\partial x}\left(t_k-\tau,\widetilde{\varphi_{m-1}}(t_k-\tau)\right)\cdot(f(t_k-\tau),\widetilde{\varphi_{m-2}}(t_k-\tau)\\
&\quad - f\left(t_k-2\tau,\widetilde{\varphi_{m-2}}(t_k-2\tau)\right))\Big]+\widetilde{r_{m,k}^{(n)}(f)}\\
&=: \widetilde{\varphi_m}(t_k)+\widetilde{r_{m,k}^{(n)}(f)},
\end{aligned}$$

$\forall m \in \mathbb{N}\,,\, m \geq 2\,,\, \forall k = \overline{n+1, q}.$

Remark 4.2. We can see that, for $k = \overline{n, 2n}$ we have $t_k - \tau \in [-\tau, 0]$ and then

$$\varphi'_{m-1}(t_k - \tau) = \Phi'(t_k - \tau) = s'(t_k - \tau) = m_{k-n}$$

for $m \in \mathbb{N}, m \geq 2$ in (4.5) and (4.6).

For the remainders we have the estimations:

$$\begin{aligned}(4.7)\qquad \left|\widetilde{r_{m,k}^{(n)}(f)}\right| &\leq \frac{\tau^4 M'''}{160n^3} + \left|\widetilde{r_{m-1,k}^{(n)}(f)}\right| \cdot \left[\tau L + \frac{\tau^2 M_2(1+2M)}{6n^2}\right] \\ &\quad + \frac{\tau^2 L M_2}{3n^2}\left|\widetilde{r_{m-1,k}^{(n)}(f)}\right| \cdot \left|\widetilde{r_{m-2,k}^{(n)}(f)}\right| \\ &\leq \frac{\tau^4 M'''}{160n^3}\left(1 + \tau L + \dots + \tau^{m-1} L^{m-1}\right) \\ &\quad + \frac{\tau^m L^{m-2} M_2 \cdot (2M+1)}{6n^2} \cdot \frac{\tau^4 M'''}{160n^3} + O\left(\frac{1}{n^5}\right) \\ &= \frac{\tau^4 M'''(1-\tau^m L^m)}{160n^3\,(1-\tau L)} + \frac{\tau^6 M''' \tau^{m-2} L^{m-2} M_2 \cdot (2M+1)}{960n^5} + O\left(\frac{1}{n^5}\right),\end{aligned}$$

$\forall m \in \mathbb{N},\ m \geq 3,\ \forall k = \overline{n+1, q}.$

For instance, if $m = 3$ we have the estimation:

$$\begin{aligned}(4.8)\qquad \left|\widetilde{r_{3,k}^{(n)}(f)}\right| &\leq \frac{\tau^4 M'''}{160n^3}\left(1 + \tau L + \tau^2 L^2\right) \\ &\quad + \frac{\tau^6 M''' M_2 \cdot (1+2\tau L)(2M+1)}{960n^5} + \frac{\tau^8 M''' M_2^2 \cdot (2M+1)^2}{5760n^7} \\ &\quad + \frac{\tau^{10}(M''')^2 M_2 L \cdot (1+\tau L)}{76800n^8} + \frac{\tau^{12}(M''')^2 M_2^2 L \cdot (2M+1)}{460800n^{10}},\end{aligned}$$

$\forall k = \overline{n+1, q}.$

We obtain the following result:

Theorem 4.3. *Considering the initial value problem (2.1) under the conditions from Corollary 2.2 and Remark 4.1, if* $f \in C^3\left([-\tau, T] \times [a, \beta]\right)$, $\tau L < 1$ *and the exact solution* φ *is approximated by the sequence* $\left(\widetilde{\varphi_m}(t_k)\right)_{k \in \mathbb{N}^*}$, $k = \overline{1, q}$ *, on the equidistant points (3.1), through the successive approximation method (4.1), combined with the quadrature rule (4.3), then the following error estimation holds :*

$$\begin{aligned}(4.9)\qquad |\varphi(t_k) - \widetilde{\varphi_m}(t_k)| &\leq \frac{\tau^m \cdot L^m}{1-\tau L} \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]} + \frac{\tau^4 M'''}{160n^3\,(1-\tau L)} \\ &\quad + \frac{\tau^6 M''' \tau^{m-2} L^{m-2} M_2 \cdot (2M+1)}{960n^5} + O\left(\frac{1}{n^5}\right), \\ &\qquad \forall m \in \mathbb{N}^*,\ m \geq 2,\ \forall k = \overline{n+1, q}.\end{aligned}$$

Proof. We have

$$|\varphi(t_k) - \widetilde{\varphi_m}(t_k)| \leq |\varphi(t_k) - \varphi_m(t_k)| + |\varphi_m(t_k) - \widetilde{\varphi_m}(t_k)|\,,\ \forall m \in \mathbb{N}^*,\ \forall k = \overline{n, q}.$$

From Banach's fixed point principle we have

$$|\varphi(t_k) - \varphi_m(t_k)| \leq \|\varphi - \varphi_m\| \leq \frac{\tau^m \cdot L^m}{1-\tau L} \cdot \|\varphi_0 - \varphi_1\|_{C[0,T]}\,,$$

$\forall m \in \mathbb{N}^*,\ \forall k = \overline{n+1,q}$. Also,

$$|\varphi_m(t_k) - \widetilde{\varphi_m}(t_k)| \leq \left|\widetilde{r_{m,k}^{(n)}}(f)\right|, \quad \forall m \in \mathbb{N}^*,\ m \geq 2,\ \forall k = \overline{n+1,q}.$$

Using the remainder estimation (4.7), the proof is completed. □

If F''' is Lipschitzian we can use a recent formula of N.S. Barnett and S.S. Dragomir from Corollary 1 in [2],

$$\int_a^b F(t)dt = \frac{(b-a)}{2}[F(a)+F(b)] - \frac{(b-a)^2}{12}[F'(b) - F'(a)] + R(F),$$

with $|R(F)| \leq \frac{L(b-a)^5}{720}$, where L is the Lipschitz constant. A composite quadrature formula can be easy obtained, considering an uniform partition of $[a,b]$ with the knots $t_i = a + i \cdot \frac{b-a}{n}$, $i = \overline{0,n}$ and the step $h = \frac{b-a}{n}$,

$$\int_a^b F(t)dt = \frac{(b-a)}{2n}\sum_{i=0}^{n-1}[F(t_i) + F(t_{i+1})] - \frac{(b-a)^2}{12n^2}[F'(a) - F'(b)] + R_n(F),$$

having the remainder estimation,

$$|R_n(F)| \leq \frac{L(b-a)^5}{720n^4}.$$

Here, we use these formulas for $F(t) = f(t, x(t))$ and obtain,

$$\begin{aligned}
(4.10) \qquad \int_{t_k-\tau}^{t_k} f(t,x(t))dt &= \int_{t_k-\tau}^{t_k} F(t)dt \\
&= \frac{\tau}{2n}\sum_{i=0}^{n-1}[F(t_{k+i}-\tau) + F(t_{k+i+1}-\tau)] \\
&\quad - \frac{\tau^2}{12n^2}[F'(t_k) - F'(t_k-\tau)] + R_n(F), \quad \forall k = \overline{n+1,q}.
\end{aligned}$$

with

$$|R_n(F)| \leq \frac{\tau^5 L_3}{720n^4},$$

if $\exists L_3 > 0$ such that $|F'''(u) - F'''(v)| \leq L_3|u-v|,\ \forall u,v \in [-\tau, T]$. From (4.7) and (4.3) we see that the relations (4.5) and (4.6) remains unchanged in this case, and for the remainders the estimations (4.7) becomes:

$$\left|r_{1,k}^{(n)}(f)\right| \leq \frac{\tau^5 L_3}{720n^4}, \qquad \forall k = \overline{n+1,q}$$

$$\left|\widetilde{r_{2,k}^{(n)}}(f)\right| \leq \frac{\tau^5 L_3}{720n^4}\left[1 + \tau L + \frac{\tau^2 M_2(1+2M)}{6n^2}\right], \qquad \forall k = \overline{n+1,q}$$

$$\begin{aligned}
(4.11) \qquad \left|\widetilde{r_{m,k}^{(n)}}(f)\right| &\leq \frac{\tau^5 L_3}{720n^4} + \left|\widetilde{r_{m-1,k}^{(n)}}(f)\right| \cdot \left[\tau L + \frac{\tau^2 M_2(1+2M)}{6n^2}\right] \\
&\quad + \frac{\tau^2 L M_2}{3n^2}\left|\widetilde{r_{m-1,k}^{(n)}}(f)\right| \cdot \left|\widetilde{r_{m-2,k}^{(n)}}(f)\right| \\
&\leq \frac{\tau^5 L_3(1-\tau^m L^m)}{720n^4(1-\tau L)} + \frac{\tau^7 L_3 \tau^{m-2} L^{m-2} M_2 \cdot (2M+1)}{4320n^6} + O\left(\frac{1}{n^6}\right),
\end{aligned}$$

$\forall m \in \mathbb{N}^*,\ \forall k = \overline{n+1,q}$.

For instance, the estimation (4.8) becomes:

$$\begin{aligned}\left|\widetilde{r_{3,k}^{(n)}}(f)\right| &\le \frac{\tau^5 L_3}{720n^4} + \left|\widetilde{r_{2,k}^{(n)}}(f)\right| \cdot \left[\tau L + \frac{\tau^2 M_2(1+2M)}{6n^2}\right] + \frac{\tau^2 L M_2}{3n^2}\left|\widetilde{r_{2,k}^{(n)}}(f)\right| \cdot \left|r_{1,k}^{(n)}(f)\right| \\ &\le \frac{\tau^5 L_3}{720n^4}\left(1+\tau L+\tau^2 L^2\right) \\ &\quad + \frac{\tau^7 L_3 M_2 \cdot (1+2\tau L)(2M+1)}{4320n^6} + \frac{\tau^9 L_3 M_2^2 \cdot (2M+1)^2}{25920n^8} \\ &\quad + \frac{\tau^{12}(L_3)^2 M_2 L \cdot (1+\tau L)}{1825200n^{10}} + \frac{\tau^{14}(L_3)^2 M_2^2 L \cdot (2M+1)}{10951200n^{12}}, \quad \forall k = \overline{n+1,q}.\end{aligned}$$

In this way we obtain the following result:

Theorem 4.4. *If* $f \in C^3([-\tau,T]\times[\alpha,\beta]), \Phi \in C^3[-\tau,0]$, *having,*

$$\begin{aligned}\Phi'''(0) = \lim_{t>0,t\to 0}\frac{d}{dt}\Bigg[&\frac{\partial f}{\partial t}(t,\varphi(t)) + \frac{\partial f}{\partial x}(t,\varphi(t))\varphi'(t) \\ &-\frac{\partial f}{\partial t}\left(t-\tau,\varphi(t-\tau)\right) - \frac{\partial f}{\partial x}\left(t-\tau,\varphi(t-\tau)\right)\varphi'(t-\tau)\Bigg],\end{aligned}$$

and the functions $\frac{\partial^3 f}{\partial t^3}, \frac{\partial^3 f}{\partial t \partial x^2}$ *and* $\frac{\partial^3 f}{\partial x^3}$ *are Lipschitzian in* t *and* x*, then* $\varphi \in C^3[-\tau,T]$, F''' *is Lipschitzian with a Lipschitz constant* $L_3 > 0$ *and the following estimation holds:*

$$\begin{aligned}|\varphi(t_k) - \widetilde{\varphi_m}(t_k)| \le \frac{\tau^m \cdot L^m}{1-\tau L}\cdot\|\varphi_0-\varphi_1\|_{C[0,T]} + \frac{\tau^5 L_3}{160n^3(1-\tau L)} \\ + \frac{\tau^7 L_3 \tau^{m-2} L^{m-2} M_2 \cdot (2M+1)}{4320n^6} + O\left(\frac{1}{n^6}\right), \\ \forall m \in \mathbb{N}^*, \quad m \ge 2, \quad \forall k = \overline{n+1,q}.\end{aligned}$$

Proof. From Remark 4.1, we infer that $\varphi \in C^3[-\tau,T]$ and because $f \in C^3([-\tau,T]\times[\alpha,\beta])$ we see that $F \in C^3[-\tau,T]$ and $\frac{\partial f}{\partial x}$ is Lipschitzian in t and x. For this reason there exist $L_{01}, L'_{01} > 0$ such that

$$\left|\frac{\partial f}{\partial x}(u,x) - \frac{\partial f}{\partial x}(v,x)\right| \le L_{01}|u-v|$$
$$\left|\frac{\partial f}{\partial x}(u,x) - \frac{\partial f}{\partial x}(u,y)\right| \le L'_{01}|x-y|,$$

$\forall u,v \in [-\tau,T], \forall x,y \in [\alpha,\beta]$. Since $\frac{\partial^3 f}{\partial t^3}$ is Lipschitzian in t and x there exist $L_{30}, L'_{30} > 0$ such that

$$\begin{aligned}&\left|\frac{\partial^3 f}{\partial t^3}(u,x(u)) - \frac{\partial^3 f}{\partial t^3}(v,x(v))\right| \\ &\quad\le \left|\frac{\partial^3 f}{\partial t^3}(u,x(u)) - \frac{\partial^3 f}{\partial t^3}(v,x(u))\right| + \left|\frac{\partial^3 f}{\partial t^3}(v,x(u)) - \frac{\partial^3 f}{\partial t^3}(v,x(v))\right| \\ &\quad\le L_{30}|u-v| + L'_{30}|x(u)-x(v)| \\ &\quad\le L_{30}|u-v| + L'_{30}\cdot\|x'\|\,|u-v| \\ &\quad\le (L_{30}+2L'_{30}M)|u-v|,\end{aligned}$$

$\forall u,v\in[-\tau,T]$. We have $x'(t)=f(t,x(t))-f(t-\tau,x(t-\tau)),\ \forall t\in[0,T]$ and $\|x'\|\le 2M$. Similar, since $\frac{\partial^3 f}{\partial t\partial x^2}$ and $\frac{\partial^3 f}{\partial x^3}$ are Lipschitzian in t and x there exist $L_{12},L'_{12}>0$ and $L_{03},L'_{03}>0$ such that we have:

$$\left|\frac{\partial^3 f}{\partial t\partial x^2}(u,x(u))-\frac{\partial^3 f}{\partial t\partial x^2}(v,x(v))\right|\le (L_{12}+L'_{12}\cdot\|x'\|)\,|u-v|\,,$$
$$\left|\frac{\partial^3 f}{\partial x^3}(u,x(u))-\frac{\partial^3 f}{\partial x^3}(v,x(v))\right|\le (L_{03}+L'_{03}\cdot\|x'\|)\,|u-v|\,,$$

$\forall u,v\in[-\tau,T]$. Now, we can state that

$$\left|\frac{\partial f}{\partial x}(u,x(u))-\frac{\partial f}{\partial x}(v,x(v))\right|\le (L_{01}+L'_{01}\cdot\|x'\|)\,|u-v|\,,$$

$\forall u,v\in[-\tau,T]$, and since $x\in C^3[-\tau,T]$ we have that x' and x'' are Lipschitzian. Also, we have $F'''(t)=[f(t,x(t))]'''_t$ and

$$\begin{aligned}[f(t,x(t))]'''_t=&\frac{\partial^3 f}{\partial t^3}(t,x(t))+3\frac{\partial^3 f}{\partial t^2\partial x}(t,x(t))x'(t)\\&+3\frac{\partial^3 f}{\partial t\partial x^2}(t,x(t))\left[x'(t)\right]^2+\frac{\partial^3 f}{\partial x^3}(t,x(t))\left[x'(t)\right]^3\\&+3\frac{\partial^2 f}{\partial t\partial x}(t,x(t))x''(t)+3\frac{\partial^2 f}{\partial x^2}(t,x(t))x'(t)x''(t)+\frac{\partial f}{\partial x}(t,x(t))x'''(t).\end{aligned}$$

From all the above we can see that:

$$\begin{aligned}&|F'''(u)-F'''(v)|\\&\le (L_{30}+2L'_{30}M)\,|u-v|+3\left\|\frac{\partial^3 f}{\partial t^2\partial x}\right\|\cdot\|x'\|\,|u-v|\\&\quad+3(L_{12}+L'_{12}\cdot\|x'\|)\cdot\|x'\|^2\,|u-v|+(L_{03}+L'_{03}\cdot\|x'\|)\cdot\|x'\|^3\cdot|u-v|\\&\quad+3\left\|\frac{\partial^2 f}{\partial t\partial x}\right\|\cdot\|x'''\|\,|u-v|+3\left\|\frac{\partial^2 f}{\partial x^2}\right\|\cdot\|x''\|^2\,|u-v|\\&\quad+(L_{01}+L'_{01}\cdot\|x'\|)\cdot\|x'''\|\,|u-v|\\&\le L_3\,|u-v|\,,\quad \forall u,v\in[-\tau,T].\end{aligned}$$

Here we have

$$\begin{aligned}L_3=&L_{30}+2L'_{30}M+6M_1M_3(2M+1)+12M^2(L_{12}+2L'_{12}M)\\&+8M^3(L_{03}+2L'_{03}M)+6M_2(2M+1)\left[M_2(2M+1)+2M_1^2\right]\\&+12M_2M_1^2(2M+1)^2+2(L_{01}+2L'_{01}M)(2M+1)[M_2(2M+1)+2M_1^2]>0,\end{aligned}$$

since $\|x''\|\le 2M_1(2M+1)$ and $\|x''\|\le 2(2M+1)[M_2(2M+1)+2M_1^2]$, having for $t\in[0,T]$

$$\begin{aligned}x''(t)=&\frac{\partial f}{\partial t}(t,x(t))+\frac{\partial f}{\partial x}(t,x(t))x'(t)\\&-\frac{\partial f}{\partial t}\left(t-\tau,x(t-\tau)\right)-\frac{\partial f}{\partial x}\left(t-\tau,x(t-\tau)\right)x'(t-\tau),\end{aligned}$$

and

$$\begin{aligned} x'''(t) &= \frac{\partial^2 f}{\partial t^2}(t, x(t)) + \frac{\partial^2 f}{\partial t \partial x}(t, x(t)) x'(t) \\ &\quad + x'(t) \cdot \left[\frac{\partial^2 f}{\partial t \partial x}(t, x(t)) + \frac{\partial^2 f}{\partial x^2}(t, x(t)) x'(t) \right] \\ &\quad + x''(t) \frac{\partial f}{\partial x}(t, x(t)) - \frac{\partial^2 f}{\partial t^2}(t-\tau, x(t-\tau)) - \frac{\partial^2 f}{\partial t \partial x}(t-\tau, x(t-\tau)) \cdot x'(t-\tau) \\ &\quad + x'(t-\tau) \cdot \left[\frac{\partial^2 f}{\partial t \partial x}(t-\tau, x(t-\tau)) + \frac{\partial^2 f}{\partial x^2}(t-\tau, x(t-\tau)) \cdot x'(t-\tau) \right] \\ &\quad + x''(t-\tau) \frac{\partial f}{\partial x}(t-\tau, x(t-\tau)). \end{aligned}$$

Then, F''' is Lipschitzian, and so we can apply the quadrature rule (4.9) from [2] and we have the inequality (4.11). Using the estimation from Banach's fixed point principle we obtain the desired estimation. □

Remark 4.5. To approximate the solution φ on $[0, T]$ we can use the cubic spline of interpolation s, defined by the interpolatory conditions:

$$\begin{cases} s(t_k) = \widetilde{\varphi_m}(t_k), \quad \forall k = \overline{n+1, q} \\ s(t_n) = y_n \\ s''(t_n) = M_n = 0, s''(t_q) = M_q = 0 \end{cases}$$

for a fixed $m \in \mathbb{N}^*$, on the knots $t_k, k = \overline{n, q}$, as in [4, p. 119–128]:

$$s : [0, T] \to \mathbb{R},\ s(t) = s_i(t),\ \forall t \in [t_{i-1}, t_i], \quad \forall i = \overline{n+1, q}$$

where,

$$\begin{aligned} s_i(t) &= \frac{M_i(t-t_{i-1})^3 + M_{i-1}(t_i - t)^3}{6h_i} + \frac{(6\varphi_{i-1} - M_{i-1}h_i^2) \cdot (t_i - t)}{6h_i} \\ &\qquad + \frac{(6\varphi_i - M_i h_i^2) \cdot (t - t_{i-1})}{6h_i}, \quad \forall t \in [t_{i-1}, t_i], \quad i = \overline{n+1, q}, \end{aligned}$$

and $\varphi_k = s(t_k), M_k = s''(t_k),\ \forall k = \overline{n, q}$. The values $M_k, k = \overline{n, q}$ are obtained from the relations

$$\frac{h_i M_{i-1}}{6} + \frac{M_i(h_i + h_{i+1})}{3} + \frac{h_{i+1} M_{i+1}}{6} = \frac{\varphi_{i+1} - \varphi_i}{h_{i+1}} - \frac{\varphi_i - \varphi_{i-1}}{h_i}, \quad i = \overline{n+1, q-1},$$

where $M_n = M_q = 0$. Since $f \in C^3([-\tau, T] \times [\alpha, \beta])$ we infer that $\varphi \in C^4[0, T]$. If $\Phi \in C^4[-\tau, 0]$ and $\Phi(0) = \lim\limits_{t>0, t\to 0} \varphi(t)$ then $\varphi \in C^4[-\tau, T]$. In these hypothesis, for $t \in [0, T] \backslash \{ t_k, k = \overline{n, q}\}$, we have the following error estimation:

$$|\varphi(t) - s(t)| \le \frac{\tau^m \cdot L^m}{1 - \tau L} \cdot \|\varphi_0 - \varphi_1\| + \left| \widetilde{r_{m,k}^{(n)}}(f) \right| + \frac{5}{384} \cdot \left\| \varphi^{IV} \right\| \cdot h^4,$$

where $h = \frac{T}{q-n-1} = \frac{\tau}{n}$. We have

$$\begin{aligned} x^{IV}(t) &= \frac{\partial^3 f}{\partial t^3}(t, x(t)) + 3\frac{\partial^3 f}{\partial t^2 \partial x}(t, x(t)) \cdot x'(t) + 3\frac{\partial^3 f}{\partial t \partial x^2}(t, x(t)) \left[x'(t)\right]^2 \\ &+ \frac{\partial^3 f}{\partial x^3}(t, x(t)) \left[x'(t)\right]^3 + 3\frac{\partial^2 f}{\partial t \partial x}(t, x(t)) x''(t) \\ &+ 3\frac{\partial^2 f}{\partial x^2}(t, x(t)) \cdot x'(t) \cdot x''(t) + \frac{\partial f}{\partial x}(t, x(t)) \cdot x'''(t) \\ &- \frac{\partial^3 f}{\partial t^3}(t-\tau, x(t-\tau)) - 3\frac{\partial^3 f}{\partial t^2 \partial x}(t-\tau, x(t-\tau)) \cdot x'(t-\tau) \\ &- 3\frac{\partial^3 f}{\partial t \partial x^2}(t-\tau, x(t-\tau)) \cdot \left[x'(t-\tau)\right]^2 - \frac{\partial^3 f}{\partial x^3}(t-\tau, x(t-\tau)) \cdot \left[x'(t-\tau)\right]^3 \\ &- 3\frac{\partial^2 f}{\partial t \partial x}(t-\tau)) x''(t-\tau) - 3\frac{\partial^2 f}{\partial x^2}(t-\tau, x(t-\tau)) \cdot x'(t-\tau) x''(t-\tau) \\ &- \frac{\partial f}{\partial x}(t-\tau, x(t-\tau)) x'''(t-\tau), \end{aligned}$$

$\forall t \in [0, T]$. Since

$$\begin{gathered} \| f \| \leq M, \quad \|\varphi'\| \leq 2M, \quad \|\varphi''\| \leq 2M_1(1+2M), \\ \|\varphi'''\| \leq 2M_2(1+2M)^2 + 4M_1^2(1+2M), \end{gathered}$$

we have the following estimation:

$$\begin{aligned} \left\|\varphi^{IV}\right\| &\leq 2M_3 + 6M_3 \|\varphi'\| + 6M_3 \|\varphi'\|^2 + 2M_3 \|\varphi'\|^3 \\ &\qquad + 6M_2 \|\varphi''\| + 6M_2 \|\varphi'\| \cdot \|\varphi''\| + 2M_1 \|\varphi'''\| \\ &\leq 2M_3(1+2M)^3 + 16M_1M_2(1+2M)^2 + 8M_1^3(1+2M). \end{aligned}$$

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