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Θ-Type Derivation and Derivation Superalgebra of the Finite-Dimensional Modular Lie Superalgebra $W(m,n,l,\underline{t})$

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Abstract In this paper F always denotes a field of characteristic $p>2$. We construct the finite-dimensional modular Lie superalgebra $W(m,n,l,\underline{t})$ over a field F, define Θ-type derivation and determine the derivation superalgebra of $W(m,n,l,\underline{t})$.

Keywords modular Lie superalgebra; generator; derivation.

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1. Introduction and definition

Modular Lie superalgebra $W(m,n,\underline{t})$ was constructed in [1]. In this paper we construct modular Lie superalgebra $W(m,n,l,\underline{t})$ that extends $W(m,n,\underline{t})$, determine the derivation superalgebra of $W(m,n,l,\underline{t})$, and prove that $W(m,n,l,\underline{t})$ has a new type of derivation called Θ-type derivation. Consequently, $W(m,n,l,\underline{t})$ is not isomorphic to the known Cartan-type modular Lie superalgebras. In this paper F always denotes a field of characteristic $p>2$ and Π is the prime field of F. Then $\Pi=\{0,1,2,\ldots,p-1\}\cong Z_p$, F is a linear space over Π. Set $z_1,\ldots,z_l\in F$ such that $z_1,\cdots,z_l$ are linearly independent over Π. Put $G=\{\lambda_1z_1+\cdots+\lambda_lz_l|\ 0\le\lambda_i<p,\ i=1,\ldots,l\}$. Then G is additive subgroup of F. We define a truncated polynomial algebra $F[y_1,y_2,\ldots,y_l]$ such that $y_i^p=1$ $(i=1,\ldots,l)$. If $\lambda\in G$, then λ can be expressed as $\lambda=\lambda_1z_1+\cdots+\lambda_lz_l$. Let $y^\lambda=y_1^{\lambda_1}\cdots y_l^{\lambda_l}$. It is easy to see

$$y^\lambda y^\eta=y^{\lambda+\eta}\ \ (\forall\lambda\ ,\ \eta\in G). \tag{1.1}$$

Let Γ denote $F[y_1,y_2,\ldots,y_l]$ for short. Then $\Gamma=\mathrm{span}_F\{y^\lambda|\ \lambda\in G\}$. Let m and n be the positive integers. Set $\widetilde{\Lambda}=\Lambda(m,n,\underline{t})\otimes\Gamma$, where $\Lambda(m,n,\underline{t})$ is defined in [1, P_{21}]. Let $Z_2=\{\overline{0},\overline{1}\}$ be the residue ring of integers modulo 2. Then $\widetilde{\Lambda}$ is an associative superalgebra with a Z_2-gradation induced by the trivial Z_2-gradation of Γ and the Z_2-gradation of $\Lambda(m,n,\underline{t})$:

$$\widetilde{\Lambda}_{\overline{0}}=\Lambda(m,n,\underline{t})_{\overline{0}}\otimes\Gamma,\quad \widetilde{\Lambda}_{\overline{1}}=\Lambda(m,n,\underline{t})_{\overline{1}}\otimes\Gamma.$$

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For $f \in \Lambda(m,n,\underline{t})$, $y^\lambda \in \Gamma$, we denote $f \otimes y^\lambda$ which is the element of $\widetilde{\Lambda}$ as fy^λ for short. In $\widetilde{\Lambda}$, besides Equation (1.1), the following statements hold:

$$x^{(\alpha)}y^\lambda = y^\lambda x^{(\alpha)} \quad (\forall\, \alpha \in N_0^m,\ \forall\, \lambda \in G),$$
$$y^\lambda x_j = x_j y^\lambda \quad (\forall\, \lambda \in G,\ j = m+1,\dots,s).$$

Then $\{x^{(\alpha)}x^u y^\lambda |\ \alpha \in A(m,\underline{t}),\ u \in B(n),\ \lambda \in G\}$ is an F-basic of $\widetilde{\Lambda}$. Let $\widetilde{\Lambda}_i = \operatorname{span}_F\{x^{(\alpha)}x^u y^\lambda | \alpha \in A(m,\underline{t}),\ u \in B(n),\ \lambda \in G,\ |\alpha| + |u| = i\}$, where $A(m,\underline{t})$ is defined in [1, p21]. Obviously, $\widetilde{\Lambda} = \bigoplus_{i=0}^{\xi} \widetilde{\Lambda}_i$, where $\xi = \sum_{i=1}^n \pi_i + n$, $\pi_i = p^{t_i} - 1$, $(1 \le i \le m)$. Put $Y_0 = \{1,2,\dots,m\}$, $Y_1 = \{m+1, m+2, \dots, s\}$, $Y = Y_0 \bigcup Y_1$, $J = Y \setminus \{m\}$.

2. Theorem and proof

Lemma 2.1 *Let $D_1, D_2, \dots, D_s$ be the linear transformations of $\widetilde{\Lambda}$ such that*

$$D_i(x^{(\alpha)}x^u y^\lambda) = \begin{cases} x^{(\alpha-\varepsilon_i)}x^u y^\lambda\ , & \forall\, i \in Y_0\ , \\ x^{(\alpha)}\partial_i(x^u)y^\lambda\ , & \forall\, i \in Y_1\ , \end{cases}$$

where $\partial_i = \frac{\partial}{\partial x_i}$ are the partial derivations of $\Lambda(n)$, $\forall i \in Y_1$. Then $D_i \in \operatorname{Der}_{\overline{0}}(\widetilde{\Lambda})$, $\forall i \in Y_0$; $D_i \in \operatorname{Der}_{\overline{1}}(\widetilde{\Lambda})$, $\forall i \in Y_1$.

Proof This proof is similar to the one in [1, Lemma 2.4, p11]. □

$D_1, D_2, \dots, D_s$ are called the partial derivations of $\widetilde{\Lambda}$. Let $A = A_{\overline{0}} \oplus A_{\overline{1}}$ be the superalgebra. If $x \in A_\theta$, where $\theta \in Z_2$, then we call x the Z_2-homogeneous element and call θ the Z_2-degree of x, denoted by $\deg(x) = \theta$ and written simply as $d(x) = \theta$. By the Lemma 2.1, it is easy to see $d(D_i) = \tau(i)$, where $\tau(i)$ is defined in [1, p12]. If $f \in \widetilde{\Lambda}_\theta$, $D \in \operatorname{Der}_\mu(\widetilde{\Lambda})$, where $\theta, \mu \in Z_2$, let $(fD)(g) := fD(g)$, $\forall g \in \widetilde{\Lambda}$. It is easy to see $fD \in \operatorname{Der}_{\theta+\mu}(\widetilde{\Lambda})$ by the direct verification.

Lemma 2.2 *Let $\sum_{i=1}^s f_i D_i \in \operatorname{Der}_\theta(\widetilde{\Lambda})$, $\sum_{j=1}^s g_j D_j \in \operatorname{Der}_\mu(\widetilde{\Lambda})$, where $\theta, \mu \in Z_2$. Then*

$$[\sum_{i=1}^s f_i D_i, \sum_{j=1}^s g_j D_j] = \sum_{i,j=1}^s f_i D_i(g_j) D_j - (-1)^{\theta\mu} \sum_{i,j=1}^s g_j D_j(f_i) D_i.$$

Proof This proof is similar to the one in [1, Lemma 2.5, p13]. □

Put $W(m,n,l,\underline{t}) = \{\sum_{i=1}^s f_i D_i | f_i \in \widetilde{\Lambda}, \forall i \in Y\}$.

Proposition 2.3 *$W(m,n,l,\underline{t})$ is a subalgebra of $\operatorname{Der}(\widetilde{\Lambda})$. In particular, it is a Lie superalgebra.*

Proof This proof is similar to the one in [1, Lemma 2.6, p13]. □

We denote Lie superalgebra $W(m,n,l,\underline{t})$ by W. It is easy to see that $W = \oplus_{i=-1}^{\xi-1} W_i$ is a Z-graded Lie superalgebra, where $\xi := \sum_{i=1}^m \pi_i + n$,

$$W_i = \operatorname{span}_F\{x^{(\alpha)}x^u y^\lambda D_j \mid |\alpha| + |u| = i+1,\ j \in Y\}.$$

Proposition 2.4 *Let $\phi \in \operatorname{Der}_t(W)$, $t \ge 0$. Then there is $z \in \operatorname{Nor}_W(W) := \{x \in W | [x, W] \subseteq W\}$ such that $\phi(D_i) = \operatorname{ad} z(D_i), i = 1, 2, \dots, s$.*

Proof Since factor y^λ has no effect on the verification of this Proposition. By virtue of [1, Proposition 2.6, p33], the Proposition holds. □

Lemma 2.5 *Let $\phi \in \mathrm{Der}_t(W)$ and $t \in Z$. If $\phi(L_j) = 0$ for $j = -1, 0, \ldots, k$, where $k \geq -1$ and $k + t \geq -1$, then $\phi = 0$.*

Proof This proof is similar to the one in [1, Lemma 2.8, p34]. □

Theorem 2.6 *Let $\Theta = \{h := (h_1(y), h_2(y), \ldots, h_l(y)) |\ h_j(y) \in \Gamma,\ 1 \leq j \leq l\}$. For $\forall h \in \Theta$, we define a linear transformation of W*

$$D_h(x^{(\alpha)}x^u y^\lambda D_i) = \sum_{j=1}^{l} \lambda_j h_j(y) x^{(\alpha)} x^u y^\lambda D_i. \tag{2.2}$$

Then $D_h \in \mathrm{Der}_{\overline{0}}(W)$.

Proof From

$$\begin{aligned}
&[x^{(\alpha)}x^u y^\lambda D_i, x^{(\beta)}x^v y^\eta D_k] \\
&\quad = x^{(\alpha)}x^u y^\lambda D_i(x^{(\beta)}x^v y^\eta)D_k - (-1)^{d(x^u D_i)d(x^v D_k)} x^{(\beta)}x^v y^\eta D_k(x^{(\alpha)}x^u y^\lambda)D_i \\
&\quad = x^{(\alpha)}x^u y^{\lambda+\eta} D_i(x^{(\beta)}x^v)D_k - (-1)^{d(x^u D_i)d(x^v D_k)} x^{(\beta)}x^v y^{\lambda+\eta} D_k(x^{(\alpha)}x^u)D_i \\
&\quad = y^{\lambda+\eta}[x^{(\alpha)}x^u D_i\ ,\ x^{(\beta)}x^v D_k],
\end{aligned}$$

$$D_h([x^{(\alpha)}x^u y^\lambda D_i\ ,\ x^{(\beta)}x^v y^\eta D_k]) = \sum_{j=1}^{l}(\lambda+\eta)_j h_j(y)[x^{(\alpha)}x^u D_i\ ,\ x^{(\beta)}x^v D_k] y^{\lambda+\eta}$$

$$\begin{aligned}
[D_h(x^{(\alpha)}x^u y^\lambda D_i)\ ,\ x^{(\beta)}x^v y^\eta D_k] &= [\sum_{j=1}^{l}\lambda_j h_j(y) x^{(\alpha)}x^u y^\lambda D_i\ ,\ x^{(\beta)}x^v y^\eta D_k] \\
&= \sum_{j=1}^{l}\lambda_j h_j(y) y^{\lambda+\eta}[x^{(\alpha)}x^u D_i\ ,\ x^{(\beta)}x^v D_k],
\end{aligned}$$

$$\begin{aligned}
[x^{(\alpha)}x^u y^\lambda D_i\ ,\ D_h(x^{(\beta)}x^v y^\eta D_k)] &= [x^{(\alpha)}x^u y^\lambda D_i\ ,\ \sum_{j=1}^{l}\eta_j h_j(y) x^{(\beta)}x^v y^\eta D_k] \\
&= \sum_{j=1}^{l}\eta_j h_j(y) y^{\lambda+\eta}[x^{(\alpha)}x^u D_i\ ,\ x^{(\beta)}x^v D_k],
\end{aligned}$$

and $\lambda_j + \eta_j = (\lambda+\eta)_j$, it follows

$$\begin{aligned}
&D_h([x^{(\alpha)}x^u y^\lambda D_i\ ,\ x^{(\beta)}x^v y^\eta D_k]) \\
&\quad = [D_h(x^{(\alpha)}x^u y^\lambda D_i)\ ,\ x^{(\beta)}x^v y^\eta D_k] + [x^{(\alpha)}x^u y^\lambda D_i\ ,\ D_h(x^{(\beta)}x^v y^\eta D_k)].
\end{aligned}$$

That is, $D_h \in \mathrm{Der}_{\overline{0}}(W)$. □

Let $h = (h_1(y), h_2(y), \ldots, h_l(y)) \in \Theta$. Then D_h is called Θ-type derivation.

Lemma 2.7 *Let $D \in \mathrm{Der}(W)$. If $D(D_i) = 0, \forall\ i \in Y$. Then there is $(h_1(y), h_2(y), \ldots, h_l(y)) \in \Theta$ such that*

$$D(y^\lambda D_i) = \sum_{j=1}^{l}\lambda_j h_j(y) y^\lambda D_i, \forall \lambda \in G.$$

Proof For $\forall i, j \in Y$, applying D to $[y^\lambda D_i, D_j] = 0$ gives

$$[D(y^\lambda D_i), D_j] = 0. \tag{2.3}$$

Since $D(y^\lambda D_i) \in W$, we can suppose that

$$D(y^\lambda D_i) = \sum_{k=1}^{s} g_{ki\lambda} D_k. \tag{2.4}$$

Applying Equation (2.4) to the Equation (2.3) yields

$$0 = [D(y^\lambda D_i), D_j] = \sum_{k=1}^{s}[g_{ki\lambda} D_k, D_j] = -\sum_{k=1}^{s}(-1)^{d(g_{ki\lambda}D_k)\tau(j)} D_j(g_{ki\lambda}) D_k.$$

Hence $D_j(g_{ki\lambda}) = 0, \forall j, k \in Y$. Consequently, $g_{ki\lambda} \in \Gamma$. We may assume that $g_{ki\lambda} = g(y)_{ki\lambda}$, which yields $D(y^\lambda D_i) = \sum_{k=1}^{s} g(y)_{ki\lambda} D_k$. Let $D(x_i D_i) = \sum_{j=1}^{s} g_j D_j$, $g_j \in \widetilde{\Lambda}$. Since $[D_i, x_i D_i] = D_i$, it follows that $[D_i, D(x_i D_i)] = 0$, thus $[D_i, \sum_{j=1}^{s} g_j D_j] = 0$.

According to the above equation, we have $\sum_{j=1}^{s} D_i(g_j) D_j = 0$, therefore $D_i(g_j) = 0$, $j = 1, 2, \ldots, s$. For $[y^\lambda D_i, x_i D_i] = y^\lambda D_i$, applying D to the above equation, we have

$$[D(y^\lambda D_i), x_i D_i] + [y^\lambda D_i, D(x_i D_i)] = D(y^\lambda D_i),$$

$$[\sum_{k=1}^{s} g(y)_{ki\lambda} D_k, x_i D_i] + [y^\lambda D_i, \sum_{j=1}^{s} g_j D_j] = \sum_{k=1}^{s} g(y)_{ki\lambda} D_k,$$

and

$$g(y)_{ii\lambda} D_i = \sum_{k=1}^{s} g(y)_{ki\lambda} D_k.$$

Therefore, if $k \neq i$, then $g(y)_{ki\lambda} = 0$, that is, $D(y^\lambda D_i) = g(y)_{ii\lambda} D_i$. We write $g(y)_{ii\lambda}$ as $g(y)_{i\lambda}$ for short. Then

$$D(y^\lambda D_i) = g(y)_{i\lambda} D_i.$$

Let $h_{i\lambda}(y) = y^{-\lambda} g(y)_{i\lambda}$. Then $g(y)_{i\lambda} = y^\lambda h_{i\lambda}(y)$ and

$$D(y^\lambda D_i) = y^\lambda h_{i\lambda}(y) D_i. \tag{2.5}$$

Suppose that $D(x_i y^\eta D_j) = \sum_{k=1}^{s} b_k D_k$, $b_k \in \widetilde{\Lambda}$. Since

$$[D_i, x_i y^\eta D_j] = y^\eta D_j,$$

we obtain that

$$[D_i, D(x_i y^\eta D_j)] = D(y^\eta D_j), \quad [D_i, \sum_{k=1}^{s} b_k D_k] = y^\eta h_{j\eta}(y) D_j$$

and

$$\sum_{k=1}^{s} D_i(b_k) D_k = y^\eta h_{j\eta}(y) D_j.$$

Hence if $k \neq j$, we have $D_i(b_k) = 0$ and $D_i(b_j) = y^\eta h_{j\eta}(y)$, so $b_j = y^\eta h_{j\eta}(y) x_i$. From the above content, we get

$$D(x_i y^\eta D_j) = y^\eta h_{j\eta}(y) x_i D_j + \sum_{k \neq j} b_k D_k. \tag{2.6}$$

For $[y^\lambda D_i, x_i y^\eta D_j] = y^{\lambda+\eta} D_j$, we have

$$[D(y^\lambda D_i), x_i y^\eta D_j] + [y^\lambda D_i, D(x_i y^\eta D_j)] = D(y^{\lambda+\eta} D_j).$$

By virtue of Equations (2.5) and (2.6), we obtain that

$$[y^\lambda h_{i\lambda}(y) D_i, x_i y^\eta D_j] + [y^\lambda D_i, y^\eta h_{j\eta}(y) x_i D_j + \sum_{k\neq j} b_k D_k] = y^{\lambda+\eta} h_{j(\lambda+\eta)}(y) D_j,$$

$$y^{\lambda+\eta} h_{i\lambda}(y) D_j + y^{\lambda+\eta} h_{j\eta}(y) D_j = y^{\lambda+\eta} h_{j(\lambda+\eta)}(y) D_j$$

and

$$h_{i\lambda}(y) + h_{j\eta}(y) = h_{j(\lambda+\eta)}(y). \tag{2.7}$$

From the randomicity of λ, η, i, j, we have

$$h_{i\lambda}(y) + h_{j\lambda}(y) = h_{j2\lambda}(y) = h_{j\lambda}(y) + h_{j\lambda}(y).$$

Hence $h_{i\lambda}(y) = h_{j\lambda}(y)$, and (2.5) can be written as

$$D(y^\lambda D_i) = y^\lambda h_\lambda(y) D_i.$$

By virtue of Equation (2.7) we get

$$h_\lambda(y) + h_\eta(y) = h_{\lambda+\eta}(y), \quad \forall \lambda, \eta \in G. \tag{2.8}$$

For arbitrary $k = 1, 2, \ldots, l$, by definition of G, we obtain $z_k \in G$. According to (2.8), we have

$$h_{z_k}(y) + h_{z_k}(y) = h_{2z_k}(y) = 2h_{z_k}(y)$$

and

$$h_{2z_k}(y) + h_{z_k}(y) = h_{3z_k}(y) = 3h_{z_k}(y).$$

By induction we have

$$h_{cz_k}(y) = c h_{z_k}(y), \quad \text{where} \;\; c \in \{0, 1, \ldots, p-1\} = Z_p.$$

For arbitrary $k, j = 1, 2, \ldots, l$ and arbitrary $a, b \in Z_p$, we have

$$h_{az_k}(y) + h_{bz_j}(y) = h_{az_k + bz_j}(y) = a h_{z_k}(y) + b h_{z_j}(y).$$

For arbitrary $\lambda = \sum_{i=1}^{l} \lambda_i z_i \in G$, we obtain that

$$h_\lambda(y) = h_{\sum_{j=1}^{l} \lambda_j z_j}(y) = \sum_{j=1}^{l} \lambda_j h_{z_j}(y).$$

Let $h_j(y)$ denote $h_{z_j}(y)$ for short. Then the above equation can be written as $h_\lambda(y) = \sum_{j=1}^{l} \lambda_j h_j(y)$. Therefore,

$$D(y^\lambda D_i) = \sum_{j=1}^{l} \lambda_j h_j(y) y^\lambda D_i.$$

Lemma 2.8 *Let $\phi \in \mathrm{Der}_t(W)$, where $t \geq 0$. Then there are $z \in \mathrm{Nor}_W(W)$ and $h \in \Theta$ such that*

$$(\phi - \mathrm{ad} z - D_h)(W_{-1}) = 0.$$

Proof By virtue of Proposition 2.4, there exists $z \in \mathrm{Nor}_W(W)$ such that $\phi(D_i) = \mathrm{adz}(D_i)$ for $i = 1, 2, \ldots, s$. Let $\phi_1 = \phi - \mathrm{adz}$. Then $\phi_1(D_i) = \phi(D_i) - \mathrm{adz}(D_i) = 0$. By virtue of Lemma 2.7, there is $(h_1(y), h_2(y), \ldots, h_l(y)) \in \Theta$ such that

$$\phi_1(y^\lambda D_i) = \sum_{j=1}^{l} \lambda_j h_j(y) y^\lambda D_i, \quad i \in Y, \lambda \in G.$$

Put $\phi_2 = \phi_1 - D_h$, then

$$\phi_2(y^\lambda D_i) = \phi_1(y^\lambda D_i) - D_h(y^\lambda D_i) = 0.$$

Therefore, $\phi_2(W_{-1}) = 0$, that is, $(\phi - \mathrm{ad}z - D_h)(W_{-1}) = 0$. □

Proposition 2.9 *Let $i \in Y_0$ and r be an arbitrary positive integer. Then* $(\mathrm{adD}_i)^{p^r} \in \mathrm{Der}_{\bar{0}}(W)$. *If $r \geq t_i$, then* $(\mathrm{adD}_i)^{p^r} = 0$.

Proof This proof is similar to the one in [1, Proposition 2.9, p35]. □

Proposition 2.10 *Let $\phi \in \mathrm{Der}_t(W)$, where $t \geq 0$. Then there are $f \in W$ and $h \in \Theta$ such that $\phi = \mathrm{ad}f + D_h$.*

Proof This is a direct consequence of Lemmas 2.5 and 2.8. □

Proposition 2.11 $\mathrm{Der}_{-1}(W) = \mathrm{ad}W_{-1}$.

Proof This proof is similar to the one in [1, Proposition 3.2, p35]. □

Theorem 2.12 *Let $T := \{x^{(k\epsilon_i)} D_j | 0 \leq k \leq \pi_i, i \in Y_0, j \in Y\}$, $G(y) := \{y^\lambda D_i | \lambda \in G, i \in Y\}$, $M := \{x_i D_j | i \in Y_1, j \in Y\}$. Then W is generated by $T \cup M \cup G(y)$.*

Proof Form [1], the subalgebra generated by $T \cup M$ is

$$\{x^{(\alpha)} x^u D_i | \alpha \in A(m, \underline{t}), u \in B(n)\} \subseteq Q,$$

where Q denotes the subalgebra of W generated by $T \cup M \cup G(y)$.

(1) If $\alpha \neq \pi$, we can suppose that $\alpha_1 < \pi_1$. Consequently, $x^{(\alpha+\epsilon_1)} x^u D_i \in Q$, $\forall u \in B(n)$, we have

$$[y^\lambda D_1, x^{(\alpha+\epsilon_1)} x^u D_i] = x^{(\alpha)} x^u y^\lambda D_i \in Q.$$

(2) If $\alpha = \pi, u \neq E$, then there exists $j \in Y_1$ such that $x_j x^u \neq 0$. Hence $x^{(\alpha)} x_j x^u D_i \in Q$, therefore,

$$[y^\lambda D_j, x^{(\alpha)} x_j x^u D_i] = x^{(\alpha)} x^u y^\lambda D_i \in Q.$$

(3) If $\alpha = \pi, u = E$, then

$$[x^{(\pi)} y^\lambda D_1, x_1 x^E D_j] = x^{(\pi)} x^E y^\lambda D_j \in Q, \quad \forall j \in Y_1.$$

$$[x^E y^\lambda D_s, x_s x^{(\pi)} D_j] = x^{(\pi)} x^E y^\lambda D_j \in Q, \quad \forall j \in Y_0.$$

We conclude $Q = W$. □

Lemma 2.13 *Let $\phi \in \mathrm{Der}_{-t}(W)$, where $t > 1$. If $\phi(x^{(t\epsilon_i)}D_j) = 0, \forall i \in Y_0, j \in Y$. Then $\phi = 0$.*

Proof This proof is similar to the one in [1, Lemma 3.4, p36]. □

Proposition 2.14 *Let $t > 1$. If there is not any positive integer k such that $t = p^k$. Then* $\mathrm{Der}_{-t}(W) = 0$.

Proof This proof is similar to the one of [1, Proposition 3.5, p36]. □

Proposition 2.15 *Let $t = p^r, r > 0$. Then*

$$\mathrm{Der}_{-t}(W) = \mathrm{span}_F\{y^{\lambda_i}(\mathrm{ad}D_i)^t | i \in Y_0, \lambda_i \in G\}.$$

Proof Let $\phi \in \mathrm{Der}_{-t}(W)$. Since $\mathrm{zd}(\phi(x^{(t\epsilon_i)}D_i)) = (-t) + (t-1) = -1$, we may suppose

$$\phi(x^{(t\epsilon_i)}D_i) = \sum_{k=1}^{s} a_{ik}y^{\lambda_k}D_k, \quad i \in Y_0, \lambda_k \in G.$$

For $j \in Y \setminus \{i\}$, applying ϕ to the following equation

$$[x^{(t\epsilon_i)}D_i, x_jD_j] = 0$$

gives $a_{ij} = 0$. Consequently, $\phi(x^{(t\epsilon_i)}D_i) = a_{ii}y^{\lambda_i}D_i, \forall i \in Y_0$. Direct verification by using the equation

$$y^{\lambda_i}(\mathrm{ad}D_i)^t(x^{(\alpha)}x^uy^\lambda D_j) = y^{\lambda_i}((\mathrm{ad}D_i)^t(x^{(\alpha)}x^uy^\lambda D_j))$$

shows that $y^{\lambda_i}(\mathrm{ad}D_i)^t$ is the derivation. Put $\varphi = \phi - \sum_{i=1}^m a_{ii}y^{\lambda_i}(\mathrm{ad}D_i)^t$, for arbitrary $j \in Y_0$, we have

$$\begin{aligned}\varphi(x^{(t\epsilon_j)}D_j) &= \phi(x^{(t\epsilon_j)}D_j) - \sum_{i=1}^{m} a_{ii}y^{\lambda_i}(\mathrm{ad}D_i)^t(x^{(t\epsilon_j)}D_j)\\ &= a_{jj}y^{\lambda_j}D_j - a_{jj}y^{\lambda_j}D_j = 0.\end{aligned}$$

Applying ϕ to the equation $x^{(t\epsilon_i)}D_j = [x^{(t\epsilon_i)}D_i, x_iD_j]$ results in $\varphi(x^{(t\epsilon_i)}D_j) = 0$ for $j \in Y, j \neq i$. By virtue of Lemma 2.13, we have $\varphi = 0$. Consequently,

$$\phi = \sum_{i=1}^{m} a_{ii}y^{\lambda_i}(\mathrm{ad}D_i)^t \in \mathrm{span}_F\{y^{\lambda_i}(\mathrm{ad}D_i)^t | i \in Y_0, \lambda_i \in G\}.$$

Lemma 2.6 *Let $\Delta = \{D_h | h \in \Theta\}$ and $\Omega = \mathrm{span}_F\{y^{\lambda_i}(\mathrm{ad}D_i)^{p^{k_i}} | i \in Y_0, 1 \leq k_i \leq t_i, \lambda_i \in G\}$. Then*

(1) $\Delta \cap \Omega = \{0\}$.

(2) $(\Delta \oplus \Omega) \cap \mathrm{ad}W = \{0\}$.

Proof (1) is obvious.

(2) Suppose that $\mathrm{ad}f = D_h + \sum_{i=1}^m \sum_{j=1}^{t_i-1} a_{ij}y^{\lambda_{ij}}(\mathrm{ad}D_i)^{p^j}$. Then $\mathrm{ad}f(D_i) = 0$, $\forall i \in Y$. Noting that $f \in W$, we may suppose that $f = \sum x^{(\alpha)}x^uy^\lambda D_j$. Since

$$\begin{aligned}[f, D_i] &= [\sum x^{(\alpha)}x^uy^\lambda D_j, D_i]\\ &= \sum -(-1)^{\tau(i)d(x^{(\alpha)}x^uy^\lambda D_j)}D_i(x^{(\alpha)}x^uy^\lambda)D_j,\end{aligned}$$

we obtain that $D_i(x^{(\alpha)}x^u y^\lambda) = 0$. Consequently, $f = \sum_{j=1}^{s} g_j(y)D_j$. Since

$$\mathrm{ad}f(x_v^p D_k) = g_v(y)x_v^{p-1}D_k, \quad 1 \leq v \leq s,$$

$$(D_h + \sum_{i=1}^{m}\sum_{j=1}^{t_i-1} a_{ij}y^{\lambda_{ij}}(\mathrm{ad}D_i)^{p^j})(x_v^p D_k) = a_{v1}y^{\lambda_{v1}}D_k,$$

and

$$\mathrm{ad}f = D_h + \sum_{i=1}^{m}\sum_{j=1}^{t_i-1} a_{ij}y^{\lambda_{ij}}(\mathrm{ad}D_i)^{p^j}.$$

We obtain $g_v(y) = 0$, namely, $(\Delta \oplus \Omega) \cap \mathrm{ad}W = \{0\}$. □

By Propositions 2.10, 2.11, 2.14, 2.15 and 2.9 we obtain the following Theorem.

Theorem 2.17 $\mathrm{Der}(W) = \mathrm{ad}W \oplus \Delta \oplus \Omega$.

Theorem 2.18 *$W(m,n,l,\underline{t})$ is not isomorphic to Cartan-type modular Lie superalgebras W, S, H, HO, K.*

Proof Since

$$W(m,n,l,\underline{t}) = \mathrm{span}_F\{x^{(\alpha)}x^u y^\lambda D_i | \alpha \in A(m,\underline{t}), u \in B(n), \lambda \in G, i \in Y\},$$

we have $\dim(W(m,n,l,\underline{t})) = 2^n s p^q$, where $q = \sum_{i=1}^{m} t_i + l$. From [1,2], we know that the dimensions of modular Lie superalgeras S, H, HO are not divided by p. Therefore, $W(m,n,l,\underline{t})$ is not isomorphic to S, H, HO. From [3,4], we know that any outer derivation in W, K is nilpotent linear transformation. But there is the Θ-type outer derivation D_h, which is not nilpotent linear transformation in $W(m,n,l,\underline{t})$, where $h = (y_1, 0, \ldots, 0), y_1 \neq 0$. Therefore, $W(m,n,l,\underline{t})$ is not isomorphic to W, K. □

By Theorem 2.17, $W(m,n,l,\underline{t})$ is not isomorphic to the known Cartan-type modular Lie superalgebras.

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