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Frobenius Property of a Cosemisimple Hopf Algebra

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Abstract: Let H be a cosemisimple Hopf algebra over an algebraically closed field k of characteristic zero. We show that if H is of type $l:1+m:p+1:q$ with $p^2<q$, or of type $1:1+1:m+1:n$ in the sense of Larson and Radford, then H has the Frobenius property, that is, Kaplansky conjecture is true for these Hopf algebras.

Key words: Hopf algebra; type; Frobenius property.
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1. Introduction

In recent years, the theory of cosemisimple Hopf algebras over an algebraically closed field k of characteristic zero has been studied considerably. Many results have been found. For example, it was showed in [1, 2] that H is cosemisimple if H is semisimple and $S^2=id$. Recall that a cosemsimple Hopf algebra H is said to have the Frobenius property if the dimensions of simple H-comodules divide the dimension of H.

It was conjectured by Kaplansky in [3] that every cosemisimple Hopf algebra had the Frobenius property. This is still an open problem, although it has been shown in [4] that any coquasitriangular cosemisimple Hopf algebra H has the Frobenius property. In several recent papers, Natale in [5] has shown that every cosemisimple Hopf algebra of dimension less than 60 is semisolvable, and hence has the Frobenius property[6,7].

In this paper, we shall show that if a cosemisimple Hopf algebra H has the type $l:1+m:p+1:q$ with $p^2<q$, or the type $1:1+1:m+1:n$ in the sense of Larson and Radford[8], then H has the Frobenius property. In particular, the cosemisimple Hopf algebras of dimension 60 have the Frobenius property. Our results follow from the theory of the Grothendieck algebras of a Hopf algebra, introduced by Nichols and Richmond in [9, 10] (see Section 2).

Throughout, k will denote an algebraically closed field of characteristic zero. All vector spaces and tensor product will be over k. Our references for the theory of Hopf algebras are [11]

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or [12]. The notation for Hopf algebras is standard.

2. The Grothendieck algebra of a Hopf algebra

Let H be a Hopf algebra over a field k, and let $\mathcal{U}^H$ be the category of finite dimensional right H-comodules. Let $\mathcal{F}$ be the free abelian group on the symbols (M), where (M) denotes the isomorphism class of the object M in $\mathcal{U}^H$. Denote by $\mathcal{G}(H) = \mathcal{F}/\mathcal{F}_0$ the Grothendieck group, where $\mathcal{F}_0$ is the subgroup of $\mathcal{F}$ generated by all expressions $(M)-(L)-(N)$, where $0 \to L \to M \to N \to 0$ is a short exact sequence of finite dimensional right H-comodules. The image of the symbol corresponding to the class of $M \in \mathcal{U}^H$ in $\mathcal{G}(H)$ is denoted $[M]$. Then by Proposition 1 of [9], the group $\mathcal{G}(H)$ is a free abelian group, with standard basis $\mathcal{X}$ consisting of the images of the classes of the simple right H-comodules.

The basic elements of degree 1 of $\mathcal{G}(H)$ are of the form $[kg]$, where $g \in G(H)$, the set of group-like elements of H. Simply, we shall write $[kg]$ as g in $\mathcal{G}(H)$.

Let M and N be right H-comodules. Then $M \otimes N$ is a right H-comodule via $\rho(m\otimes n) = \sum m_0 \otimes n_0 \otimes m_1 n_1$, for all $m \in M, n \in N$.

Thus, the Grothendieck group $\mathcal{G}(H)$ is a ring with multiplication given by $[M][N] = [M\otimes N]$ for each $M, N \in \mathcal{U}^H$. Let R be any subfield of the complex numbers field $\mathbf{C}$, and let $\mathcal{G}(H)^R = \mathcal{G}(H) \otimes_{\mathbf{Z}} R$ be the R-module obtained by extending the scalars. Then $\mathcal{G}(H)^R$ is naturally an algebra over R with the basis $\mathcal{X}$.

By Proposition 8 of [9], the map $* : \mathcal{G}(H) \to \mathcal{G}(H)$ given by $[M]^* = [M^*]$ is a group homorphism and a ring anti-homomorphism. If H is cosemisimple, then "$*$" is an involution.

For any $z \in \mathcal{G}(H)$, we write $z = \sum_{x\in\mathcal{X}} m(x,z)x$, and refer to the integer $m(x,z)$ as the multiplicity of x in z. If $m(x,z) \neq 0$, we say that x is a basic component of z. Extending m to a biadditive function, we define $m(z,z') = \sum_{x\in\mathcal{X}} m(x,z)m(x,z')$ for all $z, z' \in \mathcal{G}(H)$. In particular, for $x, y \in \mathcal{X}, g \in G(H)$

$$m(g,xy) = \begin{cases} 1, & \text{if } \ y = x^*g, \\ 0, & \text{otherwise.} \end{cases}$$

There is a unique ring homomorphism $d : \mathcal{G}(H)^R \to R$ such that $d([M]) = \dim_k M$ for all $M \in \mathcal{U}^H$.

Recall that a standard subring of $\mathcal{G}(H)$ is a subring of $\mathcal{G}(H)$ which is spanned as an abelian group by a subset of the standard basis $\mathcal{X}$. Then by Theorem 6 of [9], there is a $1-1$ correspondence between standard subrings of $\mathcal{G}(H)$ and subbialgebras of H generated as algebras by their simple subcoalgebras, given by: the subbialgebra algebra B generated by its simple subcoalgebras corresponds to the standard subring spanned by $\{x_C \mid C$ is a simple subcoalgebra of $B\}$, where x_C denotes the basis element corresponding to the simple coalgebra C.

3. Main results

Let H be a finite dimensional cosemisimple Hopf algebra over the algebraically closed field k of characteristic zero. Let $d_1, d_2, \ldots, d_s$, $n_1, n_2, \ldots, n_s$ be positive integers, with $d_1 < d_2 <$

$\cdots < d_s$. Recall that H is said to have type[8]: $n_1 : d_1 + n_2 : d_2 + \cdots + n_s : d_s$ if $d_1, d_2, \ldots, d_s$ are the dimensions of the simple H-comodules and that n_i is the number of the simple H-comodules of dimension d_i.

Let x, y be basic elements in $\mathcal{X}$. We shall denote by $G[x,y]$ the subset of $G(H)$ consisting of those elements g for which $gx = y$. In particular, by Theorem 10 of [9], $G[x]$=$G[x,x] = \{g \mid m(g, xx^*) = 1\}$.

We need the following lemma, due to the dual version[5,13].

Lemma 3.1 *Let x be a basic element in $\mathcal{X}$. Then we have:*

(1) The order of $G[x]$ divides $(d(x))^2$.

(2) The order of $G(H)$ divides $n(d(x))^2$, where n is the number of the simple H-comodule of degree $d(x)$.

Proof It follows from Nichols and Zoeller Theorem[14] (see also Lemma 2.2.2 of Ref. [5]). □

Theorem 3.2 *Let H be finite dimensional cosemisimple Hopf algebra over an algebraically closed field k of characteristic 0. Suppose that H is of type $l : 1 + m : p + 1 : q$ and $p^2 < q$, then $p \mid |G(H)|$, $q \mid \dim_k H$.*

Proof Let $x_1, x_2, \ldots, x_m$ be distinct basis elements of degree p, and let y be the unique basic element of degree q. Notice that the standard subring B generated by $\{x_i, g \mid i = 1, 2, \ldots, m, g \in G(H)\}$ is exactly $\sum_{g\in G(H)} \mathbf{Z}g + \sum_{i=1}^{m} \mathbf{Z}x_i$, since $p^2 < q$. For any i, we have $x_i x_i^* = \sum_{g \in G[x_i]} g + \sum_{z\in\mathcal{X}, m(z, x_i x_i^*)>0} z$. Again since $p^2 < q$, if $m(z, x_i x_i^*) > 0$ and $z \in \mathcal{X}$, then $z \in \{x_1, x_2, \ldots, x_m\}$. By applying d, we have $p \mid |G[x_i]|$, hence $p \mid |G(H)|$.

Now for any i, $1 \le i \le m$, we claim that $x_i y = py$ and $y x_i = py$. In fact, if for any j, $m(x_j, x_i y) > 0$, then by Theorem 8 of [10], we have $m(y, x_i^* x_j) > 0$, which contradicts the fact that $p^2 < q$. Similarly, we have $y x_i = py$.

Note that $y^* = y$, so $m(x_i, y^2) = p$. Thus, there is an integer s such that

$$y^2 = \bigoplus_{g\in G(H)} g + p(x_1 + x_2 + \cdots + x_m) + sy.$$

Applying d yields $q^2 = l + mp^2 + sq$. Using the fact $\dim_k H = l + mp^2 + q^2$, we obtain $\dim_k H = 2q^2 - sq$. This gives $q \mid \dim_k H$. □

Corollary 3.3 *Let p, q be two distinct prime numbers and let H be a cosemisimple Hopf algebra with dimension $pq^2, p^2 < q$, and let H be not-cocommutative. If H has Frobenius property, then H is of type $q^2 : 1 + (p-1) : q$. In particular, if H is a coquasitriangular Hopf algebra then $G(H)$ is non-trivial.*

Proof Let x be any basic element of $\mathcal{G}(H)$. By assumption, we have $d(x) = 1, p$ or q since $(d(x))^2 < pq^2$. Let m and n be the numbers of the simple comodules of dimension p and q, respectively. We claim that $m = 0$. In fact, by the proof of Theorem 3.2, we have $p \mid |G(H)|$. This means that $|G(H)| = p$ or $|G(H)| = pq$ since H is not cocommutative. But by Lemma 3.1

$|G(H)| \mid nq^2$, which is impossible since $n < p$.

Now H is of type $l : 1 + n : q$ where $|G(H)| = l$ and $n \neq 0$ since H is not cocomutative. By the proof of Theorem 3.2 (or see Ref.[8, Corollary 3.6]) we know $q \mid l$. It follows that $l = pq, q$ or q^2. We shall rule out pq and q.

Suppose that $l = pq$. Then $pq^2 = pq + nq^2$. Therefore, $q \mid p$, a contradiction.

Suppose that $l = q$. Then $pq^2 = q + nq^2$, which is also impossible. All together, we have $|G(H)| = q^2$.

In particular, if H is coquasitriangular, then by [4], H has the Frobenius property, and the assertion follows. $\square$

Theorem 3.4 *Let H be a cosemisimple Hopf algebra over k of type $1 : 1 + 1 : n + 1 : m$. Then H has the Frobenius property.*

Proof Let x and y be the unique basic element of degree n and m, respectively. Noticing that $x = x^*$ and $d(x) \neq 1$, we obtain that $x^2 = 1 + ax + by$, for nonnegative integers a, b with $b \neq 0$. Applying d, we get $(d(x), d(y)) = 1$, i.e., there exists $s, t \in \mathbf{Z}$, such that $sd(x) + td(y) = 1$. It follows that the standard subring generated by $\{x\}$ is $\mathcal{G}(H)$. This gives that $\{1, x, y\}$ and $\{1, x, [H]\}$ are both bases of $\mathcal{G}(H)^{\mathbf{Q}}$, where $\mathbf{Q}$ is the rational numbers field. Set $x^2 = \alpha 1 + \beta x + \gamma[H]$ for some $\alpha, \beta, \gamma \in \mathbf{Q}$ and $\gamma \neq 0$. Notice that the matrix of the left multiplication by x on $\mathcal{G}(H)^{\mathbf{Q}}$ with the basis $\{1, x, y\}$ is a matrix with integer coefficients. Using the basis $\{1, x, [H]\}$ and the fact that $x[H] = d(x)[H]$, we obtain that $\beta + d(x)$ is an integer. Hence β is an integer. Noticing that $x^2 = (\alpha + \gamma)1 + (\gamma d(x) + \beta)x + \gamma d(y)y$, we have $\gamma d(x) + \beta = a, \gamma d(y) = b \in \mathbf{N}$. It follows that $\gamma d(x) \in \mathbf{Z}$. Thus, we obtain $\gamma = \gamma sd(x) + \gamma td(y) \in \mathbf{Z}$.

Now since $0 < \gamma d(y) = b = (x^2, y) = (x, xy) \leq d(y)$, we have $\gamma = 1$. Therefore, $\alpha = 0$ and $x^2 = 1 + (\beta + d(x))x + d(y)y$. It follows that $\dim_k H = 1 + d^2(x) + d^2(y) = (d(x))^2 - \beta d(x)$. This means $n = d(x) \mid \dim_k H$. Similarly, we have $m \mid \dim_k H$. $\square$

Theorem 3.5 *Let H be a cosemisimple Hopf algebra over k of dimension 60. Then H has Frobenius property.*

Proof If H has a simple comodule with dimension 7, then the type of H must be one of: $2 : 1 + 1 : 3 + 1 : 7$; $3 : 1 + 2 : 2 + 1 : 7$; $7 : 1 + 1 : 2 + 1 : 7$; $11 : 1 + 1 : 7$. But it is impossible from the Lemma 3.1 and Theorem 3.2. So the dimensions of all simple comodules of H should be $1, 2, 3, 4, 5$ or 6, which can divide 60. $\square$

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余半单 Hopf 代数的 Frobenius 性质

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摘要: 设 k 是特征为 0 的代数闭域， H 为其上的余半单 Hopf 代数. 本文证明了当 H 有型: $l:1+m:p+1:q$ (其中 $p^2<q$) 或 $1:1+1:m+1:n$ 时，它具有 Frobenius 性质. 即对此类 Hopf 代数， Kaplansky 猜想是正确的.

关键词: Hopf 代数；型； Frobenius 性质.