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Some Applications of Orthogonal Projections in Generalized Frames

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Abstract: Using operator-theoretic-methods, we give some characterizations for a dual generalized frame of a generalized frame in a separable Hilbert space H. We also prove a result concerning two strongly disjiont generalized frames.

Key words: generalized frame; dual generalized frame; strongly disjiont; orthogonal projection.
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1. Introduction

Frames have been a very important tool in wavelet analysis in the early 1980s, but the theory already has been introduced by Duffin R. J. and Schaeffer A. C.[1] in the early 1950s to deal with studying in nonharmonic Fourier series. The concept of generalized frames was introduced by Kaiser G.[2] in 1994. Generalized frame is a useful fool in wavelet analysis, it can be viewed as a "generalized discrete frame". The interested reader can consult [2,3,4] for the details and some recent development on the particular topics. Our main purpose of this paper is to study generalization of some of the results in the theory of discrete frames. The reader will notice that, although the main ideas of some properties are similar to the ones for discrete frames, their proofs are essentially different. In the note, using properties of orthogonal projection, we give some characterizations for a dual generalized frame of a generalized frame in a separable Hilbert space H. We also prove a result concerning two strongly disjiont generalized frames.

Let H be a separable Hilbert space and let (M, S, μ) be a measure space. A generalized frame[2] in H indexed M is a family of vectors $h = \{h_m \in H : m \in M\}$, if for every $f \in H$, the function $\tilde{f} : M \to \mathbf{C}$ defined by $\tilde{f}(m) = \langle f, h_m\rangle$ is measurable and there is a pair of constants $0 < A \leq B < \infty$ such that

$$A\|f\|_H^2 \leq \|\tilde{f}\|_{L^2(\mu)} \leq B\|f\|_H^2, \quad \forall f \in H. \tag{1.1}$$

The vectors $\{h_m\}_{m\in M} \subseteq H$ are called frame vectors, (1.1) is called the frame condition, and A and B are called frame bounds. The function $\tilde{f}$ will be called the transform of f with respect

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to the frame, and the map $T_h f := \tilde{f},\ \forall f \in H$ is called the analyzing operator, and the adjoint $T_h^*: L^2(\mu) \to H$ of T_h is given by

$$(T_h^* g)(m) = \int_M g(m) h_m \mathrm{d}\mu(m),\ \forall g \in L^2(\mu). \tag{1.2}$$

Here the formula (1.2)holds in the " weak " sense in H, and T_h is clearly linear and bounded. If M is at most countable and μ is the counting measure, $h = \{h_m \in H : m \in M\}$ is called a discrete frame[1].

Let $\{h_m\}_{m\in M}$ be a generalized frame in H. If the frame bounds A and B are equal to each other, then the frame is called a tight generalized frame. In this case, the frame condition reduces to $A\|f\|_H^2 = \|\tilde{f}\|_{L^2(\mu)}^2$. If $\{h_m\}_{m\in M}$ and $\{k_m\}_{m\in M}$ are two generalized frames in H and

$$f = \int_M \langle f, h_m\rangle k_m \mathrm{d}\mu(m) = \int_M \langle f, k_m\rangle h_m \mathrm{d}\mu(m), \quad \forall f \in H, \tag{1.3}$$

then we call $k = \{k_m\}_{m\in M}$ a dual generalized frame of h. In this case, h is also a dual generalized frame of k and the pair $\{h, k\}$ is called a dual pair of generalized frames.

For the reader′s convenience, we fix some notations and terminologies. For two Hilbert spaces H_1 and H_2, $B(H_1, H_2)$ and $B(H)$ denotes the Banach spaces of all bounded linear operators from H_1 into H_2 and from H into H, respectively. For $A \in B(H)$, $R(A)$ and $N(A)$ denote the range and the null space of A, respectively. An operator $A \in B(H)$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all x in H. An operator $P \in B(H)$ is said to be an orthogonal projection if $P^2 = P$ and $P = P^*$, where P^* is the adjoint of P. Of course, an orthogonal projection is a positive operator. $A \in B(H_1, H_2)$ is said to be a contraction if $\|A\| \leq 1$.

Two generalized frames $\{h_m\}_{m\in M}$ and $\{k_m\}_{m\in M}$ for the Hilbert spaces H and K, respectively, are said to be strongly disjoint (resp. strongly complementary) if $R(T_h)$ is orthogonal to $R(T_k)$ (resp. $R(T_h)$ is the orthogonal complement of $R(T_k)$). Here T_k is the analyzing operator of generalized frame $k = \{k_m\}_{m\in M}$.

2. Some basic properties

We first give a lemma that is well-known.

Lemma 2.1 *If H has a direct sum decomposition $H = H_1 \oplus H_2$, then $A \in B(H)$ is a positive operator if and only if A has the following operator matrix form*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{ii} is a positive operator on H_i, $i = 1, 2$, $A_{21} = A_{12}^* = A_{11}^{\frac{1}{2}} D A_{22}^{\frac{1}{2}}$, and D is a contraction from H_2 into H_1.

Lemma 2.2[5] *Let $\{h_m\}_{m\in M}$ be a generalized frame, and $U \in B(H)$, set $k = \{Uh_m\}_{m\in M}$, then k is a generalized frame if and only if the adjont operator U^* of U is bounded below.*

For a given generalized frame $h = \{h_m\}_{m\in M}$, composing T_h with the adjoint operator T_h^*, we get the frame operator $S : H \to H$,

$$Sf = T_h^* T_h f = \int_M \langle f, h_m\rangle h_m \mathrm{d}\mu(m),\ \forall f \in H. \tag{2.1}$$

Clearly, S is a linear operator on H. If $h = \{h_m\}_{m\in M}$ is a generalized frame with the frame bounds A and B, then

$$A\|f\|_H^2 \le \langle Sf, f\rangle \le B\|f\|_H^2,\ \ \forall f \in H.$$

Suppose that a generalized frame $\{h_m\}_{m\in M}$ is tight, we have $\langle Sf, f\rangle = A\|f\|_H^2$, for all $f \in H$. So in this case, $S = AI$, where I is the identity operator on H.

Lemma 2.3[5] *Let $h = \{h_m\}_{m\in M}$ be a family of vectors in H, and the function $\langle f, h_m\rangle, m \in M$ be measurable. Then the family of vectors $h = \{h_m\}_{m\in M}$ is a generalized frame of H if and only if the frame operator S is a positive invertible operator in $B(H)$.*

Given a generalized frame $\{h_m\}_{m\in M}$, by the formula (2.1), we have

$$f = \int_M \langle f, h_m\rangle S^{-1} h_m \mathrm{d}\mu(m) = \int_M \langle f, S^{-1}h_m\rangle h_m \mathrm{d}\mu(m),\ \forall f \in H. \tag{2.2}$$

So

$$S^{-1}f = \int_M \langle f, S^{-1}h_m\rangle S^{-1}h_m \mathrm{d}\mu(m),\ \forall f \in H, \tag{2.3}$$

and

$$f = \int_M \langle f, S^{-\frac{1}{2}}h_m\rangle S^{-\frac{1}{2}}h_m \mathrm{d}\mu(m),\ \forall f \in H. \tag{2.4}$$

Applying the operator S^{-1} to the vectors $\{h_m\}_{m\in M}$ leads to a new family of vectors $\{S^{-1}h_m\}_{m\in M}$. By Lemma 2.3, $\{S^{-1}h_m\}_{m\in M}$ is a generalized frame. From the formula (2.2), $\{S^{-1}h_m\}_{m\in M}$ is also a dual generalized frame of $\{h_m\}_{m\in M}$. In this case, $\{h_m\}_{m\in M}$ is also a dual generalized frame of $\{S^{-1}h_m\}_{m\in M}$, and by the Formula (2.3) S^{-1} is the frame operator of $\{S^{-1}h_m\}_{m\in M}$. By the Equality (2.4), $\{S^{-\frac{1}{2}}h_m\}_{m\in M}$ is a normalized tight frame.

Proposition 2.4 *Let $\{h_m\}_{m\in M}$ be a normalized tight generalized frame and A is a bounded invertible operator. Then $\{Ah_m\}_{m\in M}$ is a generalized frame.*

Proof By Lemma 2.2, the proof is obvious.

Remark 2.5 By Proposition 2.4, a generalized frame is precisely the image of a normalized tight frame under a bounded invertible operator.

Proposition 2.6 *Let $\{h_m\}_{m\in M}$ be a generalized frame on a Hilbert space H. Then there exists a unique operator $U \in B(H)$ such that*

$$f = \int_M \langle f, Uh_m\rangle h_m \mathrm{d}\mu(m),\ \ \forall f \in H,$$

and U is given by

$$U = A^*A,$$

where A is an invertible operator in $B(H,K)$ for some Hilbert space K with the property that $\{Ah_m : m \in M\}$ is a normalized tight frame. In particular, U is an invertible operator.

Proof By Remark 2.5, there exists an invertible operator in $B(H,K)$ for some Hilbert space K with the property that $\{Ah_m : m \in M\}$ is a normalized tight frame. Let $k_m = Ah_m$ and $U = A^*A \in B(H)$. Then

$$\begin{aligned}\int_M \langle f, A^*Ah_m\rangle h_m \mathrm{d}\mu(m) &= \int_M \langle Af, k_m\rangle h_m \mathrm{d}\mu(m) = \int_M \langle Af, k_m\rangle A^{-1}k_m \mathrm{d}\mu(m)\\ &= A^{-1}\int_M \langle Af, k_m\rangle k_m \mathrm{d}\mu(m) = A^{-1}Af = f.\end{aligned}$$

For uniqueness, suppose that $V \in B(H)$ satisfies $f = \int_M \langle f, Vh_m\rangle h_m \mathrm{d}\mu(m)$, $\forall f \in H$. Then

$$\begin{aligned}f &= \int_M \langle f, Vh_m\rangle h_m \mathrm{d}\mu(m) = \int_M \langle f, VA^{-1}k_m\rangle A^{-1}k_m \mathrm{d}\mu(m)\\ &= A^{-1}\int_M \langle (A^*)^{-1}V^*f, k_m\rangle k_m \mathrm{d}\mu(m) = A^{-1}(A^*)^{-1}V^*f,\end{aligned}$$

which implies that $A^{-1}(A^*)^{-1}V^* = I$, and hence $V = A^*A$.

By Lemma 2.2, $\{Uh_m\}_{m\in M}$ in Proposition 2.6 is a generalized frame, and also a dual generalized frame of $\{h_m\}_{m\in M}$.

Proposition 2.7 *Let $\{h_m\}_{m\in M}$ be a generalized frame for H with frame bounds A, B, and P be an orthogonal projection in $B(H)$. Then $\{Ph_m\}_{m\in M}$ is a generalized frame for PH with frame bounds A, B, and the frame operator of $\{Ph_m\}_{m\in M}$ is PSP.*

Proof For any $f \in PH$, $\langle f, Ph_m\rangle = \langle Pf, h_m\rangle = \widetilde{Pf}(m)$ is measurable and

$$\int_M |\langle f, Ph_m\rangle|^2 \mathrm{d}\mu(m) = \int_M |\langle Pf, h_m\rangle|^2 \mathrm{d}\mu(m) = \int_M |\langle f, h_m\rangle|^2 \mathrm{d}\mu(m).$$

This proves the first part of the Proposition. For second part, since T_h is the analyzing operator of $\{h_m\}_{m\in M}$, T_hP is an analyzing operator associated with $\{Ph_m\}_{m\in M}$. Hence, the frame operator of $\{Ph_m\}_{m\in M}$ is PSP.

3. The theorems and proofs

If $\{h_m\}_{m\in M}$ is a generalized frame for H and P is an orthogonal projection, by Proposition 2.7, $\{Ph_m\}_{m\in M}$ is a generalized frame for PH. It is natural to ask whether $(Ph_m)^* = Ph_m^*$ for all orthogonal projections P. The following theorem gives an answer to the question.

Theorem 3.1 *Let $\{h_m\}_{m\in M}$ be a generalized frame for H, let S be the frame operator of $\{h_m\}_{m\in M}$, let $S^{-1}h_m = h_m^*$, and let P be an orthogonal projection in $B(H)$. Then $Ph_m^* = (Ph_m)^*, \forall m \in M$, if and only if $PS^{-1} = S^{-1}P$.*

Proof First, we note that if $\{h_m\}_{m\in M}$ is a generalized frame for H, then the liner spanning $\vee\{h_m\}_{m\in M}$ is dense in H.

If $Ph_m^* = (Ph_m)^*, \forall m \in M$, by the definition $S^{-1}h_m = h_m^*$, we have

$$PS^{-1}h_m = Ph_m^* = (Ph_m)^* = (PSP)^{-1}Ph_m.$$

Hence, the liner spanning $\vee\{h_m\}_{m\in M}$ is dense in H, which implies $PS^{-1} = (PSP)^{-1}P$, and $PSP = PS$. By taking adjoints on both sides, we have $PSP = SP$, and hence $PS = SP$, which implies that $PS^{-1} = S^{-1}P$.

Conversely, now suppose that $S^{-1}P = PS^{-1}$. Since $f = \int_M \langle f, h_m^*\rangle h_m \mathrm{d}\mu(m)$ for all $f \in H$, we have

$$\begin{aligned} f &= \int_M \langle f, Ph_m^*\rangle Ph_m \mathrm{d}\mu(m) = \int_M \langle f, PS^{-1}h_m\rangle Ph_m \mathrm{d}\mu(m) \\ &= \int_M \langle f, S^{-1}P(Ph_m)\rangle Ph_m \mathrm{d}\mu(m), \quad \forall f \in H. \end{aligned}$$

Thus, by Proposition 2.7, $(Ph_m)^* = S^{-1}Ph_m = PS^{-1}h_m = Ph_m^*, \forall m \in M$.

Corollary 3.2 *Let $\{h_m\}_{m\in M}$ be a generalized frame for H. Then $\{h_m\}_{m\in M}$ is a tight generalized frame if and only if $(Ph_m)^* = Ph_m^*$ for all projection $P \in B(H)$.*

Theorem 3.3 *Let $\{h_m\}_{m\in M}$ be a generalized frame for H and let S be the frame operator of $\{h_m\}_{m\in M}$. Suppose that $P \in B(H)$ is an orthogonal projection. Then $\{Ph_m\}_{m\in M}$ and $\{P^\perp h_m\}_{m\in M}$ are strongly disjoint if and only if $PS = SP$.*

Proof Let T_1, T_2 be the analyzing operator associated with Ph_m, $P^\perp h_m$, respectively. First suppose that $\{Ph_m\}_{m\in M}$ and $\{P^\perp h_m\}_{m\in M}$ are strongly disjoint. By the definition of two strongly disjoint generalized frames, $T_1x \perp T_2y$ for all $x, y \in H$. However,

$$\langle T_1x, T_2y\rangle = \langle T_hPx, T_hP^\perp y\rangle = \langle P^\perp SPx, y\rangle.$$

Hence $P^\perp SP = 0$, which means $PS = SP$ since S is positive.

Conversely, Assume that $PS = SP$, then P and S have operator matrix form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$ on $H = R(P) \oplus N(P)$, and thus,

$$P^\perp SP = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies

$$\langle T_1x, T_2y\rangle = \langle P^\perp SPx, y\rangle = 0, \quad \forall x, y \in H.$$

Hence $\{Ph_m\}_{m\in M}$ and $\{P^\perp h_m\}_{m\in M}$ are strongly disjoint.

Corollary 3.4 *Let $\{h_m\}_{m\in M}$ be a generalized frame for H and let S be the frame operator of $\{h_m\}_{m\in M}$. Suppose that $P \in B(H)$ is an orthogonal projection. Then $\{Ph_m\}_{m\in M}$ and*

$\{P^{\perp}h_m\}_{m\in M}$ *are strongly complementary if and only if* $PS=SP$.

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正交射影在广义框架理论中的应用

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摘要: 本文研究了可分的 Hilbert 空间 H 中的广义框架，运用算子理论方法，研究了可分的 Hilbert 空间 H 中广义框架的性质，给出了广义框架的对偶广义框架的一些刻画，并且证明了两个广义框架是强非交的一个充分必要条件.

关键词: 广义框架；对偶广义框架；强非交；正交射影.