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Pointwise Pseudo-Orbit Tracing Property and Chaos

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Abstract In this article, we discuss the relationship between pointwise pseudo-orbit tracing property and chaotic properties such as topological mixing. When f has pointwise pseudo-orbit tracing property, we give some equal conditions of uniform positive entropy and completely positive entropy.

Keywords tracing property; chaos; mixing; uniform positive entropy; completely positive entropy.

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1. Introduction

Pseudo-orbit and its tracing skills are powerful tools in discussing dynamic systems. Pseudo-orbit tracing property has close relations with chaotic properties of system. T.Shirnomura[1] discussed the relationship between pseudo-orbit tracing property and chain transitivity and Yang[2−6] discussed the relationship between pseudo-orbit tracing property and chaotic properties. In [7], the pointwise pseudo-orbit tracing property (PPOTP for short) was defined, and it is a generalization of pseudo-orbit tracing property. As applications, the following results were proved:

Theorem A *If f has PPOTP, and for any $k \in \mathbb{N}, f^k$ is chain transitive, then f^k is topological transitive.*

Theorem B *If f has PPOTP, and for any $n \in \mathbb{N}, f^n$ is chain transitive, then f has sensitive dependence on initial conditions.*

Theorem C *If f is topological mixing, and f has PPOTP, then f has property P.*

Theorem D *Let $f : (X, d) \to (X, d)$ be a homeomorphism. Then f has PPOTP if and only if σ_f has PPOTP.*

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In this paper, for a continuous map f on a metric space X having more than one point, we go a step further in discussing the relationship between PPOTP and chaotic properties such as topological mixing. As applications, we prove the following results:

Theorem 1 *If f has PPOTP, and for any $k \in \mathbb{N}, f^k$ is chain mixing, then f^k is topological mixing.*

Theorem 2 *If X is a locally connected compact metric space, and f is a minimal map with PPOTP, then f is chaotic in the sense of Ruelle-Takens.*

Theorem 3 *If X is a compact metric space, $f : X \to X$ has PPOTP and $\Lambda(f)$, the limit set of f, is connected, then f is chaotic in the sense of Ruelle-Takens.*

Theorem 4 *If X is a connected compact metric space, $f : X \to X$ is a chain transitive map with PPOTP, then we have the following results: (1) f is chaotic in the sense of Ruelle-Takens; (2) f is topological mixing; (3) f has property P.*

Theorem 5 *If X is a connected compact metric space, $f : X \to X$ has PPOTP and the periodic points of f is dense in X, then (1) f is chaotic in the sense of Ruelle-Takens; (2) f is topological mixing; (3) f has property P.*

Theorem 6 *Let X be a compact metric space, and f be a continuous surjective map on X. If f has PPOTP, then the following conditions are equivalent: (1) f has completely positive entropy (c.p.e.); (2) f has uniform positive entropy (u.p.e.); (3) f is chaotic in the sense of Ruelle-Takens-Kato; (4) f is chain transitive and accessible; (5) f is chain mixing; (6) f is topological weakly mixing; (7) f is topological mixing; (8) f has property P; (9) f is chain transitive and for any $\delta > 0$, there are two periodic δ-pseudo-orbits whose periods are co-prime.*

Corollary *If X is a connected compact metric space, $f : X \to X$ has PPOTP and f is positively expansive, then f is topological mixing if and only if f is topological transitive and there are two periodic points whose periods are co-prime.*

Theorem 7 *Let X be a chain connected metric space and f be a surjective map on X. If f has PPOTP, then the following conditions are equivalent: (1) f is topological mixing; (2) f is weakly mixing; (3) f is chain mixing; (4) f has property P; (5) f is chaotic in the sense of Ruelle-Takens-Kato; (6) f is chaotic in the sense of Ruelle-Takens; (7) f is chain transitive and accessible; (8) f is topological transitive; (9) f is chain transitive; (10) $CR(f) = X$; (11) f is chain transitive and for any $\delta > 0$, there are two periodic δ-pseudo-orbits whose periods are co-prime.*

Corollary *Let X be a chain connected compact metric space, and f be a continuous surjective map on X. If f has PPOTP, then the condition that f has completely positive entropy is equivalent to the conditions listed in Theorem 7 and the following: (1) f has uniform positive entropy; (2) there is an invariant probability measure μ of f, such that $supp\mu = X$; (3) f has* *full measure center, that is, $M(f) = X$.*

2. The Preliminaries

If $\delta > 0$, and for any $i \in \mathbb{N}$, $0 \leq n_1 < i < n_2 \leq +\infty$, $d(f(x_{i-1}), x_i) < \delta$, then the sequence $\{x_{n_1}, \ldots, x_{n_2}\}$ is called a δ pseudo-orbit of f (or δ-chain). If for any $x, y \in X, \varepsilon > 0$, there is a finite ε-pseudo-orbit $\{x_0, x_1, \ldots, x_n\}$ of X, such that $x_0 = x, x_n = y$, then $\{x_0, x_1, \ldots, x_n\}$ is called a ε-chain from x to y, and $n+1$ is called the length of the ε-chain. If for any $\varepsilon > 0, x, y \in X$, there is a ε-chain from x to y, then f is called chain transitive. If for any $\varepsilon > 0, x, y \in X$, there is a positive integer N, such that when $n \geq N$, there is a ε-chain with length n from x to y, then f is called chain mixing.

For any nonempty open sets U and V, if there is $n > 0$, such that $f^n(U) \cap V \neq \emptyset$, then f is called topological transitive. If for any nonempty open sets U and V, there is $N > 0$, such that for any $n > N, f^n(U) \cap V \neq \emptyset$, then f is called topological mixing. Obviously, topological transitive (mixing) map is chain transitive (mixing).

Denote by $P(f)$ the set of all periodic points of f, by $W(f)$ the set of all weakly almost periodic points of f[8], by $AP(f)$ the set of all almost periodic points of f, by $CR(f)$ the set of all chain recurrent points of f and by $\Omega(f)$ the set of all non-wandering points of f. A subset of X is called the measure center of f, if it is the minimal compact absolute measure 1 set invariant of f, we denote it by $M(f)$, and we have $M(f) = \overline{W(f)}$[8]. Obviously: $P(f) \subset W(f) \subset M(f) \subset CR(f)$.

Let $x \in X$. If for any $\varepsilon > 0$, there is $\delta > 0$, such that when $d(x, y) < \delta$, $d(f^n(x), f^n(y)) < \varepsilon$ for each $n \in \mathbb{N}$, then x is called an equicontinuous point of f. If any point in X is equicontinuous point, then f is called equicontinuous. If every point in X is not equicontinuous, then we say f has sensitive dependence on initial conditions (We call f sensitive for short). If $f : X \to X$ is topological transitive and sensitive, then f is called chaotic in the sense of Ruelle-Takens. We call f accessible, if for any nonempty open sets U, V of X and any $\varepsilon > 0$, there are $x \in U, y \in V$ and $n \in \mathbb{N} \cup \{0\}$, such that $d(f^n(x), f^n(y)) \leq \varepsilon$. f is called chaotic everywhere, if f is sensitive and accessible. If f is chaotic in the sense of Ruelle-Takens and chaotic everywhere, then f is called chaotic in the sense of Ruelle-Takens-Kato.

We say that f has property P, if for any nonempty open sets U_0, U_1 of X, there is a number N, such that for any number $k \geq 2$ and any $S = \{s(1), s(2), \ldots, s(k)\} \in \{0, 1\}^k$, there is $x \in X$, such that $x \in U_{s(1)}, f^N(x) \in U_{s(2)}, \ldots, f^{(k-1)N}(x) \in U_{s(k)}$. f is said to have uniform positive entropy (u.p.e), if any cover composed of two non-dense open sets has positive entropy. f is said to have completely positive entropy (c.p.e), if any non trivial factor of (X, T) has positive entropy.

Let $x, y \in X, \varepsilon > 0$, and $\{x_0, x_1, \ldots, x_n\}$ be a sequence composed of finite points in X, $n \in \mathbb{N}$. If $x_0 = x, x_n = y$, and $d(x_i, x_{i+1}) < \varepsilon$ for $0 \leq i \leq n-1$, then $\{x_0, x_1, \ldots, x_n\}$ is called an ε-chain from x to y in X. Metric space X is called chain connected, if for any $x, y \in X$ and $\varepsilon > 0$, there is an ε chain from x to y. It is easy to see that connected metric space is chain connected, but the converse is not true. However, for compact metric space, chain connectedness is equivalent to connectedness.

Let $\varepsilon > 0, \{x_{n_1}, x_{n_1+1}, \ldots, x_{n_2}\}$ be a δ-pseudo-orbit of f. If $x \in X$, and for any $i, 0 \leq i \leq n_2 - n_1, d(f^i(x), x_{n+i}) < \varepsilon$, then the δ-pseudo-orbit $\{x_{n_1}, x_{n_1+1}, \ldots, x_{n_2}\}$ is said to be ε traced by the orbit of f on x. If for any $\varepsilon > 0$, there is $\delta > 0$, such that any δ-pseudo-orbit of f is ε traced by the orbit of f on some point in X, then f is said to have pseudo-orbit tracing property. In [7], generalizing the definition of pseudo orbit tracing property, Li Mingjun gives the definition of PPOTP. f is said to have PPOTP, if for any $\varepsilon > 0$, there is $\delta > 0$, such that for any δ pseudo-orbit $\{x_0, x_1, \ldots\}$ of f, there is nonnegative integer N, such that $\{x_N, x_{N+1}, \ldots\}$ can be ε traced by the orbit of f on some point in X. Obviously, if f is a continuous map, then f has PPOTP if and only if for any $\varepsilon > 0$, there is $\delta > 0$, such that for any δ pseudo orbit $\{x_0, x_1, \ldots\}$, there is a nonnegative integer N and $x \in X$ such that $d(f^n(x), x_n) \leq \varepsilon, n \geq N$.

By definition, f has pseudo orbit tracing property $\Rightarrow f$ has asymptotic pseudo orbit tracing property[9]$\Rightarrow f$ has PPOTP, but the converse is false. So PPOTP is a generalization of pseudo orbit tracing property strictly.

3. Proof

Lemma 1[7] *If f has PPOTP, then for any $k \in \mathbb{Z}_+, f^k$ also has PPOTP.*

Proof of Theorem 1 Let $x, y \in X$, $B(x, \varepsilon_1) = \{z \in X | d(x, z) < \varepsilon_1\}, B(y, \varepsilon_2) = \{z \in X | d(y, z) < \varepsilon_2\}$. Suppose f has PPOTP, then for any $k \in \mathbb{Z}_+, f^k$ also has PPOTP. So for any $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, there is $\delta > 0$, such that for any δ pseudo-orbit $\{x_0, x_1, \ldots\}$ of f^k, there is nonnegative integer N and $z \in X$, such that $\{x_N, x_{N+1}, \ldots\}$ is ε traced by the orbit of f^k on z. As f^k is chain mixing, there is an integer $M > 0$, such that when $n \geq M$, there are δ chain $\alpha_n = \{y_0^{(n)} = x, y_1^{(n)}, \ldots, y_{n-1}^{(n)} = y\}$ of f^k from x to y with length n and δ chain $\beta_n = \{z_0^{(n)} = y, z_1^{(n)}, \ldots, z_{n-1}^{(n)} = x\}$ of f^k from y to x with length n. Let

$$\overline{\alpha_n} = \{y_0^{(n)} = x, y_1^{(n)}, \ldots, y_{n-2}^{(n)}\}, \overline{\beta_n} = \{z_0^{(n)} = y, z_1^{(n)}, \ldots, z_{n-2}^{(n)}\},$$

$$A = \{\overline{\alpha_M}, \overline{\beta_M}, \overline{\alpha_{M+1}}, \overline{\beta_{M+1}}, \ldots\} = \{p_0, p_1, \ldots\}.$$

Then A is a δ pseudo-orbit of f^k, so there is nonnegative integer N, such that $\{p_N, p_{N+1}, \ldots\}$ is ε traced by some point $p \in X$ for f^k.

Take nonnegative integer i_0, such that p_{N+i_0} is the first element of some $\overline{\alpha_l}$. Then $p_{N+i_0} = x, p_{N+i_0+(l-1)} = y$. By the construction of A,

$$p_{N+i_0+2(l-1)} = x, p_{N+i_0+2(l-1)+l} = y;$$

$$p_{N+i_0+2(l-1)+2l} = x, p_{N+i_0+2(l-1)+2l+(l+1)} = y, \ldots$$

by and by, for $i = i_0 + 2[(l-1) + l + \cdots + (l+h)]$, $-1 \leq h \in \mathbb{Z}$, $p_{N+i} = x$, $p_{N+i+(l+h+1)} = y$. Let $j = i + (l + h + 1)$. By PPOTP, $d(f^{ki}(p), x) = d(f^{ki}(p), p_{N+i}) < \varepsilon$, and $d(f^{kj}(p), y) = d(f^{kj}(p), p_{N+j}) < \varepsilon$. So $f^{k(j-i)}(B(x, \varepsilon_1)) \cap B(y, \varepsilon_2) \neq \emptyset$. Also as $j - i = l + h + 1, -1 \leq h \in \mathbb{Z}$, there is $l \in \mathbb{N}$, such that when $m \geq l$, $f^{km}(B(x, \varepsilon_1)) \cap B(y, \varepsilon_2) \neq \emptyset$, that is to say, f^k is topological mixing. □

Lemma 2 *Let $f : X \to X$ be a surjective continuous map on compact metric space X. If f has PPOTP, then for any $\varepsilon > 0, x \in \Omega(f)$, there is $y \in X$ and $k = k(x, \varepsilon) > 0$, such that $\overline{O_{f^k}(y)} \subset U_\varepsilon(x)$.*

Proof Let $\varepsilon > 0$. As f has PPOTP, there is $0 < \delta < \frac{\varepsilon}{3}$, for any δ pseudo-orbit $\{x_0, x_1, \ldots\}$ of f, there is $N > 0$, such that $\{x_N, x_{N+1}, \ldots\}$ is $\frac{\varepsilon}{3}$ traced for f.

Since $x \in \Omega(f)$, there is $k > 0$, such that $f^k(U_{\frac{\delta}{2}}(x)) \cap U_{\frac{\delta}{2}}(x) \neq \emptyset$. So there is $z \in X$, such that $z, f^k(z) \in U_{\frac{\delta}{2}}(x)$. Let $z_{nk+i} = f^i(z), n \in \mathbb{Z}_+, 0 \le i < k$. Then $\{z_i | i \in \mathbb{Z}_+\} = \{z, f(z), \ldots, f^{k-1}(z), z, \ldots\}$ is a periodic δ chain. So there is $N > 0, y' \in X$, such that $d(f^i(y'), z_{N+i}) < \frac{\varepsilon}{3}, i \in \mathbb{Z}_+$. Suppose i_0 is the minimal positive integer such that $ki_0 - N > 0$. Put $y = f^{ki_0-N}(y')$. Then $d(f^{ki}(y), z) = d(f^{ki}(f^{ki_0-N}(y')), z) = d(f^{k(i+i_0)-N}(y'), z_{k(i+i_0)}) = d(f^{k(i+i_0)-N}(y'), z_{N+(k(i+i_0)-N)}) < \frac{\varepsilon}{3}, i \in \mathbb{Z}_+$. So $d(f^{ki}(y), x) \le d(f^{ki}(y), z) + d(z, f^k(z)) + d(f^k(z), x) < \frac{\varepsilon}{3} + \delta + \frac{\delta}{2} < \frac{5\varepsilon}{6}, \forall i \in \mathbb{Z}_+$. So $\overline{O_{f^k}(y)} \subset U_\varepsilon(x)$. □

Lemma 3 *Let X be a connected compact metric space with more than one point, and f be a surjective continuous map on X. If f is minimal,then f doesn't have PPOTP.*

Proof Let $l > 0$ be the diameter of X, $\varepsilon = \frac{l}{3}$. Suppose f has PPOTP. Since X is compact, $\Omega(f)$ is not empty. Let $x \in \Omega(f)$, by Lemma 2, there is $y \in X$ and $k > 0$ such that $\overline{O_{f^k}(y)} \subset U_\varepsilon(x)$, obviously, $X = \overline{O_{f^k}(y)} \cup \overline{O_{f^k}(f(y))} \cup \cdots \cup \overline{O_{f^k}(f^{k-1}(y))}$. By connectedness and minimality, we have $\overline{O_{f^k}(y)} = X$ and therefore $l \le 2\varepsilon$. That is a contradiction. □

Lemma 4[12] *If $n \ge 2, f : X \to X$ is topological transitive, but f^n is not topological transitive, then there is closed set $K \neq X, K^o$ (the internal of K)$\neq \emptyset$ and $m > 1$, the factor of n, such that: (1) $f^m(K) = K$; (2) $K \cup f(K) \cup \cdots \cup f^{m-1}(K) = X$; (3) $[f^i(K) \cap f^j(K)]^o = \emptyset, 0 \le i, j \le m-1, i \neq j$.*

Lemma 5[13] *Let X be a compact metric space. If $f : X \to X$ is topological transitive and equicontinuous, then f is a minimal homeomorphism.*

Lemma 6 *Let X be a local connected compact metric space with infinite points. If $f : X \to X$ is chain transitive and equicontinuous, then f does not have PPOTP.*

Proof (reduction to absurdity) Suppose f has PPOTP. Since f has equicontinuous point, by Theorem B, there is $n > 0$, such that f^n is not chain transitive. Suppose $n = \min\{l | f^l$ is not chain transitive$\}$. Since f is chain transitive, $n \ge 2$. Also as topological transitive map is chain transitive, by Theorem A, f is topological transitive, but f^n is not topological transitive. By Lemma 4, there is a proper closed subset K of X, $K^o \neq \emptyset$, and $m > 1$, the factor of n, such that $f^m(K) = K$, $K \cup f(K) \cup \cdots \cup f^{m-1}(K) = X$, and when $i \neq j$, $[f^i(K) \cap f^j(K)]^o = \emptyset$. On the other hand, as f is topological transitive and equicontinuous, by Lemma 5, $f : X \to X$ is a minimal homeomorphism. So for any $x \in K$, $\omega(x, f) = X$ and $\omega(x, f^m) \subset K$. For any $i = 0, 1, \ldots, m-1, f^i(x) \in AP(f) = AP(f^m)$, $\omega(f^i(x), f^m)$ is minimal set of f^m, and $\omega(f^i(x), f^m) = f^i(\omega(x, f^m)) \subset f^i(K)$, $X = \omega(x, f) = \bigcup_{i=0}^{m-1} \omega(f^i(x), f^m) \subset \bigcup_{i=0}^{m-1} f^i(K) = X$. As f is a homeomorphism, K is a nonempty proper closed subset of X. So $f^i(K) = (f^{-1})^{-i}(K)$ is a nonempty closed subset of X. Also as $f^m(f^i(K)) = f^i(f^m(K)) = f^i(K), \omega(f^i(x), f^m)$ is a minimal set contained in $f^i(K)$. So for any $i = 0, 1, \ldots, m-1, f^i(\omega(x, f^m)) = \omega(f^i(x), f^m) = f^i(K)$. When $i \neq j$, $f^i(K) \cap f^j(K) = \emptyset$. Put $g = f^m$. Then $K = \omega(x, g), g|_K : K \to K$ is a minimal homeomorphism, K is an open and closed set of X. As a subset of X, K must be a local connected compact metric space containing infinite points. Thus K has only finite connected components, therefore, there is at least one connected component $G(= G^o)$ which has more than one point. As $g|_K : K \to K$ is minimal, there is $l \in \mathbb{N}$, such that $g^l(G) \cap G \neq \emptyset$. So $g^l(G) \subset G$. As g is a homeomorphism, $(g^l)^{-1}(G)$ is also connected, and $(g^l)^{-1}(G) \cap G \neq \emptyset$. So $(g^l)^{-1}(G) \subset G$ and $g^l(G) = G, g^l|_G : G \to G$ is a surjective equicontinuous homeomorphism. Since X is compact and local connected, X has only finite connected components $\{G_0, G_1, \ldots, G_h\}$. Because when $0 \leq i, j \leq m-1, i \neq j$, $f^i(K) \cap f^j(K) = \emptyset$, G is a component of X. Take $G = G_0$ and $\eta = \min\{d(G, G_i) | i = 1, 2, \ldots, h\}$. Since $G_0, G_1, \ldots, G_h$ are all closed sets, $\eta > 0$. By Lemma 1, $g^l = f^{ml} : X \to X$ has PPOTP. For any $0 < \varepsilon < \eta$, there exists $\delta > 0$, such that for any $\{x_0, x_1, \ldots\}$, the δ pseudo-orbit of g^l in G, there is $N \in \mathbb{N}$ and $t \in X$, such that $\{x_N, x_{N+1}, \ldots\}$ is ε traced by t for g^l. So there is $k \in \mathbb{N}$ such that $y = g^k(x) \in G$ and $\omega(y, g^l) \subset G$. Hence $K \supset \bigcup_{i=0}^{l-1} g^i(G) \supset \bigcup_{i=0}^{l-1} g^i(\omega(y, g^l)) = \omega(y, g) = K$. As $g|_K$ is homeomorphism, $\omega(y, g^l) = G$. Also as $y = g^k(x) = f^{nk}(x) \in AP(f) = AP(g^l)$, $g^l|_G : G \to G$ is a minimal surjective homeomorphism, which is a contradiction with Lemma 3. □

Proof of Theorem 2 Since f is a minimal homeomorphism, f is topological transitive. If f is not chaotic in the sense of Ruelle-Takens, then there is at least an equicontinuous point. By Corollary 2 in [13], a minimal map having equicontinuous point must be equicontinuous. By Lemma 6, f does not have PPOTP, leading to a contradiction. □

Lemma 7[2] *If $f : X \to X$ is a surjective continuous map on compact metric space X and $\Lambda(f) = \bigcup_{x \in X} \omega(x, f)$ is connected, then for any $n \in \mathbb{N}$, f^n is chain transitive on X.*

Proof of Theorem 3 The conclusion follows from Lemma 7, Theorem A and Theorem B.

Lemma 8[1] *If X is connected, and $f : X \to X$ is chain transitive, then f is chain mixing.*

Proof of Theorem 4 By Lemma 8, f is chain mixing, so for any $n \in \mathbb{N}, f^n$ is chain transitive. By Theorem B, if f has PPOTP, then f is sensitive. By Theorem A, f is chaotic in the sense of Ruelle-Takens. By Theorem 1, f is topological mixing, and by Theorem C, f has Property P.□

Lemma 9 *If $f : X \to X$ is a continuous map on compact metric space X and $\overline{P(f)} = X$ is connected, then f is chain transitive on X.*

Proof For any $\varepsilon > 0$, let $R_\varepsilon(x) = \{y \in X |$ there is a ε chain from x to $y\}$. Then for any $y \in R_\varepsilon(x)$, there is a ε chain $\{x_0 = x, x_1, \ldots, x_{n-1}, x_n = y\}$ from x to y and $d(f(x_{n-1}), y) < \varepsilon$. So there exists a neighborhood U_y of y, such that $U_y \subset B(f(x_{n-1}), \varepsilon)$. Thus for any $y' \in U_y, \{x_0, x_1, \ldots, x_{n-1}, y'\}$ is a ε chain from x to y', $U_y \subset R_\varepsilon(x)$, that is, $R_\varepsilon(x)$ is open. Let $R^\varepsilon(x) = \{y \in X |$ there is a ε chain from y to $x\}$. For any $y \in R^\varepsilon(x)$, there is a ε chain $\{y_0, y_1, \ldots, y_n\}$, where $y_0 = y, y_n = x$, such that $d(f(y_0), y_1) < \varepsilon$. As f is continuous, there is an open neighborhood U_{y_0} of y_0, such that $f(U_{y_0}) \subset B(y, \varepsilon)$. So $U_{y_0} \subset R^\varepsilon(x)$ and $R^\varepsilon(x)$ is also an open set. For any $x \in X = \overline{P(f)}$, let $E(x, \varepsilon) = R_\varepsilon(x) \cap R^\varepsilon(x) \cap \overline{P(f)}$. Then x must be a chain recurrent point of f ($P(f) \subset CR(f)$, and $CR(f)$ is closed), $x \in R_\varepsilon(x) \cap R^\varepsilon(x)$. So $E(x, \varepsilon)$ must be a nonempty open set of X. If $y \notin E(x, \varepsilon)$, then $E(x, \varepsilon) \cap E(y, \varepsilon) = \emptyset$. If $y \in E(x, \varepsilon)$, then $E(x, \varepsilon) = E(y, \varepsilon)$ and $E(x, \varepsilon) = X - \bigcup_{y \in X \setminus \underline{E(x,\varepsilon)}} E(y, \varepsilon)$. It is obvious that $E(x, \varepsilon)$ is also a closed set in X. As $X = \overline{P(f)}$ is connected, $\overline{P(f)} = E(x, \varepsilon)$. Since ε is arbitrary, $\overline{P(f)}$ is contained in a chain component of f, namely, f is chain transitive in $X(= \overline{P(f)})$. □

Lemma 10[11] *If X is a compact metric space, then we have the following results: property P$\Rightarrow$ uniformly positive entropy $\Rightarrow$ topological weakly mixing. Completely positive entropy$\Rightarrow$ there is $\mu \in M(X, f)$ such that* $\text{supp}\mu = X$.

Thereinto, $M(X, f)$ is the set of invariant possibility measures of f on X, $\text{supp}\mu$ is the support of measure μ.

Proof of Theorem 5 By Lemma 9, $\overline{P(f)}$ is contained in a chain component of f, that is, f is chain transitive on $X(= \overline{P(f)})$. So by Theorem 4, Theorem 5 is proved. □

Proof of Theorem 6

(5) $\Rightarrow$ (7). By Theorem 1.

(7) $\Rightarrow$ (6). Obviously.

(6) $\Rightarrow$ (5). By Lemma 2.3 in [3].

So (5), (6) and (7) are equivalent.

(7) $\Rightarrow$ (8). By Theorem C.

(8) $\Rightarrow$ (2). By Lemma 10.

(2) $\Rightarrow$ (6). By Lemma 10.

(6) $\Rightarrow$ (7). Since (6) and (7) are equivalent,

(2),(6),(7) and (8) are equivalent.

By [3] Lemma 2.2, (5) and (9) are equivalent. So (2),(5),(6),(7),(8) and (9) are equivalent.

The equivalence of (1), (2), (3) and (4) is proved below.

(1) $\Rightarrow$ (2). By Theorem 1 in [6] and the equivalence of (5) and (2).

(2) $\Rightarrow$ (3). By the equivalence of (2) and (6) and Theorem 2 in [6].

(3) $\Rightarrow$ (4). Obviously.

(4) $\Rightarrow$ (1). By [6] Lemma 2, the equivalence of (5) and (2), and the fact that u.p.e. implies c.p.e. □

Proof of Corollary of Theorem 6 $\Rightarrow$. It is easy to see that f is topological transitive. Let c be the positively expansive constant of f. For any $0 < \varepsilon < \frac{c}{2}$, since f has PPOTP, there is $\delta > 0$, such that for any δ pseudo-orbit $\{x_0, x_1, \ldots\}$ of f, there is $N > 0$, such that $\{x_N, x_{N+1}, \ldots\}$ can be ε traced by some point in X. By Lemma 2.2 in [3], there are periodic δ chain $\alpha = \{x_i\}_0^{+\infty}$ with period m and periodic δ chain $\beta = \{y_i\}_0^{+\infty}$ with period n, where m and n are co-prime. So there exist $N > 0$ and $x \in X$ such that $x\varepsilon$ traces $\{x_N, x_{N+1}, \ldots\}$, and there exist $M > 0$ and $y \in X$ such that $y\varepsilon$ traces $\{y_M, y_{M+1}, \ldots\}$. So for any $i \in \mathbb{Z}_+$, $d(f^i(x), x_{N+i}) < \varepsilon$ and

$d(f^i(f^n(x)), f^i(x)) \leq d(f^{i+n}(x), x_{N+i+n}) + d(x_{N+i}, f^i(x)) < 2\varepsilon < c$. Therefore $f^n(x) = x$. Analogously, $f^m(y) = y$ and m, n are co-prime.

$\Leftarrow$. By Lemma 2.2 in [3], f is chain mixing and by Lemma 5, f is topological mixing.

Proof of Theorem 7 (1) $\Rightarrow$ (2). Obviously;

(2) $\Rightarrow$ (3). by Lemma 2.3 in [3];

(3) $\Rightarrow$ (10). Obviously;

(10) $\Rightarrow$ (1). By [6] Lemma 3, Lemma 4 and Theorem 1.

(3) $\Leftrightarrow$ (11). By Lemma 2.2 in [3].

So (1),(2),(3),(10),(11) are equivalent.

(1) $\Rightarrow$ (4). By Theorem C;

(4) $\Rightarrow$ (5). By Lemma 5 and Theorem 2 in [6];

(5) $\Rightarrow$ (6) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10). Obviously;

(5) $\Rightarrow$ (7) $\Rightarrow$ (8). Obviously.

(10) $\Rightarrow$ (1). By the equivalence of (1) and (10).

So (1), (4–10) are equivalent.

Therefore, the properties listed in Theorem 7 are equivalent. □

Proof of Corollary of Theorem 7 The conclusion follows from Theorem 6, Theorem 7, Lemma 10 and the fact that for any $\mu \in M(X, f)$, supp$\mu \subset M(f) \subset CR(f)$. □

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