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单纯形上的 q-Stancu 多项式的最优逼近阶

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摘 要 构造了单纯形上的多元 q-Stancu 多项式, 它是著名的 Bernstein 多项式和 Stancu 多项式的推广. 建立该类多项式逼近连续函数的上、下界估计, 进而给出其对连续函数的最优逼近阶 (饱和阶) 及其特征刻画. 此外, 还研究了该类多项式逼近连续函数的饱和类.

关键词 q-Stancu 多项式; 逼近阶; 单纯形
MR(2000) 主题分类 41A35, 41A25
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Optimal Approximation Order for q-Stancu Operators Defined on a Simplex

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Abstract In this paper, a kind of multivariate linear operators, q-Stancu operators, are defined on a simplex, which generalize the famous Bernstein operators and Stancu operators. By establishing the upper and lower bound estimations of approximation order, the optimal approximation order (saturation order) and the characteristics of these operators approximating the continuous functions are characterized. Moreover, the saturation class of approximation is confirmed.

Keywords q-Stancu polynomials; approximation order; simplex
MR(2000) Subject Classification 41A35, 41A25
Chinese Library Classification O174.41

1 引言及主要结果

令 N, R 分别是自然数集和实数集, 且 $N_0 = N \cup \{0\}$. 设

$$S = S_d = \left\{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in R^d \,|\, x_i \geq 0,\, |\mathbf{x}| \leq 1\right\}$$

为 R^d $(d \in N)$ 中的单纯形. 对任意的 $\mathbf{x} \in S$, $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in N_0^d$, 定义

$$|\mathbf{x}| = \sum_{i=1}^{d} x_i, \quad \mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \ldots x_d^{k_d}.$$

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令 $q \ge 1$, 对任意 $n \in N_0$, 定义 q- 数 $[n]_q$ 及其 q- 阶乘 $[n]_q!$ 如下

$$[n]_q := 1 + q + \cdots + q^{n-1} \ \ (n \in N), \ \ [0]_q := 0; \ \ [n]_q! := [1]_q[2]_q \cdots [n]_q \ (n \in N), \ \ [0]_q! := 1.$$

对任意 $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in N_0^d$, $n \in N_0$, 定义 q- 数 $[\mathbf{k}]_q$ 及 q- 二项式系数分别为

$$[\mathbf{k}]_q = [(k_1, k_2, \ldots, k_d)]_q := ([k_1]_q, [k_2]_q, \ldots, [k_d]_q)$$

和

$$\begin{bmatrix} n \\ \mathbf{k} \end{bmatrix}_q := \frac{[n]_q!}{[k_1]_q![k_2]_q! \cdots [k_d]_q![n - |\mathbf{k}|]_q!}.$$

显然, 当 $q = 1$, $k \in N_0$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad [\mathbf{k}]_1 = \mathbf{k} = (k_1, k_2, \ldots, k_d), \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

对任意 $\mathbf{x} = (x_1, x_2, \ldots, x_d)$, $\mathbf{y} = (y_1, y_2, \ldots, y_d)$, 令 $d(\mathbf{x}, \mathbf{y})$ 为 $\mathbf{x}$ 和 $\mathbf{y}$ 的欧几里得距离, 即

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_d - y_d)^2}.$$

设 V_S 为表示 S 的边的方向的单位向量的集合, $e_i = (0, \ldots, 0, \overset{\text{ith}}{1}, 0, \ldots, 0)$ 表示 R^d 中的单位向量. 对于 $\mathbf{x} \in S$, $\xi \in V_S$, 定义如下权函数

$$\varphi_\xi(\mathbf{x}) = \sqrt{\inf_{\mathbf{x}+\lambda\xi \in S, \, \lambda \ge 0} d(\mathbf{x}, \mathbf{x} + \lambda\xi) \inf_{\mathbf{x}-\lambda\xi \in S, \, \lambda \ge 0} d(\mathbf{x}, \mathbf{x} - \lambda\xi)},$$

从而

$$\varphi_\xi^2(\mathbf{x}) = \begin{cases} x_i(1 - |\mathbf{x}|), & \xi = e_i, \ \ 1 \le i \le d, \\ 2x_i x_j, & \xi = (e_i - e_j)/\sqrt{2}, \ \ 1 \le i < j \le d. \end{cases}$$

定义微分算子

$$P(D) = \sum_{\xi \in V_S} \varphi_\xi^2(\mathbf{x}) \Big(\frac{\partial}{\partial x_i}\Big)^2,$$

则

$$P(D) = \sum_{i=1}^{d} x_i(1 - |\mathbf{x}|) \Big(\frac{\partial}{\partial x_i}\Big)^2 + \sum_{1 \le i < j \le d} x_i x_j \Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big)^2.$$

令 $C(S)$ 为 S 上的全体连续函数构成的集合. 对于 $f \in C(S)$, 定义 K- 泛函

$$K(f, t) = \inf_{g \in C^3(S)} \Big\{ ||f - g|| + t^2 ||P(D)g|| + t^3 \sup_{\xi \in V_S} \Big|\Big| \varphi_\xi^3 \Big(\frac{\partial}{\partial x_i}\Big)^3 g \Big|\Big| \Big\},$$

其中 $0 < t < t_0$, t_0 为一给定的正常数. 对于 $q \ge 1$, $f \in C(S)$, 我们定义 d 维 q-Stancu 多项式为

$$M_{n,q} f = M_n(f, q; \mathbf{x}) = \sum_{|\mathbf{k}| \le n-s} P_{n-s,\mathbf{k}} \Big\{ (1 - |\mathbf{x}|) f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big) + \sum_{i=1}^{d} x_i f\Big(\frac{[\mathbf{k}]_q + [s]_q e_i}{[n]_q}\Big) \Big\}, \qquad (1)$$

其中

$$P_{n-s,\mathbf{k}} = \begin{bmatrix} n - s \\ \mathbf{k} \end{bmatrix}_q \mathbf{x}^{\mathbf{k}} (1 - |\mathbf{x}|)^{n-s-|\mathbf{k}|}, \ \ \mathbf{x} \in S, \ \ n \in N, \ \ s \in Z, \ \ 0 \le s \le \sqrt{n}/2.$$

显然, 当 $q = 1$ 时, $M_{n,q} f$ 就是 Stancu 多项式, 当 $q = 1$, $s = 0$ 或 $s = 1$ 时, $M_{n,q} f$ 就退化为著名的 Bernstein 多项式. 由于我们引入了参数 q, 并且只要求 $q \ge 1$, 可以看出 q-Stancu 多项式比 Stancu 多项式和 Bernstein 多项式具有更广泛的代表性. 近年来, 关于经典 Bernstein 多项式的一个重要研究方向就是采用各种方法对其加以改进 [1–10]. 1981 年, Stancu [3] 定义了 Stancu 多项式用来改进传统的 Bernstein 多项式. 从此之后, Stancu 多项式引起了许多学者的极大兴趣,

很多关于 Stancu 多项式的研究成果相继产生 [3−8]. 文 [6] 研究了 Stancu 多项式的对称性并且证明了其保持函数的 Lipschitz 性质. 利用最佳逼近作为工具, 文 [7] 证明了 Stancu 多项式对连续函数的逼近性质, 刻画了逼近速度及逼近阶. 文 [8] 研究了多元 Stancu 多项式与连续模之间的关系.

我们知道算子的饱和阶的确定及饱和类的刻画是十分重要的, 只有确定了一个算子的饱和阶和饱和类, 才能真正澄清其逼近性能. 本文通过引入参数, 定义一类比 Stancu 多项式更广泛的 q-Stancu 多项式, 给出该类多项式逼近的上、下界估计, 进而给出其对连续函数的最优逼近阶 (饱和阶) 及其特征刻画. 此外还研究了该类多项式逼近连续函数的饱和类. 本文主要结果如下：

定理 1　设 $f(\mathbf{x})\in C(S), s\in N_0$. 若 s 是小于 $n/2$ 的常数, 或者 $s=s(n)$ 与 n 有关但小于 $\sqrt{n}/2$, 则有

(i) 逼近上界估计
$$||M_{n,q}f-f||\le CK(f,[n]_q^{-\frac{1}{2}}),\tag{2}$$

(ii) 逼近下界估计
$$K(f,[n]_q^{-\frac{1}{2}})\le C[n]_q^{-\frac{3}{2}}\sum_{i=1}^{n}i^{\frac{1}{2}}||M_{i,q}f-f||,\tag{3}$$

(iii) 饱和类估计
$$||M_{n,q}f-f||=O([n]_q^{-\alpha})\Longleftrightarrow K(f,t)=O(t^{2\alpha}),\quad 0<\alpha<1,\tag{4}$$

(iv) 饱和阶估计
$$||M_{n,q}f-f||=o([n]_q^{-1})\Longleftrightarrow f\in\Pi_1,\tag{5}$$

其中 Π_m 是阶数不超过 m 的多项式全体构成的集合, C 是与 $f,n,\mathbf{x}$ 无关的正常数, 但其值可能在上下文中不同.

2　基本引理

为了证明我们的结论, 需要给出以下引理.

引理 1　设 $q\ge 1$, 对任意 $m,n\in N_0$, $m\ge n$, 直接计算可得

(i) $m\le [m]_q$, (ii) $[m]_q+[n]_q\le[m+n]_q$, (iii) $[m]_q+1\le[m+1]_q$, (iv) $[m-n]_q\le[m]_q-[n]_q$.

定义单纯形 S 到其自身的变换 $T_i,\ i=1,2,\dots,d$,
$$T_i(\mathbf{x})=\mathbf{u},\quad \mathbf{u}=(x_1,x_2,\dots,x_{i-1},1-|\mathbf{x}|,x_{i+1},\dots,x_d),\quad \mathbf{x}\in S,$$
容易看出 $T_i^2=I$ (I 是单位变换),
$$\begin{cases}\dfrac{\partial}{\partial u_l}=\dfrac{\partial}{\partial x_l}-\dfrac{\partial}{\partial x_j},\quad l\ne j,\quad l,\qquad j=1,2,\dots,d,\\[2mm] \dfrac{\partial}{\partial u_j}=-\dfrac{\partial}{\partial x_j},\qquad\qquad\qquad j=1,2,\dots,d.\end{cases}$$

利用上述关系, 采用文 [6, 定理 2] 的方法可得:

引理 2　若 $f_{T_i}(\mathbf{u})=f(T_i\mathbf{x})$, 则
$$M_n(f,q;\mathbf{x})=M_n(f_{T_i},q;T_i\mathbf{x}),\quad M_n(f,q;T_i\mathbf{x})=M_n(f_{T_i},q;\mathbf{x}).\tag{6}$$

引理 3　若 $\xi\in V_S$, $v=0,1$, $f\in C^r(S)$, $r\in N_0$, 则
$$\Big|\Big|\varphi_\xi^{r+v}\Big(\frac{\partial}{\partial\xi}\Big)^{r+v}M_{n,q}f\Big|\Big|\le C[n]_q^{\frac{v}{2}}\Big|\Big|\varphi_\xi^{r}\Big(\frac{\partial}{\partial\xi}\Big)^{r}f\Big|\Big|.\tag{7}$$

证明 因 $\xi=e_j$ $(j=1,2,\dots,i-1,i+1,\dots,d)$ 的情形与 $\xi=e_i,\ i\in\{1,2,\dots,d\}$ 的情形是对称的, 故仅须证明 $\xi=e_i$ 时, (7) 式成立即可. 若 $\xi=(e_i-e_j)/\sqrt{2}$, 令 $\eta=e_j$. 由 (6) 式, 可得

$$\begin{aligned}\Big\|\varphi_\xi^{r+v}(\mathbf{x})\Big(\frac{\partial}{\partial\xi}\Big)^{r+v}M_n(f,q;\mathbf{x})\Big\|&=\Big\|\varphi_\eta^{r+v}(T_i\mathbf{x})\Big(\frac{\partial}{\partial\eta}\Big)^{r+v}M_n(f_{T_i},q;T_i\mathbf{x})\Big\|\\&=\Big\|\varphi_\eta^{r+v}(\mathbf{u})\Big(\frac{\partial}{\partial\eta}\Big)^{r+v}M_n(f_{T_i},q;\mathbf{u})\Big\|\\&\le C[n]_q^{\frac{v}{2}}\Big\|\varphi_\eta^{r}(\mathbf{u})\Big(\frac{\partial}{\partial\eta}\Big)^{r}f_{T_i}(\mathbf{u})\Big\|\\&=C[n]_q^{\frac{v}{2}}\Big\|\varphi_\xi^{r}\Big(\frac{\partial}{\partial\xi}\Big)^{r}f\Big\|.\end{aligned}$$

对任意 $\mathbf{x}\in R^d$, 当 $|\mathbf{k}|<0$ 或 $|\mathbf{k}|>n$ 时, 设 $P_{n,\mathbf{k}}(\mathbf{x})\equiv 0$, 则

$$\frac{\partial}{\partial x_i}P_{n,\mathbf{k}}(\mathbf{x})=[n]_q\big(P_{n-1,\mathbf{k}-e_i}(\mathbf{x})-P_{n-1,\mathbf{k}}(\mathbf{x})\big).$$

对于函数 $f\in C^r(S)$, 定义函数沿方向 e 步长为 h 的 r 阶前向差分为

$$\triangle_{he}^r f(\mathbf{x})=\begin{cases}\displaystyle\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}f(\mathbf{x}+ihe), & \mathbf{x},\mathbf{x}+rhe\in S,\\ 0, & \text{其它}.\end{cases}$$

由 (1) 可知

$$\begin{aligned}M_n(f,q;\mathbf{x})=&\sum_{|\mathbf{k}|\le n-s}(1-|\mathbf{x}^*|)P_{n-s,\mathbf{k}}(\mathbf{x})f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)+x_iP_{n-s,\mathbf{k}}(\mathbf{x})\triangle_{\frac{[s]_q}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\\&+\sum_{\substack{j=1\\j\ne i}}^{d}x_j\sum_{|\mathbf{k}|\le n-s}P_{n-s,\mathbf{k}}(\mathbf{x})f\Big(\frac{[\mathbf{k}]_q+[s]_qe_j}{[n]_q}\Big),\end{aligned}$$

其中 $\mathbf{x}^*=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)\in S_{d-1}$, $|\mathbf{x}^*|=\sum_{\substack{j=1\\j\ne i}}^{d}x_j$. 直接计算可得

$$\begin{aligned}\Big(\frac{\partial}{\partial x_i}\Big)^{r+v}M_n(f,q;\mathbf{x})=&([n-s]_q)\cdots([n-s-r+1]_q)(1-|\mathbf{x}^*|)\\&\times\sum_{|\mathbf{k}|\le n-s-r}\triangle^r_{\frac{1}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big(\frac{\partial}{\partial x_i}\Big)^{v}P_{n-s-r,\mathbf{k}}(\mathbf{x})\\&+r([n-s]_q)\cdots([n-s-r+2]_q)\\&\times\sum_{|\mathbf{k}|\le n-s-r+1}\triangle^{r-1}_{\frac{1}{[n]_q}e_i}\Big(\triangle_{\frac{[s]_q}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big)\Big(\frac{\partial}{\partial x_i}\Big)^{v}P_{n-s-r,\mathbf{k}}(\mathbf{x})\\&+([n-s]_q)\cdots([n-s-r+1]_q)\\&\times\sum_{|\mathbf{k}|\le n-s-r}\triangle^{r}_{\frac{1}{[n]_q}e_i}\Big(\triangle_{\frac{[s]_q}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big)\Big(\frac{\partial}{\partial x_i}\Big)^{v}(x_iP_{n-s-r,\mathbf{k}}(\mathbf{x}))\\&+([n-s]_q)\cdots([n-s-r+1]_q)\\&\times\sum_{\substack{j=1\\j\ne i}}^{d}x_j\sum_{|\mathbf{k}|\le n-s-r}\triangle^r_{\frac{1}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q+[s]_qe_j}{[n]_q}\Big)\Big(\frac{\partial}{\partial x_i}\Big)^{v}P_{n-s-r,\mathbf{k}}(\mathbf{x})\\:=&\sum_{i=1}^{4}I_i.\end{aligned}\tag{8}$$

至此, 采用文 [9, 定理 4.1] 的方法可证明 (7) 在 $s=0$ 时成立. 下面只须证明 $s>0$ 的情形. 由于

$$\begin{aligned}
&\Big|\triangle^{r-1}_{\frac{1}{[n]_q}e_i}\Big(\triangle_{\frac{[s]_q}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big)\Big|\\
&\quad=\Big|\int_0^{\frac{1}{[n]_q}}\cdots\int_0^{\frac{1}{[n]_q}}\int_0^{\frac{[s]_q}{[n]_q}}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}+e_i\sum_{j=1}^r t_j\Big)dt_1dt_2\cdots dt_r\Big|\\
&\quad\le\Big\|\varphi^r_{e_i}\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big\|\int_0^{\frac{1}{[n]_q}}\cdots\int_0^{\frac{1}{[n]_q}}\int_0^{\frac{[s]_q}{[n]_q}}\Big(\Big(\frac{[k_i]_q}{[n]_q}+\sum_{j=1}^r t_j\Big)\Big(1-\frac{[|\mathbf{k}|]_q}{[n]_q}-\sum_{j=1}^r t_j\Big)\Big)^{-\frac{r}{2}}dt_1dt_2\cdots dt_r\\
&\quad\le\Big\|\varphi^r_{e_i}\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big\|[n]_q^{1-r}\int_0^{\frac{[s]_q}{[n]_q}}\Big(\Big(\frac{[k_i]_q}{[n]_q}+t\Big)\Big(1-\frac{[|\mathbf{k}|]_q}{[n]_q}-\frac{[r-1]_q}{[n]_q}-t\Big)\Big)^{-\frac{r}{2}}dt\\
&\quad\le\Big\|\varphi^r_{e_i}\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big\|[n]_q^{1-r}\Big(\int_0^{\frac{[s]_q}{[n]_q}}\Big(\frac{[k_i]_q}{[n]_q}+t\Big)^{-\frac12}\Big(1-\frac{[|\mathbf{k}|]_q}{[n]_q}-\frac{[r-1]_q}{[n]_q}-t\Big)^{-\frac12}dt\Big)^r,
\end{aligned}$$

于是, 对于 $|\mathbf{k}|\le n-s-r+1$, 有

$$\begin{aligned}
&\int_0^{\frac{[s]_q}{[n]_q}}\Big(\frac{[k_i]_q}{[n]_q}+t\Big)^{-\frac12}\Big(1-\frac{[|\mathbf{k}|]_q}{[n]_q}-\frac{[r-1]_q}{[n]_q}-t\Big)^{-\frac12}dt\\
&\quad\le\Big(1-\frac{[|\mathbf{k}|]_q}{[n]_q}-\frac{[r-1]_q}{[n]_q}-\frac{[s]_q}{2[n]_q}\Big)^{-\frac12}\int_0^{\frac{[s]_q}{[n]_q}}\Big(\frac{[k_i]_q}{[n]_q}+t\Big)^{-\frac12}dt\\
&\qquad+\Big(\frac{[k_i]_q}{[n]_q}+\frac{[s]_q}{2[n]_q}\Big)^{-\frac12}\int_{\frac{[s]_q}{2[n]_q}}^{\frac{[s]_q}{[n]_q}}\Big(1-\frac{[|\mathbf{k}|]_q}{[n]_q}-\frac{[r-1]_q}{[n]_q}-t\Big)^{-\frac12}dt\\
&\quad\le\sqrt2[s]_q[n]_q^{-1}\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac12}\Big(\frac{[n]_q}{[k_i]_q+[s]_q/2}\Big)^{\frac12}\\
&\qquad+\sqrt2[s]_q(2[n]_q)^{-1}\Big(\frac{[n]_q}{[n]_q+[s]_q}\Big)^{\frac12}\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q-[s]_q/2}\Big)^{\frac12}\\
&\quad\le4[s]_q[n]_q^{-1}\Big(\frac{[n]_q}{[k_i]_q+[s]_q}\Big)^{\frac12}\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac12},
\end{aligned}$$

即对于 $|\mathbf{k}|\le n-s-r+1$, 得

$$\Big|\triangle^{r-1}_{\frac{1}{[n]_q}e_i}\Big(\triangle_{\frac{[s]_q}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big)\Big|\le C[s]_q[n]_q^{-r}\Big(\frac{[n]_q}{[k_i]_q+[s]_q}\Big)^{\frac{r}{2}}\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac{r}{2}}.$$

特别地, 对于 $|\mathbf{k}|\le n-s-r$, 成立

$$\Big|\triangle^r_{\frac{1}{[n]_q}e_i}\Big(f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big)\Big|\le C[n]_q^{-r}\Big(\frac{[n]_q}{[k_i]_q+1}\Big)^{\frac{r}{2}}\Big(\frac{[n]_q}{[n]_q+[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac{r}{2}}.$$

同理, 对于 $|\mathbf{k}|\le n-s-r$, $i=1,2,\ldots,j-1,j+1,\ldots,d$, 有

$$\Big|\triangle^r_{\frac{1}{[n]_q}e_i}\Big(f\Big(\frac{[\mathbf{k}]_q+[s]_qe_i}{[n]_q}\Big)\Big)\Big|\le C[n]_q^{-r}\Big(\frac{[n]_q}{[k_i]_q+[s]_q}\Big)^{\frac{r}{2}}\Big(\frac{[n]_q}{[n]_q-[s]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac{r}{2}},$$

$$\Big|\triangle^r_{\frac{1}{[n]_q}e_j}\Big(f\Big(\frac{[\mathbf{k}]_q+[s]_qe_j}{[n]_q}\Big)\Big)\Big|\le C[n]_q^{-r}\Big(\frac{[n]_q}{[k_j]_q+[s]_q}\Big)^{\frac{r}{2}}\Big(\frac{[n]_q}{[n]_q-[s]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac{r}{2}}.$$

若 $v=0$, 由

$$\begin{aligned}
\varphi^{2r}_{e_i}(\mathbf{x})P_{n-r-s,\mathbf{k}}(\mathbf{x})=&\frac{[k_i+1]_q\cdots[k_i+r]_q[n-s-|\mathbf{k}|-r+1]_q\cdots[n-s-|\mathbf{k}|]_q}{[n-s-r+1]_q\cdots[n-s+r]_q}\\
&\times P_{n+r-s,\mathbf{k}+re_i}(\mathbf{x}),
\end{aligned}$$

及

$$A(n,\mathbf{k}):=\frac{[k_i+1]_q\cdots[k_i+r]_q[n-s-|\mathbf{k}|-r+1]_q\cdots[n-s-|\mathbf{k}|]_q}{[n-s-r+1]_q\cdots[n-s+r]_q}\times\Big(\frac{[n]_q}{[k_i+1]_q}\Big)^r\Big(\frac{[n]_q}{[n-|\mathbf{k}|-r+1]_q}\Big)^r\le C,$$

可得

$$\begin{aligned}|\varphi_{e_i}^r(\mathbf{x})I_1|&\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|\sum_{|\mathbf{k}|\le n-s-r}(x_i(1-|\mathbf{x}|))^{\frac r2}\Big(\frac{[n]_q}{[k_i]_q+1}\Big)^{\frac r2}\\&\quad\times\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac r2}P_{n-r-s,\mathbf{k}}(\mathbf{x})\\&\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|\Big\{\sum_{|\mathbf{k}|\le n-s-r}A(n,\mathbf{k})P_{n-r-s,\mathbf{k}+re_i}(\mathbf{x})\Big\}^{\frac12}\\&\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|,\end{aligned}\tag{9}$$

$$\begin{aligned}|\varphi_{e_i}^r(\mathbf{x})I_2|&\le C[s]_q[n]_q^{-1}\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|\sum_{|\mathbf{k}|\le n-s-r+1}(x_i(1-|\mathbf{x}|))^{\frac r2}\\&\quad\times\Big(\frac{[n]_q}{[k_i]_q+1}\Big)^{\frac r2}\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^{\frac r2}P_{n-r-s,\mathbf{k}}(\mathbf{x})\\&\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|\Big\{\sum_{|\mathbf{k}|\le n-s-r+1}P_{n-s+r+1,\mathbf{k}+re_i}(\mathbf{x})\\&\quad\times\frac{[k_i+1]_q\cdots[k_i+r]_q[n-s-|\mathbf{k}|-r+1]_q\cdots[n-s-|\mathbf{k}|]_q}{[n-s-r+1]_q\cdots[n-s+r]_q}\\&\quad\times\Big(\frac{[n]_q}{[k_i]_q+[s]_q}\Big)^r\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^r\Big\}^{\frac12}\\&\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|.\end{aligned}\tag{10}$$

类似地, 可得

$$\begin{aligned}|\varphi_{e_i}^r(\mathbf{x})I_3|&\le([n-s]_q)\cdots([n-s-r+1]_q)\sum_{|\mathbf{k}|\le n-s-r}P_{n-s-r,\mathbf{k}}(\mathbf{x})\varphi_{e_i}^r(\mathbf{x})\\&\quad\times\Big\{\Big|\triangle^r_{\frac{1}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q+[s]_qe_i}{[n]_q}\Big)\Big|+\Big|\triangle^r_{\frac{1}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big|\Big\}\\&\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|,\end{aligned}\tag{11}$$

$$\begin{aligned}|\varphi_{e_i}^r(\mathbf{x})I_4|&\le\sum_{\substack{j=1\\j\ne i}}^d x_j([n-s]_q)\cdots([n-s-r+1]_q)\sum_{|\mathbf{k}|\le n-s-r}\Big|\triangle^r_{\frac{1}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q+[s]_qe_i}{[n]_q}\Big)\Big|P_{n-s-r,\mathbf{k}}(\mathbf{x})\\&\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|.\end{aligned}\tag{12}$$

因此, 由 (8)–(12) 式, 得

$$\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r M_{n,q}f\Big\|\le C\Big\|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^r f\Big\|,\quad i=1,2,\ldots,d.\tag{13}$$

采用文 [10, (3.8) 式] 的方法, 可得

$$B(n,\mathbf{k}):=\varphi_{e_i}^{-2}(\mathbf{x})\sum_{|\mathbf{k}|\le n-s-r}\big([k_i]_q(1-|\mathbf{x}|)-([n-|\mathbf{k}|-s]_q-r)x_i\big)^2P_{n-s-r,\mathbf{k}}(\mathbf{x})$$
$$\le([n]_q-[s]_q-r)\Big(1-\sum_{\substack{j=1\\ j\ne i}}^d x_j\Big).\tag{14}$$

若 $v=1$, 则由

$$\frac{\partial}{\partial x_i}P_{n,\mathbf{k}}(\mathbf{x})=\frac{k_i(1-|\mathbf{x}|)-(n-|\mathbf{k}|)x_i}{x_i(1-|\mathbf{x}|)}P_{n,\mathbf{k}}(\mathbf{x})\le\frac{[k_i]_q(1-|\mathbf{x}|)+[|\mathbf{k}|]_qx_i}{x_i(1-|\mathbf{x}|)}P_{n,\mathbf{k}}(\mathbf{x})$$

及 (14) 式, 得

$$|\varphi_{e_i}^{r+1}(\mathbf{x})I_1|\le C\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|(B(n,\mathbf{k}))^{\frac12}$$
$$\times\Big\{(x_i(1-|\mathbf{x}|))^r\sum_{|\mathbf{k}|\le n-s-r}P_{n-s-r,\mathbf{k}}(\mathbf{x})\Big(\frac{[n]_q}{[k_i]_q+1}\Big)^r\Big(\frac{[n]_q}{[n]_q-[|\mathbf{k}|]_q-[r+1]_q}\Big)^r\Big\}^{\frac12}$$
$$\le C[n]_q^{\frac12}\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|.\tag{15}$$

同理可得

$$|\varphi_{e_i}^{r+1}(\mathbf{x})I_2|\le C[n]_q^{\frac12}\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|,\quad|\varphi_{e_i}^{r+1}(\mathbf{x})I_4|\le C[n]_q^{\frac12}\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|.$$

此外

$$|\varphi_{e_i}^{r+1}(\mathbf{x})I_3|\le([n-s]_q)\cdots([n-s-r+1]_q)\sum_{|\mathbf{k}|\le n-s-r}\Big\{\Big|\triangle^r_{\frac{1}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q+[s]_qe_i}{[n]_q}\Big)\Big|$$
$$+\Big|\triangle^r_{\frac{1}{[n]_q}e_i}f\Big(\frac{[\mathbf{k}]_q}{[n]_q}\Big)\Big|\Big\}\Big(P_{n-s-r,\mathbf{k}}(\mathbf{x})+\Big|\frac{\partial}{\partial x_i}P_{n-s-r,\mathbf{k}}(\mathbf{x})\Big|\Big)$$
$$\le C\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|+C[n]_q^{\frac12}\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|\le C[n]_q^{\frac12}\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|.$$

因此

$$\Big|\Big|\varphi_{e_i}^{r+1}\Big(\frac{\partial}{\partial x_i}\Big)^{r+1}M_{n,q}f\Big|\Big|\le C[n]_q^{\frac12}\Big|\Big|\varphi_{e_i}^r\Big(\frac{\partial}{\partial x_i}\Big)^rf\Big|\Big|,\ i=1,2,\ldots,d.\tag{16}$$

综合 (13) 和 (16) 式, 引理 3 证毕.

引理 4 令 Π_m 为阶数不超过 m 的全体多项式空间. 若 $P\in\Pi_m$, $m\le\sqrt{n}$, 则

$$\Big|M_n(P,q;\mathbf{x})-P(\mathbf{x})-\frac{1}{2[n]_q}\Big(1+\frac{[s]_q([s]_q-1)}{[n]_q}\Big)\sum_{\xi\in V_S}\varphi_\xi^2(\mathbf{x})\Big(\frac{\partial}{\partial x_i}\Big)^2P(\mathbf{x})\Big|\le C[n]_q^{-2}m^4||P||.$$

证明 采用文 [9, 定理 5.1] 的方法即可证明引理 4, 此处略去细节.

对于 $f\in C(S)$, 定义另一 K- 泛函

$$K_{r,S}(f,t^r)=\inf_{g\in C^r(S)}\Big\{||f-g||+t^r\sup_{\xi\in V_S}\Big|\Big|\varphi_\xi^r\Big(\frac{\partial}{\partial x_i}\Big)^rg\Big|\Big|\Big\}.\tag{17}$$

利用 (17) 式, 可得下面的引理 5.

引理 5 令 $E_m(f)$ 为函数 f 的最佳逼近, $P_m\in\Pi_m$, $m=[\sqrt{[n]_q}]$ (“$[x]$” 表示 x 的取整函数), 使得 $||P_m-f||\le CE_m(f)$. 若 $f\in C(S)$, 则

$$\Big|\Big|M_{n,q}P_m-P_m-\frac{1}{2[n]_q}\Big(1+\frac{[s]_q([s]_q-1)}{[n]_q}\Big)P(D)P_m\Big|\Big|\le CK_{3,S}(f,[n]_q^{-\frac32}).\tag{18}$$

证明 选择 $P_j \in \Pi_j$, 使得 $||P_j - f|| \le CE_j(f)$. 利用

$$P_m = P_m - P_{2^l} + \sum_{j=1}^{l}(P_{2^j} - P_{2^{j-1}}) + P_1,\ l = \max\{j \,|\, 2^j < m\},\quad M_{n,q}P_1 - P_1 = P(D)P_1 = 0,$$

及引理 4, 可得

$$\begin{aligned} J(n) :=& \Big\| M_{n,q}P_m - P_m - \frac{1}{2[n]_q}\Big(1 + \frac{[s]_q([s]_q - 1)}{[n]_q}\Big)P(D)P_m \Big\| \\ \le& C[n]_q^{-2}\Big(m^4||P_m - P_{2^l}|| + \sum_{j=1}^{l} 2^{4j}||P_{2^j} - P_{2^{j-1}}||\Big) \\ \le& C[n]_q^{-2}\Big(m^4 E_{2^l}(f) + \sum_{j=1}^{l} 2^{4(j+1)}E_{2^j}(f)\Big) \\ \le& C[n]_q^{-2}\sum_{j=1}^{l} 2^{4(j+1)}E_{2^j}(f). \end{aligned}$$

另外, 采用文 [11, 定理 1.1] 及文 [9, (3.10) 式] 的方法, 得知

$$E_n(f) \le CK_{a,S}\Big(f, \frac{C_1}{[n]_q^a}\Big), \tag{19}$$

其中 C_1 是一固定的常数. 令 $a = 3$, 由 $m = [\sqrt{[n]_q}]$, $2^l \le m \le 2^{l+1}$, 得

$$\begin{aligned} J(n) \le& C[n]_q^{-2}\sum_{j=1}^{l} 2^{4(j+1)}K_{3,S}(f, 2^{-3j}) \le C[n]_q^{-2}\sum_{j=1}^{l} 2^{3l+j}K_{3,S}(f, 2^{-3l}) \\ \le& C[n]_q^{-2}2^{3l}K_{3,S}(f, 2^{-3l})\sum_{j=1}^{l} 2^j \le C[n]_q^{-2}2^{4l}K_{3,S}(f, 2^{-3l}) \\ \le& CK_{3,S}(f, [n]_q^{-\frac{3}{2}}). \end{aligned}$$

引理 5 证毕.

3 定理 1 的证明

(2) 的证明 由文 [12], 可知

$$||P(D)P_m|| \le Cm^2K^*(f, m^{-1}), \tag{20}$$

其中 $K^*(f,t) = \inf_{g\in C^2(S)}\{||f - g|| + t^2||P(D)g||\}$. 利用 (17)–(20) 式, 得

$$\begin{aligned} ||M_{n,q}f - f|| \le& \Big\| M_{n,q}P_m - P_m - \frac{1}{2[n]_q}\Big(1 + \frac{[s]_q([s]_q - 1)}{[n]_q}\Big)P(D)P_m \Big\| \\ &+ \Big\| \frac{1}{2[n]_q}\Big(1 + \frac{[s]_q([s]_q - 1)}{[n]_q}\Big)P(D)P_m \Big\| \\ \le& CK_{3,S}(f, [n]_q^{-\frac{3}{2}}) + CK^*\Big(f, \frac{1}{[n]_q}\Big(1 + \frac{[s]_q([s]_q - 1)}{[n]_q}\Big)^{\frac{1}{2}}\Big) + 2||f - P_m|| \\ \le& CK_{3,S}(f, [n]_q^{-\frac{3}{2}}) + CK^*(f, [n]_q^{-\frac{1}{2}}) + CE_m(f) \\ \le& C\big(K_{3,S}(f, [n]_q^{-\frac{3}{2}}) + K^*(f, [n]_q^{-\frac{1}{2}})\big) \le CK(f, [n]_q^{-\frac{1}{2}}). \end{aligned}$$

至此, (2) 证毕.

(3) 的证明 定义迭代算子 $M_{n,q}^k f = M_{n,q} M_{n,q}^{k-1} f,\ k \geq 1,\ M_{n,q}^0 f \equiv f$. 由 (17) 式, 得

$$K_{3,S}(f,t^4) \leq ||f - M_{k,q}^4 f|| + t^4 \sup_{\xi\in V_S} \Big|\Big|\varphi_\xi^4 \Big(\frac{\partial}{\partial\xi}\Big)^4 M_{k,q}^4 f\Big|\Big|,$$

$$||f - M_{k,q}^4 f|| = \Big|\Big| \sum_{j=0}^{3} (M_{k,q}^{j+1} f - M_{k,q}^j f)\Big|\Big| \leq \sum_{j=0}^{3} ||M_{k,q}^j (M_{k,q} f - f)|| \leq C||M_{k,q} f - f||.$$

反复利用引理 3, 得

$$\sup_{\xi\in V_S} \Big|\Big|\varphi_\xi^4 \Big(\frac{\partial}{\partial\xi}\Big)^4 M_{n,q}^4 f\Big|\Big| \leq C \sup_{\xi\in V_S} \Big|\Big|\varphi_\xi^4 \Big(\frac{\partial}{\partial\xi}\Big)^4 f\Big|\Big|,\quad \sup_{\xi\in V_S} \Big|\Big|\varphi_\xi^4 \Big(\frac{\partial}{\partial\xi}\Big)^4 M_{n,q}^4 f\Big|\Big| \leq C[n]_q^2 ||f||.$$

因此

$$K_{4,S}(f,t^4) \leq C\big(||M_{k,q} f - f|| + t^4 k^2 K_{4,S}(f,k^{-2})\big). \tag{21}$$

采用文 [13, 469 页] 或文 [14, 定理 9.3.6] 的方法, 并利用 (21) 式, 可得

$$K_{4,S}(f,t^4) \leq Ct^\rho \Big(\sum_{1\leq k\leq t^{-2}} k^{\frac{\rho}{2}-1} ||M_{k,q} f - f|| + ||f||\Big),\quad \rho\in(0,4).$$

由文献 [9, (3.16) 式], 可得

$$K_{r,S}(f,t^r) \leq C\Big\{t^r \sum_{1\leq k\leq t^{-1}} k^{r-1} K_{r+1,S}(f,k^{-r-1}) + t^r ||f||\Big\}.$$

因此

$$\begin{aligned} K_{3,S}(f,t^3) &\leq Ct^3 \Big(\sum_{1\leq k\leq t^{-1}} k^{2-\rho} \sum_{1\leq l\leq k^2} l^{\frac{\rho}{2}-1} ||M_{l,q} f - f|| + ||f||\Big) \\ &\leq Ct^3 \Big(\sum_{1\leq l\leq t^{-2}} l^{\frac{\rho}{2}-1} ||M_{l,q} f - f|| \sum_{k\geq\sqrt{l}} k^{2-\rho} + ||f||\Big). \end{aligned}$$

取 $3 < \rho < 4$, 则

$$K_{3,S}(f,t^3) \leq Ct^3 \Big(\sum_{1\leq l\leq t^{-2}} l^{\frac{1}{2}} ||M_{l,q} f - f|| + ||f||\Big). \tag{22}$$

对于给定的 $n\in N$, 选择 $n_0 \in N$ 满足 $\frac{[n]_q}{2} < n_0 < [n]_q$, 使得

$$||M_{n_0,q} f - f|| = \min_{\frac{[n]_q}{2} < k < [n]_q} ||M_{k,q} f - f||.$$

如果 $P_m \in \Pi_m$, $||P_m - f|| = E_m(f)$, $m = [\sqrt{n_0}]$, 那么

$$K(f,m^{-1}) \leq ||f - P_m|| + m^{-2} ||P(D)P_m|| + m^{-3} \sup_{\xi\in V_S} \Big|\Big|\varphi_\xi^3 \Big(\frac{\partial}{\partial\xi}\Big)^3 P_m\Big|\Big|. \tag{23}$$

由文 [11, (4.4) 式], 可得

$$m^{-3} \max_{\xi\in V_S} \Big|\Big|\varphi_\xi^3 \Big(\frac{\partial}{\partial\xi}\Big)^3 P_m\Big|\Big| \leq K_{3,S}(f,m^{-3}).$$

利用 (18) 和 (19) 式, 得

$$\begin{aligned} \Big(\frac{1}{2m}\Big)^2 ||P(D)P_m|| &\leq \frac{1}{m}\Big(1 + \frac{[s]_q([s]_q - 1)}{[n]_q}\Big) ||P(D)P_m|| \\ &\leq C\big(K_{3,S}(f,m^{-3}) + ||M_{n_0,q} P_m - P_m||\big) \\ &\leq C\big(K_{3,S}(f,m^{-3}) + 2E_m(f) + ||M_{n_0,q} f - f||\big) \\ &\leq C\big(K_{3,S}(f,m^{-3}) + ||M_{n_0,q} f - f||\big). \end{aligned}$$

在 (22) 式中取 $t=m^{-1}$, 并利用 (23) 式, 可得

$$K(f,[n]_q^{-\frac{1}{2}})\le K(f,m^{-1})\le C\bigg(n_0^{-\frac{3}{2}}\sum_{l=1}^{n_0}l^{\frac{1}{2}}||M_{l,q}f-f||+||M_{n_0,q}f-f||+n_0^{-\frac{3}{2}}\bigg).$$

因为

$$||M_{n_0,q}f-f||\le\frac{2}{[n]_q}\sum_{l=n/2}^{n}||M_{l,q}f-f||\le C[n]_q^{-\frac{3}{2}}\sum_{l=1}^{n}l^{\frac{1}{2}}||M_{l,q}f-f||,$$

所以

$$K(f,[n]_q^{\frac{1}{2}})\le C[n]_q^{-1}\sum_{l=1}^{n}l^{\frac{1}{2}}||M_{l,q}f-f||+C[n]_q^{-\frac{3}{2}}||f||.$$

利用 $\big(\frac{\partial}{\partial\xi}\big)^2P_1=0,\ K(f-P_1,t)=K(f,t),\ E_1(f)\le||M_{1,q}f-f||$, 可得

$$K(f,[n]_q^{-\frac{1}{2}})\le C[n]_q^{-\frac{3}{2}}\sum_{l=1}^{n}l^{\frac{1}{2}}||M_{n,q}f-f||.$$

(3) 证毕.

(4) 的证明 综合 (2) 和 (3) 立即可得 $||M_{n,q}f-f||=O([n]_q^{-\alpha})$ 当且仅当 $K(f,[n]_q^{-\frac{1}{2}})=O([n]_q^{-\alpha}),\ 0<\alpha<1$. (4) 证毕.

(5) 的证明 充分性是明显的, 下证必要性. 若 $||M_{n,q}f-f||=o([n]_q^{-\alpha})$, 则由 (3) 得知 $K(f,[n]_q^{-\frac{1}{2}})=o([n]_q^{-1})$. 这意味着存在 $f_\eta\in C^3(S)$, 使得

$$||f_\eta-f||\to0,\quad||P(D)f_\eta||\to0,\quad[n]_q^{-\frac{3}{2}}||\varphi_\xi^3\Big(\frac{\partial}{\partial\xi}\Big)^rf_\eta||\to0,\quad\eta\to0.$$

由 $E_n(f)\le CK_{a,S}\big(f,\frac{C_1}{[n]_q^a}\big)$, 可知 $E_n(f)=0$. 由此得 $f\in\Pi_1$. (5) 证毕.

参 考 文 献

[1] Kageyama Y., A new class of modified Bernstein operators, *J. Approx. Theory*, 1999, **101**(1): 121–147.

[2] Robert P. B., Generalized Bernstein polynomials and symmetric functions, *Adv. in Comp. Math.*, 2002, **28**: 17–39.

[3] Stancu D. D., in: Hämmerlin, G. (Ed.), Proc. Conf. Math. Res. Inst., *Oberwolfach*, 1981, 241–251.

[4] Gonska H., Meier J., Quantitative theorems on approximation by Bernstein-Stancu operators. *Calcolo*, 1984, **21**(4): 317–335.

[5] Walz G., Trigonometric Brzier and Stancu polynomials over intervals and triangles, *Computer Aided Geom. Design*, 1997, **14**: 393–397.

[6] Yang R. Y., Xiong J. Y., Cao F. L., Multivariate Stancu operators defined on a simplex, *Applied Math. Comput.*, 2003, **46**: 189–198.

[7] Cao F. L., Xu Z. B., Stancu polynomials defined on a simplex and best polynomial approximation, *Acta Mathematica Sinica, Chinese Series*, 2003, **46**(1): 189–196.

[8] Cao F. L., Multivariate Stancu polynomials and modulus of continuity, *Acta Mathematica Sinica, Chinese Series*, 2005, **48**(1): 51–62.

[9] Ditzian Z., Zhou X. L., Optimal approximation class for multivariate Bernstein operator, *Pacific J. Math.*, 1993, **158**(1): 93–120.

[10] Chen W., Ditzian Z., Ivanov K., Strong converse inequality for the Bernstein-Durrmeyer operator, *J. Approx. Theory*, 1993, **75**(1): 25–43.

[11] Ditzian Z., Polynomial approximation in $L_p(S)$ for $p>0$, *Constr. Approx.*, 1996, **12**(2): 241–269.

[12] Zhou X. L., Degree of approximation associated with some Elliptic operators and its applications, *Approx. Theory and Its Appl.*, 1995, **11**, 9–29.

[13] Totik V., An interpolation theorem and its application to positive operators, *Pacific J. Math.*, 1984, **111**: 447–481.

[14] Ditian Z., Totik V., Moduli of smoothness, Berlin, New York: Springer-Verlag, 1987.