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一类非线性偏积分微分方程二阶差分全离散格式*

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摘要 给出了数值求解一类非线性偏积分微分方程的二阶全离散差分格式. 采用了二阶向后差分格式，积分项的离散利用了 Lubich 的二阶卷积求积公式，给出了稳定性的证明、误差估计及收敛性的结果.

关键词 偏积分微分方程，分数次计算，卷积积分，差分格式，二阶全离散.

MR(2000) 主题分类号 65M60, 45K05, 65D32

1 引 言

我们将研究下面一类非线性偏积分微分方程数值解的有限差分格式

$$u_t(x,t)+u(x,t)u_x(x,t)-\int_0^t \beta(t-s)u_{xx}(x,s)\mathrm{d}s=f(x,t), \tag{1.1}$$

(其中核 $\beta(t)=\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}$, 在 $t=0$ 点是奇异的) $0<x<1,\ 0<t<T$, 满足如下边界条件

$$u(0,t)=u(1,t)=0,\quad 0\le t\le T \tag{1.2}$$

和初始条件

$$u(x,0)=v(x),\quad 0\le x\le 1. \tag{1.3}$$

比较方程 (1.1) 与著名的 Burgers 方程

$$u_t+uu_x=u_{xx}, \tag{1.4}$$

我们能更好的了解其性质. (1.4) 中时刻 t 弹性项的作用由 $u_{xx}(x,t)$ 给出，而 (1.1) 中时刻 t 值必须考虑整个历史 $u_{xx}(x,s)$, $0\le s\le t$, 因此在 (1.1) 中记忆积分项能被当作粘弹性力. 此

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类问题经常出现在非牛顿力的流体中. 从这种意义上来讲, (1.1) 提供了一个包含欧拉导数且带有粘弹性力的简单模型方程, 就像 Burgers 方程提供了一个研究现实情况下, 包含欧拉导数和弹性力的简单模型.

问题 (1.1)–(1.3) 常出现在带有粘弹性流体模型及带有记忆功能的热传导物质. 它的齐次方程曾被 Sanz–Serna[1] 研究, 它是介于标准热传导 (抛物) 方程和波动 (双曲) 方程之间的一类方程. 实际上, 积分算子 $I^{\frac{1}{2}}$ 将每一个 (局部可积的) 函数 $f(t)$ $(t>0)$ 映射为如下函数

$$(I^{\frac{1}{2}}f)(t)=\int_0^t (t-s)^{-\frac{1}{2}}f(s)\mathrm{d}s, \tag{1.5}$$

满足下列性质 [1]

$$(I^{\frac{1}{2}}(I^{\frac{1}{2}}f))(t)=\pi\int_0^t f(s)\mathrm{d}s. \tag{1.6}$$

因此 $\pi^{-\frac{1}{2}}I^{\frac{1}{2}}$ 能看成不定积分算子的平方根, 通过运用分数次计算的理论 [2], 我们能定义微分算子 $D=\frac{\mathrm{d}}{\mathrm{d}t}$ 的平方根 $D^{\frac{1}{2}}$

$$D^{\frac{1}{2}}D^{\frac{1}{2}}\int_0^t f(s)\mathrm{d}s=f(t), \quad \pi^{-\frac{1}{2}}D^{\frac{1}{2}}I^{\frac{1}{2}}=\text{恒等算子}.$$

在 (1.1) 的齐次方程两边运用 $D^{\frac{1}{2}}$ 可得

$$D^{\frac{1}{2}}Du=\pi^{\frac{1}{2}}u_{xx}, \tag{1.7}$$

因此方程 (1.1) 的齐次方程可被看作介于我们熟悉的方程 $Du=au_{xx}$ 与 $D^2u=bu_{xx}$ (a,b 为正常数) 之间的一类方程.

近年来, 国内外有很多人研究了这类方程. 其齐次方程先后有陈传淼、 Thomée 和 Wahlbin[3] 采用向后 Euler 格式, 空间方向采用线性有限元, 积分项通过内积求积技巧进行离散, 得到解的正则性条件及误差估计. Mclean, Thomée[4] 使用了 Euler 和二阶向后差分格式, 空间方向用 Galerkin 有限元方法, 并给出了问题 (1.1)–(1.3) 的正则性估计. Sanz–Serna[1] 也研究了这类问题, 在时间方向, 他采用了向后 Euler 格式和一阶卷积求积逼近积分项, 对光滑与非光滑的初始值导出了相应的误差估计. 徐大 [5] 考虑了 Euler 和 Crank-Nicolson 格式和一阶、二阶卷积求积, 得到了带权的误差估计. López-Marcos[6] 研究了此类非线性的积分微分方程, 采用了一阶时间全离散差分格式. 由于时间离散必须保留前面所有的值, 故要求大量的内存, 为了克服这些困难, 黄元清 [7] 提出了一种迭代格式, 减少了大量的计算和内存要求. Sloan, Thomée[8] 建议减少求积区间, 使用高阶的求积公式.

López-Marcos[6] 研究了此类非线性的积分微分方程, 时间方向使用向后 Euler 格式、空间方向使用一阶卷积求积公式. 本文中是文 [6] 的延伸, 本文我们运用二阶向后差分格式进行时间离散, 空间方向采用有限二阶差分格式, 对积分项采用二阶卷积求积, 在非线性项处理上模拟 López-Marcos[6] 的技术. 由于方程的解在 $t=0$ 处不光滑, 导致误差估计在整个过程中都不能达到时间的二阶精度, 但其所得的结果相对于 [6] 有了明显的提高, 而且使用的证明方法与文 [6] 完全不同.

全文中, 我们假设 (见文 [6] 中的 (1.7)), 对 $0\le t\le T,\ 0\le x\le 1$,

$$\begin{aligned} &|u_{tt}(x,t)|\le ct^{-\frac{1}{2}}, \quad &&|u_{ttt}(x,t)|\le ct^{-\frac{3}{2}},\\ &|u_{xxt}(x,0)|\le c, &&|u_{xxtt}(x,t)|\le ct^{-\frac{1}{2}}, \end{aligned} \tag{1.8}$$

注 1　对充分光滑 $v(x)$ 及 $f(x,t)$, (1.1)–(1.3) 存在唯一的解，并且满足下面的正则性 [1]

$$u\in C([0,T];H^2\bigcap H_0^1),\quad u_t\in C([0,T];L_2)\bigcap L_1([0,T];H^2\bigcap H_0^1),\quad u_{tt}\in L_1([0,T];L_2).$$

注 2　此类齐次问题的正则性估计已被证明 [4], 它由半模的形式给出

$$|v|_r=\|A^{\frac{r}{2}}v\|,\quad r\in R,\ \text{其中}\ A=-\frac{\partial^2}{\partial x^2},$$

$$|u(x,t)|_{r+2\theta}\le c(\alpha)t^{-(\alpha+1)\theta}|v|_r,\quad t\ge 0,\ 0\le\theta\le 1. \tag{1.9}$$

类似地，时间导数满足

$$|D_t^m u(x,t)|_{r+2\theta}\le c(m,\alpha)t^{-(\alpha+1)\theta-m}|v|_r,\quad t\ge 0,\ -1\le\theta\le 1. \tag{1.10}$$

如果在 (1.9), (1.10) 中，选取适当的 θ 和 r, 我们能得到如下正则性估计 ($\|\cdot\|_0$ 为连续的 L^2 模)(见文 [5] 中的 (7.12)): 对 $0\le t\le T,\ 0\le x\le 1$,

$$\begin{aligned}&\|\ u_{tt}(x,t)\ \|_0\le ct^{-\frac{1}{2}},\quad &&\|\ u_{ttt}(x,t)\ \|_0\le ct^{-\frac{3}{2}},\\ &\|\ u_{xxt}(x,0)\ \|_0\le c,\quad &&\|\ u_{xxtt}(x,t)\ \|_0\le ct^{-\frac{1}{2}}.\end{aligned} \tag{1.11}$$

全文结构按如下安排：第 2 节，我们介绍数值格式；第 3 节讨论数值格式的稳定性；第 4 节给出误差估计.

2　数值格式

我们给出如下网格 $x_j=jh,\ j=0,1,\cdots,J$, 其中 $h=\frac{1}{J}$(J 是正整数), 时间步长记为 k, 给出时间的一个划分 $t_n=nk,\ n=0,1,\cdots,N,\ (N=[\frac{T}{k}])$, 记 U_j^n 近似 $u(x_j,t_n)$, 定义二阶向后差分为

$$\overline{D}_t^{(2)}U_j^n=k^{-1}\Big(\frac{3}{2}U_j^n-2U_j^{n-1}+\frac{1}{2}U_j^{n-2}\Big). \tag{2.1}$$

我们介绍以下二阶积分近似，运用二阶卷积积分公式 (见文 [2, 5])

$$\int_0^{t_n}\beta(t_n-s)\varphi(s)\mathrm{d}s\approx q_n(\varphi)=k^{\frac{1}{2}}\sum_{p=0}^{n}\beta_p\varphi^{n-p}+w_{n0}\varphi^0, \tag{2.2}$$

其中 β_p 是下列级数的系数

$$\widehat{\beta}\Big(\frac{(1-z)(3-z)}{2}\Big)=\Big(\frac{(1-z)(3-z)}{2}\Big)^{-\frac{1}{2}}=\sum_{p=0}^{\infty}\beta_p z^p, \tag{2.3}$$

采用校正积分权 w_{n0} 以保证积分能达到二阶精度，因此积分公式 (2.2) 对多项式 $P(x)=1$ 准确成立，即

$$k^{\frac{1}{2}}\sum_{p=0}^{n}\beta_p+w_{n0}(k)=\int_0^{t_n}\beta(t_n-s)\mathrm{d}s=2\Big(\frac{t_n}{\pi}\Big)^{\frac{1}{2}}. \tag{2.4}$$

定义以下差分记号

$$\delta^2 V_j = V_{j+1} - 2V_j + V_{j-1}, \quad \varDelta V_j = V_{j+1} - V_{j-1},$$

则 (1.1)–(1.3) 的离散格式可以写成如下形式

当 $n \geq 2$ 时

$$k^{-1}\left(\frac{3}{2}U_j^n - 2U_j^{n-1} + \frac{1}{2}U_j^{n-2}\right) + \frac{U_{j+1}^n + U_j^n + U_{j-1}^n}{3}\frac{\varDelta U_j^n}{2h} - q_n\left(\frac{1}{h^2}\delta^2 U_j\right) = f_j^n,$$

即

$$\begin{aligned}&k^{-1}\Big(\frac{3}{2}U_j^n - 2U_j^{n-1} + \frac{1}{2}U_j^{n-2}\Big) + \frac{U_{j+1}^n + U_j^n + U_{j-1}^n}{3}\frac{U_{j+1}^n - U_{j-1}^n}{2h}\\ &-\Big(\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^{n}\beta_p\delta^2 U_j^{n-p} + \frac{\omega_{n0}}{h^2}\delta^2 U_j^0\Big) = f_j^n, \quad j = 1, 2, \cdots, (J-1),\ n = 2, 3, \cdots, N;\end{aligned} \tag{2.5 a}$$

当 $n = 1$ 时

$$k^{-1}(U_j^1 - U_j^0) + \frac{U_{j+1}^1 + U_j^1 + U_{j-1}^1}{3}\frac{\varDelta U_j^1}{2h} - q_1\Big(\frac{1}{h^2}\delta^2 U_j\Big) = f_j^1,$$

即

$$\begin{aligned}&k^{-1}(U_j^1 - U_j^0) + \frac{U_{j+1}^1 + U_j^1 + U_{j-1}^1}{3}\frac{U_{j+1}^1 - U_{j-1}^1}{2h}\\ &-\Big(\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\delta^2 U_j^{1-p} + \frac{\omega_{10}}{h^2}\delta^2 U_j^0\Big) = f_j^1, \quad j = 1, 2, \cdots, (J-1),\end{aligned} \tag{2.5 b}$$

其中

$$U_0^n = U_J^n = 0, \quad n = 0, 1, \cdots, N. \tag{2.6}$$

$$U^0 = (U_1^0, U_2^0, \cdots, U_{J-1}^0). \tag{2.7}$$

注意到对于 uu_x 项的离散, 我们使用了熟悉的基于分片线性检验函数的 Galerkin 方法, 众所周知它要优于其它的选择, 如 $\frac{1}{2h}U_j\varDelta U_j$ 或 $\frac{1}{4h}\varDelta(U_j)^2$, 它经常用下面这种记号代替

$$(U_{j+1}^n + U_j^n + U_{j-1}^n)\varDelta U_j^n = U_j^n\varDelta U_j^n + \varDelta(U_j^n)^2. \tag{2.8}$$

为了后面的需要, 我们收集了差分计算的一些记号和结论. 我们将用记号 $U^n(n = 0, 1, \cdots, N)$ 表示 $\mathbb{R}^{J-1}$ 中的向量, 即 U^n 表示向量 $(U_1^n, U_2^n, \cdots, U_{J-1}^n)$. 在我们的分析过程中, 如遇到 U_0 和 U_J, 我们定义 $U_0 = U_J = 0$. 如 $(V_1, V_2, \cdots, V_{J-1})$, $(W_1, W_2, \cdots, W_{J-1})$ 为 $\mathbb{R}^{J-1}$ 中的向量, 则我们规定

$$\begin{aligned}&\varDelta_+ V_j = V_{j+1} - V_j, \quad \varDelta_- V_j = V_j - V_{j-1}, \quad T_+ V_j = V_{j+1},\\ &T_- V_j = V_{j-1}, \quad (VW)_j = V_j W_j, \quad 1 \leq j \leq (J-1),\\ &\|V\|_\infty = \max_{1\leq j\leq (J-1)}|V_j|, \quad \langle V, W\rangle = \sum_{j=1}^{J-1} hV_j W_j, \quad \|V\|^2 = \langle V, V\rangle.\end{aligned} \tag{2.9}$$

容易验证下面的等式成立 (见文 [6] 中的 (2.6)–(2.9))

$$\langle V, \varDelta W\rangle = -\langle \varDelta V, W\rangle, \tag{2.10}$$

$$\langle \delta^2 V, W\rangle = -\sum_{j=0}^{J-1} h(\varDelta_+ V_j)(\varDelta_+ W_j), \tag{2.11}$$

$$\langle V\varDelta V + \varDelta (V)^2, V\rangle = 0, \tag{2.12}$$

$$\langle \varDelta (VW), V\rangle = \frac{1}{2}\langle T_+ V\varDelta_+ W + T_- V\varDelta_- W, V\rangle. \tag{2.13}$$

3　稳定性

这一节我们将给出离散格式 (2.5)–(2.7) 的稳定性.

首先由 Cauchy 不等式我们很容易得到下述引理.

引理 3.1　当 $U_0 = U_J = 0$ 时，我们有

1) 当 $n \geq 1$ 时,

$$\langle \delta^2 U^n, U^n\rangle \leq 0.$$

2) 当 $0 \leq n \leq N,\ 0 \leq m \leq N$ 时,

$$| \langle \delta^2 U^m, U^n\rangle | \leq 4 \parallel U^m \parallel \parallel U^n \parallel .$$

证

1)

$$\begin{aligned}
\langle \delta^2 U^n, U^n\rangle &= \sum_{j=1}^{J-1} h\delta^2 U_j^n U_j^n = \sum_{j=1}^{J-1} h(U_{j+1}^n - 2U_j^n + U_{j-1}^n)U_j^n \\
&= \sum_{j=1}^{J-1} hU_{j+1}^n U_j^n + \sum_{j=1}^{J-1} hU_{j-1}^n U_j^n - 2\|U^n\|^2 \\
&\leq \sum_{j=1}^{J-1} h\left((U_{j+1}^n)^2 + \frac{(U_j^n)^2}{2}\right) + \sum_{j=1}^{J-1} h\left((U_{j-1}^n)^2 + \frac{(U_j^n)^2}{2}\right) - 2 \parallel U^n \parallel^2 \\
&\leq 2 \parallel U^n \parallel^2 - 2 \parallel U^n \parallel^2 = 0.
\end{aligned}$$

2)

$$\begin{aligned}
\left(\left|\sum_{j=1}^{J-1} hU_{j+1}^m U_j^n\right|\right)^2 &\leq \left(\sum_{j=1}^{J-1} h\left|U_{j+1}^m\right|\left|U_j^n\right|\right)^2 \leq \left(\sum_{j=1}^{J-1} h(U_{j+1}^m)^2\right)\left(\sum_{j=1}^{J-1} h(U_j^n)^2\right) \\
&\leq \parallel U^m \parallel^2 \parallel U^n \parallel^2,
\end{aligned}$$

同理可得

$$\left(\left|\sum_{j=1}^{J-1} hU_{j-1}^m U_j^n\right|\right)^2 \leq \parallel U^m \parallel^2 \parallel U^n \parallel^2,$$

故

$$\begin{aligned}|\langle\delta^2U^m,U^n\rangle| &= \Big|\sum_{j=1}^{J-1}h\delta^2U_j^mU_j^n\Big| = \Big|\sum_{j=1}^{J-1}h(U_{j+1}^m-2U_j^m+U_{j-1}^m)U_j^n\Big|\\ &\le \Big|\sum_{j=1}^{J-1}hU_{j+1}^mU_j^n\Big|+\Big|\sum_{j=1}^{J-1}hU_{j-1}^mU_j^n\Big|+2\parallel U^m\parallel\parallel U^n\parallel\\ &\le 4\parallel U^m\parallel\parallel U^n\parallel.\end{aligned}$$

证毕.

引理 3.2 如果实数序列 $\{a_0,a_1,\cdots,a_n,\cdots\}$ 满足 $\widehat{a}(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ 在开球域 $D=\{z\in\mathcal{C}:|z|<1\}$ 解析的. 则对任意正整数 N, 任意序列 $(V^0,V^1,V^2,\cdots,V^N)\in\mathbb{R}^{N+1}$ 有

$$\sum_{n=0}^{N}\Big(\sum_{p=0}^{n}a_pV^{n-p}\Big)V^n\ge 0,$$

当且仅当

$$\mathrm{Re}\widehat{a}(z)\ge 0,\quad 对\ z\in D. \tag{3.1}$$

证明见文 [5] 或文 [8] 中的命题.

容易验证 $\beta(t)$ 为正类型核，即 $\mathrm{Re}\widehat{\beta}(s)\ge 0$, 对于 $\mathrm{Re}s>0$. 当 $|z|<1$ 时，我们有 $\mathrm{Re}(\frac{3}{2}-2z+\frac{1}{2}z^2)>0$ (证明见文 [9] 中的引理 4.1), 因此，我们得到

$$\mathrm{Re}\widehat{\beta}\Big(\frac{(1-z)(3-z)}{2}\Big)=\mathrm{Re}\widehat{\beta}\Big(\frac{3}{2}-2z+\frac{1}{2}z^2\Big)\ge 0,\quad 对\ |z|<1,$$

发生函数 (2.3) 满足引理 3.2 的条件 (3.1).

下面给出格式的稳定性.

定理 3.3 如果 U_j^n 按照 (2.5)–(2.7) 定义，当 $k=O(h^2)$ 时，那么对于 $N\ge 1$

$$\parallel U^N\parallel\le C(T)\parallel U^0\parallel+3\sum_{n=1}^{N}k\parallel f^n\parallel. \tag{3.2}$$

证 当 $m=1,2$ 时，记 $\varDelta_mU_j^n=U_j^n-U_j^{n-m}$. 由定义，我们很容易验证

$$\begin{aligned}2\langle\varDelta_mU^n,U^n\rangle &= 2\sum_{j=1}^{J-1}h(U_j^n-U_j^{n-m})U_j^n\\ &= \sum_{j=1}^{J-1}h((U_j^n)^2-(U_j^{n-m})^2+(U_j^n-U_j^{n-m})^2)\\ &= \varDelta_m\parallel U^n\parallel^2+\parallel\varDelta_mU^n\parallel^2,\end{aligned}$$

$$\frac{3}{2}U_j^n-2U_j^{n-1}+\frac{1}{2}U_j^{n-2}=2\varDelta_1U_j^n-\frac{1}{2}\varDelta_2U_j^n. \tag{3.3}$$

故

$$
\begin{aligned}
&k\Big\langle k^{-1}\Big(\frac{3}{2}U^n-2U^{n-1}+\frac{1}{2}U^{n-2}\Big),U^n\Big\rangle\\
=&\Big\langle 2\varDelta_1U^n-\frac{1}{2}\varDelta_2U^n,U^n\Big\rangle\\
=&2\langle\varDelta_1U^n,U^n\rangle-\frac{1}{2}\langle\varDelta_2U^n,U^n\rangle\\
=&\varDelta_1\parallel U^n\parallel^2+\parallel\varDelta_1U^n\parallel^2-\frac{1}{4}(\varDelta_2\parallel U^n\parallel^2+\parallel\varDelta_2U^n\parallel^2)\\
=&\varDelta_1\parallel U^n\parallel^2-\frac{1}{4}\varDelta_2\parallel U^n\parallel^2+\parallel\varDelta_1U^n\parallel^2-\frac{1}{4}\parallel\varDelta_2U^n\parallel^2.
\end{aligned}
\tag{3.4}
$$

由 (3.4), 对 $2\le n\le N$ 求和得

$$
\begin{aligned}
&\sum_{n=2}^{N}(\varDelta_1\parallel U^n\parallel^2-\frac{1}{4}\varDelta_2\parallel U^n\parallel^2)\\
=&\sum_{n=2}^{N}\Big(\parallel U^n\parallel^2-\parallel U^{n-1}\parallel^2-\frac{1}{4}\parallel U^n\parallel^2+\frac{1}{4}\parallel U^{n-2}\parallel^2\Big)\\
=&\sum_{n=2}^{N}\Big(\frac{3}{4}\parallel U^n\parallel^2-\parallel U^{n-1}\parallel^2+\frac{1}{4}\parallel U^{n-2}\parallel^2\Big)\\
=&\frac{3}{4}\parallel U^N\parallel^2-\frac{1}{4}\parallel U^{N-1}\parallel^2-\frac{3}{4}\parallel U^1\parallel^2+\frac{1}{4}\parallel U^0\parallel^2,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=2}^{N}\Big(\parallel\varDelta_1U^n\parallel^2-\frac{1}{4}\parallel\varDelta_2U^n\parallel^2\Big)\\
=&\sum_{n=2}^{N}\Big(\parallel\varDelta_1U^n\parallel^2-\frac{1}{4}\parallel\varDelta_1U^n+\varDelta_1U^{n-1}\parallel^2\Big)\\
\ge&\sum_{n=2}^{N}\Big(\parallel\varDelta_1U^n\parallel^2-\frac{1}{4}(\parallel\varDelta_1U^n\parallel+\parallel\varDelta_1U^{n-1}\parallel)^2\Big)\\
\ge&\sum_{n=2}^{N}\Big(\parallel\varDelta_1U^n\parallel^2-\frac{1}{2}(\parallel\varDelta_1U^n\parallel^2+\parallel\varDelta_1U^{n-1}\parallel^2)\Big)\\
\ge&\frac{1}{2}\sum_{n=2}^{N}(\parallel\varDelta_1U^n\parallel^2-\parallel\varDelta_1U^{n-1}\parallel^2)\\
=&\frac{1}{2}(\parallel\varDelta_1U^N\parallel^2-\parallel\varDelta_1U^1\parallel^2).
\end{aligned}
\tag{3.5}
$$

因此对 $2\le n\le N$ 有

$$
\begin{aligned}
&k\langle k^{-1}(U^1-U^0),U^1\rangle+k\sum_{n=2}^{N}\Big\langle k^{-1}\Big(\frac{3}{2}U^n-2U^{n-1}+\frac{1}{2}U^{n-2}\Big),U^n\Big\rangle\\
\ge&\frac{1}{2}(\parallel U^1\parallel^2-\parallel U^0\parallel^2+\parallel\varDelta_1U^1\parallel^2)+\frac{1}{2}(\parallel\varDelta_1U^N\parallel^2-\parallel\varDelta_1U^1\parallel^2)\\
&+\Big(\frac{3}{4}\parallel U^N\parallel^2-\frac{1}{4}\parallel U^{N-1}\parallel^2-\frac{3}{4}\parallel U^1\parallel^2+\frac{1}{4}\parallel U^0\parallel^2\Big)\\
\ge&\frac{3}{4}\parallel U^N\parallel^2-\frac{1}{4}\parallel U^{N-1}\parallel^2-\frac{1}{4}\parallel U^1\parallel^2-\frac{1}{4}\parallel U^0\parallel^2.
\end{aligned}
\tag{3.6}
$$

当 $n\geq 2$ 时，在 (2.5 a) 的两边同时乘以 hU_j^n, 并对 j $(1\leq j\leq (J-1))$ 求和，对于非线性项使用 (2.8), 可得

$$\begin{aligned}&\Big\langle \frac{3}{2}U^n-2U^{n-1}+\frac{1}{2}U^{n-2},U^n\Big\rangle+\Big\langle U^n\varDelta U^n+\varDelta(U^n)^2,U^n\Big\rangle\\&\quad-\frac{k^{\frac{3}{2}}}{h^2}\sum_{p=0}^{n}\beta_p\langle\delta^2U^{n-p},U^n\rangle-\frac{k}{h^2}\omega_{n0}\langle\delta^2U^0,U^n\rangle\\&=k\langle f^n,U^n\rangle.\end{aligned}\tag{3.7}$$

当 $n=1$ 时，在式 (2.5 b) 的两边同时乘以 hU_j^1, 并对 $j(1\leq j\leq (J-1))$ 求和，同样对于非线性项使用 (2.8), 可得

$$\begin{aligned}&\langle U^1-U^0,U^1\rangle+\langle U^1\varDelta U^1+\varDelta(U^1)^2,U^1\rangle\\&\quad-\frac{k^{\frac{3}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\langle\delta^2U^{1-p},U^1\rangle-\frac{k}{h^2}\omega_{10}\langle\delta^2U^0,U^1\rangle\\&=k\langle f^1,U^1\rangle.\end{aligned}\tag{3.8}$$

故当 $N\geq 2$ 时，由 (3.7), (3.8), 利用 (2.12) 可得

$$\begin{aligned}&\langle U^1-U^0,U^1\rangle+\sum_{n=2}^{N}\Big\langle\frac{3}{2}U^n-2U^{n-1}+\frac{1}{2}U^{n-2},U^n\Big\rangle\\&=\frac{k^{\frac{3}{2}}}{h^2}\sum_{n=2}^{N}\sum_{p=0}^{n}\beta_p\langle\delta^2U^{n-p},U^n\rangle+\frac{k}{h^2}\sum_{n=2}^{N}\omega_{n0}\langle\delta^2U^0,U^n\rangle\\&\quad+\frac{k^{\frac{3}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\langle\delta^2U^{1-p},U^1\rangle+\frac{k}{h^2}\omega_{10}\langle\delta^2U^0,U^1\rangle+\sum_{n=1}^{N}k\langle f^n,U^n\rangle\\&=\frac{k^{\frac{3}{2}}}{h^2}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\langle\delta^2U^{n-p},U^n\rangle+\frac{k}{h^2}\sum_{n=1}^{N}\omega_{n0}\langle\delta^2U^0,U^n\rangle+\sum_{n=1}^{N}k\langle f^n,U^n\rangle.\end{aligned}$$

由 (2.11), 先交换求和次序，再对每个固定的 j, 由引理 3.2 可将上式等号的右边第一项化为

$$\begin{aligned}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\langle\delta^2U^{n-p},U^n\rangle&=-\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\sum_{j=0}^{J-1}h\varDelta_+U_j^{n-p}\varDelta_+U_j^n\\&=-h\sum_{j=0}^{J-1}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\varDelta_+U_j^{n-p}\varDelta_+U_j^n\\&=-h\sum_{j=0}^{J-1}\Big(\sum_{n=0}^{N}\sum_{p=0}^{n}\beta_p\varDelta_+U_j^{n-p}\varDelta_+U_j^n-\beta_0(\varDelta_+U_j^0)^2\Big)\\&\leq\beta_0\sum_{j=0}^{J-1}h(\varDelta_+U_j^0)^2\leq 2\beta_0\parallel U^0\parallel^2.\end{aligned}\tag{3.9}$$

因此由 (3.6) 及引理 3.1 中 2) 和上式可得

$$\begin{aligned}
&\frac{3}{4}\parallel U^N\parallel^2-\frac{1}{4}\parallel U^{N-1}\parallel^2-\frac{1}{4}\parallel U^1\parallel^2-\frac{1}{4}\parallel U^0\parallel^2\\
\leq&\frac{k^{\frac{3}{2}}}{h^2}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\langle\delta^2U^{n-p},U^n\rangle+\frac{k}{h^2}\sum_{n=1}^{N}\omega_{n0}\langle\delta^2U^0,U^n\rangle+\sum_{n=1}^{N}k\langle f^n,U^n\rangle\\
\leq&\frac{2k^{\frac{3}{2}}}{h^2}\beta_0\parallel U^0\parallel^2+\frac{k}{h^2}\sum_{n=1}^{N}|\omega_{n0}||\langle\delta^2U^0,U^n\rangle|+\sum_{n=1}^{N}k|\langle f^n,U^n\rangle|\\
\leq&\frac{2k^{\frac{3}{2}}}{h^2}\beta_0\parallel U^0\parallel^2+\frac{4k}{h^2}\sum_{n=1}^{N}|\omega_{n0}|\parallel U^0\parallel\parallel U^n\parallel+\sum_{n=1}^{N}k\parallel f^n\parallel\parallel U^n\parallel,
\end{aligned}$$

整理得

$$\begin{aligned}
\parallel U^N\parallel^2\leq&\frac{1}{3}(\parallel U^{N-1}\parallel^2+\parallel U^1\parallel^2+\parallel U^0\parallel^2)\\
&+\frac{8k^{\frac{3}{2}}}{3h^2}\beta_0\parallel U^0\parallel^2+\frac{16k}{3h^2}\sum_{n=1}^{N}|\omega_{n0}|\parallel U^0\parallel\parallel U^n\parallel+\frac{4}{3}\sum_{n=1}^{N}k\parallel f^n\parallel\parallel U^n\parallel.
\end{aligned}$$

假设 $\parallel U^M\parallel=\max\limits_{0\leq n\leq N}\parallel U^n\parallel$, 则

$$\begin{aligned}
\parallel U^M\parallel^2\leq&\frac{1}{3}(\parallel U^{M-1}\parallel^2+\parallel U^1\parallel^2+\parallel U^0\parallel^2)+\frac{8k^{\frac{3}{2}}}{3h^2}\beta_0\parallel U^0\parallel\parallel U^M\parallel\\
&+\frac{16k}{3h^2}\sum_{n=1}^{M}|\omega_{n0}|\parallel U^0\parallel\parallel U^M\parallel+\frac{4}{3}\sum_{n=1}^{M}k\parallel f^n\parallel\parallel U^M\parallel\\
\leq&\frac{1}{3}(\parallel U^M\parallel^2+\parallel U^1\parallel^2+\parallel U^0\parallel^2)+\frac{8k^{\frac{3}{2}}}{3h^2}\beta_0\parallel U^0\parallel\parallel U^M\parallel\\
&+\frac{16k}{3h^2}\sum_{n=1}^{N}|\omega_{n0}|\parallel U^0\parallel\parallel U^M\parallel+\frac{4}{3}\sum_{n=1}^{N}k\parallel f^n\parallel\parallel U^M\parallel.
\end{aligned}$$

进一步得到

$$\begin{aligned}
\parallel U^M\parallel^2\leq&\frac{1}{2}(\parallel U^1\parallel^2+\parallel U^0\parallel^2)+\frac{4k^{\frac{3}{2}}}{h^2}\beta_0\parallel U^0\parallel\parallel U^M\parallel\\
&+\frac{8k}{h^2}\sum_{n=1}^{N}|\omega_{n0}|\parallel U^0\parallel\parallel U^M\parallel+2\sum_{n=1}^{N}k\parallel f^n\parallel\parallel U^M\parallel\\
\leq&\frac{1}{2}(\parallel U^1\parallel+\parallel U^0\parallel)\parallel U^M\parallel+\frac{4k^{\frac{3}{2}}}{h^2}\beta_0\parallel U^0\parallel\parallel U^M\parallel\\
&+\frac{8k}{h^2}\sum_{n=1}^{N}|\omega_{n0}|\parallel U^0\parallel\parallel U^M\parallel+2\sum_{n=1}^{N}k\parallel f^n\parallel\parallel U^M\parallel.
\end{aligned}\tag{3.10}$$

从而得到当 $N \geq 2$ 时

$$
\begin{aligned}
\| U^N \| \leq \| U^M \| &\leq \frac{1}{2}(\| U^1 \| + \| U^0 \|) + \frac{4k^{\frac{3}{2}}}{h^2}\beta_0 \| U^0 \| \\
&+\frac{8k}{h^2}\sum_{n=1}^{N}|\omega_{n0}| \| U^0 \| +2\sum_{n=1}^{N} k \| f^n \| \\
&\leq \frac{1}{2} \| U^1 \| + \left(\frac{1}{2} + \frac{4k^{\frac{3}{2}}}{h^2}\beta_0 + \frac{8k}{h^2}\sum_{n=1}^{N}|\omega_{n0}|\right) \| U^0 \| +2\sum_{n=1}^{N} k \| f^n \| .
\end{aligned} \tag{3.11}
$$

当 $N=1$ 时，由 (3.8), 利用 (2.12) 可得

$$
\begin{aligned}
\| U^1 \|^2 &= \langle U^0, U^1\rangle + \frac{k^{\frac{3}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\langle \delta^2 U^{1-p}, U^1\rangle + \frac{k}{h^2}\omega_{10}\langle \delta^2 U^0, U^1\rangle + k\langle f^1, U^1\rangle \\
&\leq \| U^0 \| \| U^1 \| +4\left(\frac{k^{\frac{3}{2}}}{h^2}\beta_1 + \frac{k}{h^2}|\omega_{10}|\right) \| U^0 \| \| U^1 \| +k \| f^1 \| \| U^1 \| .
\end{aligned}
$$

上述不等式是由基本不等式及引理 3.1 中 1) $\langle \delta^2 U^1, U^1\rangle \leq 0$ 得. 因此

$$
\| U^1 \| \leq \| U^0 \| +4\left(\frac{k^{\frac{3}{2}}}{h^2}\beta_1 + \frac{k}{h^2}|\omega_{10}|\right) \| U^0 \| +k \| f^1 \| . \tag{3.12}
$$

由于 $w_{n0} = O(k^{\frac{1}{2}}n^{-\frac{1}{2}})$ $(1 \leq n \leq N)$(见文 [2] 中的定理 2.4.(1)), 可得

$$
\sum_{n=1}^{N}|w_{n0}| \leq C\sum_{n=1}^{N}(k^{\frac{1}{2}}n^{-\frac{1}{2}}) \leq C(Nk)^{\frac{1}{2}} \leq C(T). \tag{3.13}
$$

由 (3.1), (3.12) 及 (3.13) 即得定理 3.3 成立. 证毕.

4 误差估计

这一节我们将研究离散格式 (2.5) – (2.7) 的误差估计.

首先我们给出 $\varepsilon(u_{xx})(t_n) = q_n(u_{xx}) - I^{\frac{1}{2}}(u_{xx})(t_n)$ 的界，其中 $q_n(\varphi)$ 按 (2.2) 定义.

引理 4.1 如果 $\beta(t) = \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}$, 则对 $n \geq 1$ 有

$$
\begin{aligned}
|\varepsilon(\varphi)(t_n)| \leq Ck^2 t_n^{-\frac{1}{2}}|\varphi_t(0)| + Ck^{\frac{3}{2}}\int_{t_{n-1}}^{t_n}|\varphi_{tt}(s)|\mathrm{d}s \\
+Ck^2\int_0^{t_{n-1}}(t_n-s)^{-\frac{1}{2}}|\varphi_{tt}(s)|\mathrm{d}s.
\end{aligned} \tag{4.1}
$$

证 见文 [5] 中的引理 7.2.

引理 4.2 若 u_{xx} 是 $0 \leq t \leq T$ 上实的、连续可微的函数，且 u_{xxtt} 在 $0 < t < T$ 上连续可积，则存在一个正常数 C 仅仅依赖于 T, 满足

$$
|\varepsilon(u_{xx})(t_n)| = |q_n(u_{xx}) - I^{\frac{1}{2}}(u_{xx})(t_n)| \leq Ck\left(\frac{k}{n}\right)^{\frac{1}{2}}, \quad 1 \leq n \leq N. \tag{4.2}
$$

证　从假设 (1.8), 我们得到

$$k^2 t_n^{-\frac{1}{2}}|u_{xxt}(x,0)| \le Ck^2(nk)^{-\frac{1}{2}} \le Ck\Big(\frac{k}{n}\Big)^{\frac{1}{2}}. \tag{4.3}$$

当 $n=1$ 时,

$$\begin{aligned} & k^{\frac{3}{2}}\int_0^{t_1}|u_{xxtt}(x,s)|\mathrm{d}s \\ & \le Ck^{\frac{3}{2}}\int_0^{t_1}s^{-\frac{1}{2}}\mathrm{d}s \le Ck^{\frac{3}{2}}t_1{}^{\frac{1}{2}} \le Ck^2 \\ & \le C(T)k\Big(\frac{k}{n}\Big)^{\frac{1}{2}}; \end{aligned} \tag{4.4.1}$$

当 $n\ge 2$ 时,

$$\begin{aligned} & k^{\frac{3}{2}}\int_{t_{n-1}}^{t_n}|u_{xxtt}(x,s)|\mathrm{d}s \\ & \le Ck^{\frac{3}{2}}\int_{t_{n-1}}^{t_n}s^{-\frac{1}{2}}\mathrm{d}s \le Ck^{\frac{3}{2}}k\xi^{-\frac{1}{2}} \le Ck^{\frac{5}{2}}t_{n-1}^{-\frac{1}{2}} \\ & \le C(T)k\Big(\tfrac{k}{(n-1)}\Big)^{\frac{1}{2}}, \end{aligned} \tag{4.4.2}$$

其中第二个不等式运用了积分中值定理 $(t_{n-1}\le\xi\le t_n)$;

$$\begin{aligned} & k^2\int_0^{t_{n-1}}(t_n-s)^{-\frac{1}{2}}|u_{xxtt}(x,s)|\mathrm{d}s \\ & \le Ck^2\int_0^{t_{n-1}}(t_n-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\mathrm{d}s \le Ck^2\int_0^{t_n}(t_n-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\mathrm{d}s \\ & \le Ck^2\int_0^1(1-x)^{-\frac{1}{2}}x^{-\frac{1}{2}}\mathrm{d}x \le CB\Big(\frac{1}{2},\frac{1}{2}\Big)k^2 \\ & \le C(T)k\Big(\frac{k}{n}\Big)^{\frac{1}{2}}. \end{aligned} \tag{4.5}$$

由 (4.3)–(4.5) 及引理 4.1 即得. 证毕.

引理 4.3　(离散型 Gronwall 引理) 如果 ω_n 是一非负的实数序列，满足

$$\omega_n \le \alpha_n + \sum_{s=0}^{n-1}\beta_s\omega_s, \quad n\ge 0,$$

其中 α_n 是非降序列， α_n 非负， $\beta_s\ge 0$, 那么

$$\omega_n \le \alpha_n \exp\Big(\sum_{s=0}^{n-1}\beta_s\Big), \quad n\ge 0.$$

证　见文 [8] 中的第 1055 页.

定理 4.4 假设 u 为问题 (1.1)–(1.3) 的解，$(U^0,U^1,\cdots,U^N)(N=[\frac{T}{k}])$ 是满足 (2.5)–(2.7) 的数值解，对充分光滑的 $v(x)$ 及 $f(x,t)$, u 满足 $u\in C((0,T];H^2\cap H_0^1)\cap C^3((0,T])$, 进一步假设问题 (1.1)–(1.3) 的解满足假设条件 (1.8), 当 $k=O(h^2)$ 时

$$\max_{0\le n\le N}\parallel U^n-u^n\parallel=C(T)(\parallel U^0-u^0\parallel+k^{\frac{3}{2}}+h^2). \tag{4.6}$$

证 令 $e_j^n=U_j^n-u_j^n$, 其中 $u_j^n=u(x_j,t_n)$. 由于

$$\begin{aligned}&U_j^n\varDelta U_j^n+\varDelta(U_j^n)^2\\ =&(e_j^n+u_j^n)\varDelta(e_j^n+u_j^n)+\varDelta(e_j^n+u_j^n)^2\\ =&e_j^n\varDelta e_j^n+\varDelta(e_j^n)^2+u_j^n\varDelta u_j^n+\varDelta(u_j^n)^2+u_j^n\varDelta e_j^n+2\varDelta(e_j^nu_j^n)+e_j^n\varDelta u_j^n,\end{aligned}$$

则当 $n\ge 2$ 时

$$\begin{aligned}&k^{-1}\Big(\frac{3}{2}e_j^n-2e_j^{n-1}+\frac{1}{2}e_j^{n-2}\Big)+\frac{e_{j+1}^n+e_j^n+e_{j-1}^n}{3}\frac{e_{j+1}^n-e_{j-1}^n}{2h}-\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^n\beta_p\delta^2e_j^{n-p}-\frac{\omega_{n0}}{h^2}\delta^2e_j^0\\ =&f_j^n-k^{-1}\Big(\frac{3}{2}u_j^n-2u_j^{n-1}+\frac{1}{2}u_j^{n-2}\Big)+\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^n\beta_p\delta^2u_j^{n-p}+\frac{\omega_{n0}}{h^2}\delta^2u_j^0\\ &-(u_j^n\varDelta u_j^n+\varDelta(u_j^n)^2)-(u_j^n\varDelta e_j^n+2\varDelta(e_j^nu_j^n)+e_j^n\varDelta u_j^n),\end{aligned}$$

而

$$u_t(x_j,t_n)+u(x_j,t_n)u_x(x_j,t_n)-\int_0^{t_n}\beta(t_n-s)u_{xx}(x_j,s)\mathrm{d}s=f(x_j,t_n),$$

故

$$\begin{aligned}&k^{-1}\Big(\frac{3}{2}e_j^n-2e_j^{n-1}+\frac{1}{2}e_j^{n-2}\Big)+\frac{1}{6h}(e_j^n\varDelta e_j^n+\varDelta(e_j^n)^2)-\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^n\beta_p\delta^2e_j^{n-p}-\frac{\omega_{n0}}{h^2}\delta^2e_j^0\\ =&\tau_{1j}^n-\tau_{2j}^n+\frac{1}{6h}(\tau_{3j}^n-\tau_{4j}^n-\tau_{5j}^n),\end{aligned} \tag{4.7}$$

其中

$$\begin{aligned}&\tau_{1j}^n=u_t(x_j,t_n)-k^{-1}\Big(\frac{3}{2}u_j^n-2u_j^{n-1}+\frac{1}{2}u_j^{n-2}\Big),\\ &\tau_{2j}^n=\int_0^{t_n}\beta(t_n-s)u_{xx}(x_j,s)\mathrm{d}s-\Big(\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^n\beta_p\delta^2u_j^{n-p}+\frac{\omega_{n0}}{h^2}\delta^2u_j^0\Big),\\ &\tau_{3j}^n=u(x_j,t_n)u_x(x_j,t_n)-(u_j^n\varDelta u_j^n+\varDelta(u_j^n)^2),\\ &\tau_{4j}^n=u_j^n\varDelta e_j^n+\varDelta(e_j^nu_j^n),\\ &\tau_{5j}^n=e_j^n\varDelta u_j^n+\varDelta(e_j^nu_j^n),\end{aligned}$$

同理当 $n=1$ 时

$$
\begin{aligned}
&k^{-1}(e_j^1-e_j^0)+\frac{1}{6h}(e_j^1\Delta e_j^1+\Delta(e_j^1)^2)-\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\delta^2 e_j^{1-p}-\frac{\omega_{10}}{h^2}\delta^2 e_j^0\\
&=f_j^1-k^{-1}(u_j^1-u_j^0)+\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\delta^2 e_j^{1-p}+\frac{\omega_{10}}{h^2}\delta^2 u_j^0\\
&=\tau_{1j}^1-\tau_{2j}^1+\frac{1}{6h}(\tau_{3j}^1-\tau_{4j}^1-\tau_{5j}^1),
\end{aligned}
\tag{4.8}
$$

其中

$$
\begin{aligned}
&\tau_{1j}^1=u_t(x_j,t_1)-k^{-1}(u_j^1-u_j^0),\\
&\tau_{2j}^1=\int_0^{t_1}\beta(t_1-s)u_{xx}(x_j,s)\mathrm{d}s-\left(\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\delta^2 u_j^{1-p}+\frac{\omega_{10}}{h^2}\delta^2 u_j^0\right),\\
&\tau_{3j}^1=u(x_j,t_1)u_x(x_j,t_1)-(u_j^1\Delta u_j^1+\Delta(u_j^1)^2),\\
&\tau_{4j}^1=u_j^1\Delta e_j^1+\Delta(e_j^1u_j^1),\\
&\tau_{5j}^1=e_j^1\Delta u_j^1+\Delta(e_j^1u_j^1).
\end{aligned}
$$

当 $n\ge 2$ 时，在 (4.7) 的两边同时乘以 he_j^n, 并对 j $(1\le j\le J-1)$ 求和可得

$$
\begin{aligned}
&\left\langle k^{-1}\left(\frac{3}{2}e^n-2e^{n-1}+\frac{1}{2}e^{n-2}\right),e^n\right\rangle+\frac{1}{6h}\langle(e^n\Delta e^n+\Delta(e^n)^2),e^n\rangle\\
&-\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^{n}\beta_p\langle\delta^2 e^{n-p},e^n\rangle-\frac{\omega_{n0}}{h^2}\langle\delta^2 e^0,e^n\rangle\\
&=\langle\tau_1^n,e^n\rangle-\langle\tau_2^n,e^n\rangle+\frac{1}{6h}\langle\tau_3^n-\tau_4^n-\tau_5^n,e^n\rangle.
\end{aligned}
\tag{4.9}
$$

当 $n=1$ 时，在 (4.8) 的两边同时乘以 he_j^1, 并对 j $(1\le j\le J-1)$ 求和可得

$$
\begin{aligned}
&\langle k^{-1}(e^1-e^0),e^1\rangle+\frac{1}{6h}\langle(e^1\Delta e^1+\Delta(e^1)^2),e^1\rangle\\
&-\frac{k^{\frac{1}{2}}}{h^2}\sum_{p=0}^{1}\beta_p\langle\delta^2 e^{1-p},e^1\rangle-\frac{\omega_{10}}{h^2}\langle\delta^2 e^0,e^1\rangle\\
&=\langle\tau_1^1,e^1\rangle-\langle\tau_2^1,e^1\rangle+\frac{1}{6h}\langle\tau_3^1-\tau_4^1-\tau_5^1,e^1\rangle.
\end{aligned}
\tag{4.10}
$$

引理 4.5 当 $U_0=U_J=0$ 时，有

$$\langle\tau_4^n,e^n\rangle=0. \tag{4.11}$$

证

$$
\begin{aligned}
\langle\tau_4^n,e^n\rangle&=\sum_{j=1}^{J-1}h(u_j^n\Delta e_j^n+\Delta(e_j^nu_j^n))e_j^n\\
&=\sum_{j=1}^{J-1}h[u_j^n(e_{j+1}^n-e_{j-1}^n)+(e_{j+1}^nu_{j+1}^n-e_{j-1}^nu_{j-1}^n)]e_j^n
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{j=1}^{J-1} h[(u_j^n+u_{j+1}^n)e_{j+1}^n-(u_j^n+u_{j-1}^n)e_{j-1}^n]e_j^n \\
&=\sum_{j=1}^{J-1} h(u_j^n+u_{j+1}^n)e_{j+1}^n e_j^n-\sum_{j=1}^{J-1} h(u_j^n+u_{j-1}^n)e_{j-1}^n e_j^n \\
&=\sum_{j=1}^{J-1} h(u_j^n+u_{j+1}^n)e_{j+1}^n e_j^n-\sum_{j=0}^{J-2} h(u_j^n+u_{j+1}^n)e_{j+1}^n e_j^n \\
&=h(u_{J-1}^n+u_J^n)e_J^n e_{J-1}^n-h(u_0^n+u_1^n)e_1^n e_0^n \\
&=0.
\end{aligned}
$$

证毕.

引理 4.6 当 $U_0=U_J=0$ 时，有

$$
|\langle \tau_5^n, e^n\rangle| \le Ch \parallel e^n \parallel^2 . \tag{4.12}
$$

证 由式 (2.13), 可得

$$
\langle \Delta(e^n u^n), e^n\rangle=\frac{1}{2}\langle T_+e^n\Delta_+u^n+T_-e^n\Delta_-u^n, e^n\rangle,
$$

由边界条件 $U_0=U_J=0$ 和 Cauchy-Schwarz 不等式，可得

$$
\begin{aligned}
&|\langle T_+e^n\Delta_+u^n, e^n\rangle| \le \parallel \Delta_+u^n \parallel_\infty |\langle T_+e^n, e^n\rangle| \\
=&\parallel \Delta_+u^n \parallel_\infty \left|\sum_{j=1}^{J-1} he_{j+1}^n e_j^n\right| \\
\le&\parallel \Delta_+u^n \parallel_\infty \sum_{j=1}^{J-1} h\frac{(e_{j+1}^n)^2+(e_j^n)^2}{2} \\
=&\frac{1}{2}\parallel \Delta_+u^n \parallel_\infty \left(\sum_{j=1}^{J-1} h(e_{j+1}^n)^2+\parallel e^n \parallel^2\right) \\
\le&\frac{1}{2}\parallel \Delta_+u^n \parallel_\infty \left(\sum_{j=2}^{J} h(e_j^n)^2+\parallel e^n \parallel^2\right) \\
\le&\parallel \Delta_+u^n \parallel_\infty \parallel e^n \parallel^2 .
\end{aligned}
$$

同理可得

$$
|\langle T_-e^n\Delta_-u^n, e^n\rangle| \le \parallel \Delta_-u^n \parallel_\infty \parallel e^n \parallel^2,
$$

故

$$
\begin{aligned}
|\langle \tau_5^n, e^n\rangle| &= |\langle e^n\Delta u^n+\Delta(e^n u^n), e^n\rangle| \\
&=|\langle e^n\Delta u^n, e^n\rangle+\frac{1}{2}\langle T_+e^n\Delta_+u^n+T_-e^n\Delta_-u^n, e^n\rangle| \\
&\le \parallel \Delta u^n \parallel_\infty \parallel e^n \parallel^2+\frac{1}{2}(\parallel \Delta_+u^n \parallel_\infty+\parallel \Delta_-u^n \parallel_\infty)\parallel e^n \parallel^2 \\
&\le Ch \parallel e^n \parallel^2,
\end{aligned}
$$

其中最后一个不等式是由于 u 的光滑性．证毕．

故由 (2.12) 和 (4.12), (4.9)–(4.10) 可分别化为

$$
\begin{aligned}
&\Big\langle k^{-1}\Big(\frac{3}{2}e^n-2e^{n-1}+\frac{1}{2}e^{n-2}\Big),e^n\Big\rangle\\
=&\frac{k^{\frac12}}{h^2}\sum_{p=0}^{n}\beta_p\langle\delta^2e^{n-p},e^n\rangle+\frac{\omega_{n0}}{h^2}\langle\delta^2e^0,e^n\rangle\\
&+\langle\tau_1^n,e^n\rangle-\langle\tau_2^n,e^n\rangle+\frac{1}{6h}\langle\tau_3^n-\tau_5^n,e^n\rangle
\end{aligned}\tag{4.13}
$$

和

$$
\begin{aligned}
&\langle k^{-1}(e^1-e^0),e^1\rangle\\
=&\frac{k^{\frac12}}{h^2}\sum_{p=0}^{1}\beta_p\langle\delta^2e^{1-p},e^1\rangle+\frac{\omega_{10}}{h^2}\langle\delta^2e^0,e^1\rangle\\
&+\langle\tau_1^1,e^1\rangle-\langle\tau_2^1,e^1\rangle+\frac{1}{6h}\langle\tau_3^1-\tau_5^1,e^1\rangle.
\end{aligned}\tag{4.14}
$$

故当 $N\geq 2$ 时，由 (4.13)–(4.14) 可得

$$
\begin{aligned}
&\langle e^1-e^0,e^1\rangle+\sum_{n=2}^{N}\Big\langle\frac{3}{2}e^n-2e^{n-1}+\frac{1}{2}e^{n-2},e^n\Big\rangle\\
=&\frac{k^{\frac32}}{h^2}\sum_{p=0}^{1}\beta_p\langle\delta^2e^{1-p},e^1\rangle+\frac{k^{\frac32}}{h^2}\sum_{n=2}^{N}\sum_{p=0}^{n}\beta_p\langle\delta^2e^{n-p},e^n\rangle\\
&+\sum_{n=1}^{N}\frac{k\omega_{n0}}{h^2}\langle\delta^2e^0,e^n\rangle+\sum_{n=1}^{N}k\Big(\langle\tau_1^n,e^n\rangle-\langle\tau_2^n,e^n\rangle+\frac{1}{6h}\langle\tau_3^n-\tau_5^n,e^n\rangle\Big)\\
=&\frac{k^{\frac32}}{h^2}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\langle\delta^2e^{n-p},e^n\rangle+\sum_{n=1}^{N}\frac{k\omega_{n0}}{h^2}\langle\delta^2e^0,e^n\rangle\\
&+\sum_{n=1}^{N}k\Big(\langle\tau_1^n,e^n\rangle-\langle\tau_2^n,e^n\rangle+\frac{1}{6h}\langle\tau_3^n-\tau_5^n,e^n\rangle\Big).
\end{aligned}\tag{4.15}
$$

由 (3.5)–(3.8) 和 (3.11) 可得

$$
\begin{aligned}
&\frac{3}{4}\parallel e^N\parallel^2-\frac{1}{4}\parallel e^{N-1}\parallel^2-\frac{1}{4}\parallel e^1\parallel^2-\frac{1}{4}\parallel e^0\parallel^2\\
\leq&\frac{2k^{\frac32}}{h^2}\beta_0\parallel e^0\parallel^2+\frac{4k}{h^2}\sum_{n=1}^{N}|\omega_{n0}|\parallel e^0\parallel\parallel e^n\parallel\\
&+\sum_{n=1}^{N}k\Big(\parallel\tau_1^n\parallel+\parallel\tau_2^n\parallel+\frac{1}{6h}\parallel\tau_3^n\parallel\Big)\parallel e^n\parallel+\frac{1}{6h}\sum_{n=1}^{N}k|\langle\tau_5^n,e^n\rangle|.
\end{aligned}
$$

假设 $\parallel e^M\parallel=\max\limits_{0\leq n\leq N}\parallel e^n\parallel$, 则

$$
\parallel e^M\parallel^2\leq\frac{1}{3}(\parallel e^{M-1}\parallel^2+\parallel e^1\parallel^2+\parallel e^0\parallel^2)
$$

$$
\begin{aligned}
&+\frac{8k^{\frac{3}{2}}}{3h^2}\beta_0 \| e^0 \|\| e^M \| + \frac{16k}{3h^2}\sum_{n=1}^{M}|\omega_{n0}| \| e^0 \|\| e^M \| \\
&+\frac{4}{3}\sum_{n=1}^{M}k\Big(\| \tau_1^n \| + \| \tau_2^n \| + \frac{1}{6h} \| \tau_3^n \| \Big) \| e^M \| + C\sum_{n=1}^{M}k \| e^n \|^2 \\
\le\ & \frac{1}{3}(\| e^M \|^2 + \| e^1 \|\| e^M \| + \| e^0 \|\| e^M \|) \\
&+\frac{8k^{\frac{3}{2}}}{3h^2}\beta_0 \| e^0 \|\| e^M \| + \frac{16k}{3h^2}\sum_{n=1}^{M}|\omega_{n0}| \| e^0 \|\| e^M \| \\
&+\frac{4}{3}\sum_{n=1}^{M}k\Big(\| \tau_1^n \| + \| \tau_2^n \| + \frac{1}{6h} \| \tau_3^n \| \Big) \| e^M \| + C\sum_{n=1}^{M}k \| e^n \|\| e^M \|,
\end{aligned}
$$

因此，当 $N\ge 2$ 时

$$
\begin{aligned}
\| e^N \| \le\ & \| e^M \| \\
\le\ & \frac{1}{2}(\| e^1 \| + \| e^0 \|) \\
&+\frac{4k^{\frac{3}{2}}}{h^2}\beta_0 \| e^0 \| + \frac{8k}{h^2}\sum_{n=1}^{N}|\omega_{n0}| \| e^0 \| \\
&+2\sum_{n=1}^{M}k\Big(\| \tau_1^n \| + \| \tau_2^n \| + \frac{1}{6h} \| \tau_3^n \| \Big) + C\sum_{n=1}^{N}k \| e^n \|,
\end{aligned} \tag{4.16}
$$

由离散型 Gronwall 引理 (引理 4.3), 可得

$$
\begin{aligned}
\| e^N \| \le\ & \| e^M \| \\
\le\ & C(T)\Bigg(\frac{1}{2}(\| e^1 \| + \| e^0 \|) \\
&+\frac{4k^{\frac{3}{2}}}{h^2}\beta_0 \| e^0 \| + \frac{8k}{h^2}\sum_{n=1}^{N}|\omega_{n0}| \| e^0 \| \\
&+2\sum_{n=1}^{N}k(\| \tau_1^n \| + \| \tau_2^n \| + \frac{1}{6h} \| \tau_3^n \|)\Bigg).
\end{aligned} \tag{4.17}
$$

引理 4.7 假如方程 (1.1)–(1.3) 满足正则性条件 (1.8), 则对 $n\ge 1$

$$
\sum_{n=1}^{N}k \| \tau_1^n \| \le Ck^{\frac{3}{2}}. \tag{4.18}
$$

证 由带积分余项的 Taylor 展开可得，对一切 x $(0\le x\le 1)$ 均有：
当 $n\ge 3$ 时

$$
u(x,t_{n-1}) = u(x,t_n) - ku_t(x,t_n) + \frac{1}{2!}(-k)^2u_{tt}(x,t_n) + \frac{1}{2!}\int_{t_n}^{t_{n-1}} u_{ttt}(x,s)(t_{n-1}-s)^2\mathrm{d}s,
$$

$$
u(x,t_{n-2}) = u(x,t_n) - 2ku_t(x,t_n) + \frac{1}{2!}(-2k)^2u_{tt}(x,t_n) + \frac{1}{2!}\int_{t_n}^{t_{n-2}} u_{ttt}(x,s)(t_{n-2}-s)^2\mathrm{d}s.
$$

由上述两式可得对 $1 \le j \le J-1$,

$$\begin{aligned}
|\tau_{1j}^n| &= \Big|u_t(x,t_n) - k^{-1}\Big(\frac{3}{2}u^n - 2u^{n-1} + \frac{1}{2}u^{n-2}\Big)\Big| \\
&= \frac{1}{k}\frac{1}{2!}\Big|\int_{t_n}^{t_{n-1}} u_{ttt}(x,s)(t_{n-1}-s)^2\mathrm{d}s - \frac{1}{4}\int_{t_n}^{t_{n-2}} u_{ttt}(x,s)(t_{n-2}-s)^2\mathrm{d}s\Big| \\
&\le \frac{1}{k}\Big(k^2\int_{t_{n-1}}^{t_n}|u_{ttt}(x,s)|\mathrm{d}s + \frac{1}{4}(2k)^2\int_{t_{n-2}}^{t_n}|u_{ttt}(x,s)|\mathrm{d}s\Big) \\
&\le Ck\int_{t_{n-2}}^{t_n}|u_{ttt}(x,s)|\mathrm{d}s.
\end{aligned}$$

当 $n=2$ 时

$$\begin{aligned}
u(x,t_1) &= u(x,t_2) - ku_t(x,t_2) + \int_{t_2}^{t_1} u_{tt}(x,s)(t_1-s)\mathrm{d}s, \\
u(x,t_0) &= u(x,t_2) - 2ku_t(x,t_2) + \int_{t_2}^{t_0} u_{tt}(x,s)(t_0-s)\mathrm{d}s.
\end{aligned}$$

由上述两式可得对 $1 \le j \le J-1$,

$$\begin{aligned}
|\tau_{1j}^2| &= \Big|u_t(x,t_2) - k^{-1}\Big(\frac{3}{2}u^2 - 2u^1 + \frac{1}{2}u^0\Big)\Big| \\
&= \frac{1}{k}\Big|\int_{t_2}^{t_1} u_{tt}(x,s)(t_1-s)\mathrm{d}s - \frac{1}{2}\int_{t_2}^{t_0} u_{tt}(x,s)(t_0-s)\mathrm{d}s\Big| \\
&\le \frac{1}{k}\Big(k\int_{t_1}^{t_2}|u_{tt}(x,s)|\mathrm{d}s + \frac{1}{2}(2k)\int_{t_0}^{t_2}|u_{tt}(x,s)|\mathrm{d}s\Big) \\
&\le C\int_0^{2k}|u_{tt}(x,s)|\mathrm{d}s.
\end{aligned}$$

当 $n=1$ 时

$$u(x,t_0) = u(x,t_1) + ku_t(x,t_1) + \int_{t_1}^{t_0} u_{tt}(x,s)(t_1-s)^2\mathrm{d}s.$$

故对 $1 \le j \le J-1$,

$$\begin{aligned}
|\tau_{1j}^1| &= |u_t(x,t_1) - k^{-1}(u^1 - u^0)| \\
&= \frac{1}{k}\Big|\int_{t_1}^{t_0} u_{tt}(x,s)(t_1-s)\mathrm{d}s\Big| \\
&\le C\int_0^k |u_{tt}(x,s)|\mathrm{d}s.
\end{aligned}$$

对一切 $1 \le j \le J-1$, $n \ge 3$, 由假设条件 (1.8), 均有

$$\begin{aligned}
\|\ \tau_1^n\ \|^2 &\le Ck^2\Big\|\int_{t_{n-2}}^{t_n}|u_{ttt}(x,s)|\mathrm{d}s\Big\|^2 \\
&= Ck^2\sum_{j=1}^{J-1}h\Big(\int_{t_{n-2}}^{t_n}|u_{ttt}(x_j,s)|\mathrm{d}s\Big)^2
\end{aligned}$$

$$
\begin{aligned}
&= Ck^2 \sum_{j=1}^{J-1} h \int_{t_{n-2}}^{t_n} |u_{ttt}(x_j, s)| \mathrm{d}s \int_{t_{n-2}}^{t_n} |u_{ttt}(x_j, y)| \mathrm{d}y \\
&= Ck^2 \int_{t_{n-2}}^{t_n} \int_{t_{n-2}}^{t_n} \sum_{j=1}^{J-1} h |u_{ttt}(x_j, s)| |u_{ttt}(x_j, y)| \mathrm{d}s \mathrm{d}y \\
&\le Ck^2 \int_{t_{n-2}}^{t_n} \int_{t_{n-2}}^{t_n} \sum_{j=1}^{J-1} h s^{-\frac{3}{2}} y^{-\frac{3}{2}} \mathrm{d}s \mathrm{d}y \\
&\le Ck^2 \Big(\int_{t_{n-2}}^{t_n} s^{-\frac{3}{2}} \mathrm{d}s \Big)^2,
\end{aligned}
$$

即

$$
\| \tau_1^n \| \le Ck \int_{t_{n-2}}^{t_n} s^{-\frac{3}{2}} \mathrm{d}s.
$$

同理可得

$$
\| \tau_1^1 \| \le C \int_0^k s^{-\frac{1}{2}} \mathrm{d}s, \quad \| \tau_1^2 \| \le C \int_0^{2k} s^{-\frac{1}{2}} \mathrm{d}s.
$$

因此很容易得到

$$
\sum_{n=1}^{N} k \| \tau_1^n \| \le \sum_{n=3}^{N} k^2 \int_{t_{n-2}}^{t_n} s^{-\frac{3}{2}} \mathrm{d}s + k \int_0^k s^{-\frac{1}{2}} \mathrm{d}s + k \int_0^{2k} s^{-\frac{1}{2}} \mathrm{d}s \le Ck^{\frac{3}{2}}.
$$

证毕.

引理 4.8 对 $n \ge 1$, 我们有

$$
\begin{aligned}
&\sum_{n=1}^{N} k \| \tau_2^n \| \le C(T)(h^2 + k^2), \\
&\sum_{n=1}^{N} \frac{k}{6h} \| \tau_3^n \| \le C(T) h^2.
\end{aligned} \tag{4.19}
$$

证 很显然, 对一切 $0 \le t \le T$ 和 $1 \le j \le J-1$ 有如下一致估计 (参见文 [6] 中的 (4.13)–(4.14))

$$
\begin{aligned}
&h^{-2} \delta^2 u(x_j, t) - u_{xx}(x_j, t) = O(h^2), \\
&\frac{1}{6h} \tau_{3j}^n = \frac{1}{6h} ((u_j \varDelta u_j + \varDelta (u_j)^2) - u(x_j, t) u_x(x_j, t)) = O(h^2).
\end{aligned} \tag{4.20}
$$

故由引理 4.2, 对一切 $1 \le n \le N$ 和 $1 \le j \le J-1$ 有

$$
\begin{aligned}
|\tau_{2j}^n| &= \Big| \int_0^{t_n} \beta(t_n - s) u_{xx}(x_j, s) \mathrm{d}s - q_n \Big(\frac{1}{h^2} \delta^2 u_j \Big) \Big| \\
&\le \Big| \int_0^{t_n} \beta(t_n - s) (u_{xx}(x_j, s) \mathrm{d}s - q_n \Big(u_{xx}(x_j, \cdot) \Big) \Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|q_n(u_{xx}(x_j,\cdot))-q_n\Big(\frac{1}{h^2}\delta^2u_j\Big)\Big|\\
&\le Ck\Big(\frac{k}{n}\Big)^{\frac12}+Ch^2\Big(k^{\frac12}\sum_{p=0}^{n}\beta_p+w_{n0}(k)\Big)\\
&\le Ck\Big(\frac{k}{n}\Big)^{\frac12}+Ch^2\Big(2\Big(\frac{t_n}{\pi}\Big)^{\frac12}\Big)\\
&\le C(T)\Big(h^2+k\Big(\frac{k}{n}\Big)^{\frac12}\Big).
\end{aligned}
$$

故

$$
\sum_{n=1}^{N}k\parallel\tau_2^n\parallel\le\sum_{n=1}^{N}k\Big(\sum_{j=1}^{J-1}h(\tau_{2j}^n)^2\Big)^{\frac12}\le\sum_{n=1}^{N}k\Big(h^2+k\Big(\frac{k}{n}\Big)^{\frac12}\Big)\le C(T)(h^2+k^2),
$$

$$
\sum_{n=1}^{N}\frac{k}{6h}\parallel\tau_3^n\parallel\le\sum_{n=1}^{N}\frac{k}{6h}\Big(\sum_{j=1}^{J-1}h(\tau_{3j}^n)^2\Big)^{\frac12}\le\sum_{n=1}^{N}k\Big(\sum_{j=1}^{J-1}h\Big(\frac{1}{6h}\tau_{3j}^n\Big)^2\Big)^{\frac12}\le C(T)h^2.
$$

证毕.

当 $N=1$ 时，由 (4.14) 得

$$
\begin{aligned}
\parallel e^1\parallel^2&=\langle e^0,e^1\rangle+\frac{k^{\frac32}}{h^2}\sum_{p=0}^{1}\beta_p\langle\delta^2e^{1-p},e^1\rangle+\frac{k\omega_{10}}{h^2}\langle\delta^2e^0,e^1\rangle\\
&\quad+k\langle\tau_1^1,e^1\rangle-k\langle\tau_2^1,e^1\rangle+\frac{k}{6h}\langle\tau_3^1-k\tau_5^1,e^1\rangle\\
&\le\parallel e^0\parallel\parallel e^1\parallel+\frac{4k^{\frac32}}{h^2}\beta_0\parallel e^0\parallel\parallel e^1\parallel+\frac{4k|\omega_{10}|}{h^2}\parallel e^0\parallel\parallel e^1\parallel\\
&\quad+k\Big(\parallel\tau_1^1\parallel\parallel e^1\parallel+\parallel\tau_2^1\parallel\parallel e^1\parallel+\frac{1}{6h}\parallel\tau_3^1\parallel\parallel e^1\parallel+C\parallel e^1\parallel^2\Big),
\end{aligned}\tag{4.21}
$$

因此

$$
\begin{aligned}
\parallel e^1\parallel\le&\parallel e^0\parallel+\frac{4k^{\frac32}}{h^2}\beta_0\parallel e^0\parallel+\frac{4k|\omega_{10}|}{h^2}\parallel e^0\parallel\\
&+k\parallel\tau_1^1\parallel+k\parallel\tau_2^1\parallel+\frac{k}{6h}\parallel\tau_3^1\parallel,
\end{aligned}\tag{4.22}
$$

而

$$
\begin{aligned}
&\parallel\tau_1^1\parallel\le C\int_0^k s^{-\frac12}\mathrm{d}s\le Ck^{\frac32},\\
&\parallel\tau_1^2\parallel\le C\int_0^{2k} s^{-\frac12}\mathrm{d}s\le C(T)(h^2+k^2).
\end{aligned}\tag{4.23}
$$

由 (4.17)–(4.19), (4.22)–(4.23), 即证.

注 3　对格式作稍微的调整能够使它适合类似于 (1.1) 的弱奇异方程，其中积分项由

$$
(I^\alpha u_{xx})(x,t)=\int_0^t(t-s)^{\alpha-1}u_{xx}(x,s)\mathrm{d}s,\quad 0\le\alpha\le1\tag{4.24}
$$

代替，通过平行的分析就能得出上述方程数值解的稳定性和收敛性.

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A SECOND ORDER FULLY DISCRETE DIFFERENCE SCHEME FOR A NONLINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract In this paper, the second order fully discrete difference method for a nonlinear partial integro-differential equation is considered. The integral term is treated by means of the second order convolution quadrature raised by Lubich. By the second order backward difference scheme, the stability and the error estimate are given.

Key words Partial integro-differential equation, fractional calculus, convolution quadrature, finite difference scheme, second order fully discrete scheme.