Constructing Common Quadratic Lyapunov Functions for a Class of Stable Matrices

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Abstract Narendra and Balakrishnan proposed a way to construct a common quadratic Lyapunov function (CQLF)[1], when a set of stable matrices are commutative. The purpose of this paper is to generalize the method to non-commutative and non-solvable case. A modified constructing algorithm is proposed and certain conditions are provided to assure the resulting matrix being a CQLF. Next, the problem discussed is when a stable matrix can be added to a set of matrices with CQLF to construct a new CQLF for the enlarged set.

Key words Switched system, common quadratic Lyapunov function, Lie algebra.

1 Introduction

Consider a switched linear system:

$$\dot{x} = A_{\sigma(t)}x \tag{1}$$

where $\sigma(t) : [0, +\infty) \to \Lambda$ is an right continuous function, $\Lambda = \{1, 2, \cdots, N\}$.

To assure the stability of switched system (1) under arbitrary switching, a common Lyapunov function is sufficient. A quadratic Lyapunov function, $x^{\mathrm{T}}Px$, with positive definite matrix $P > 0$ (or briefly, P) is called a CQLF of $\{A_\lambda | \lambda \in \Lambda\}$ if

$$PA_\lambda + A_\lambda^{\mathrm{T}}P < 0, \quad \forall \lambda \in \Lambda \tag{2}$$

[1] considerd a set of commutative stable matrices $\{A_1, \cdots, A_N\}$ and proposed a method to construct CQLF, which is described in the following: Choose a positive definite matrix P_0 and define $P_i > 0$, $i = 1, \cdots, N$, recursively by

$$P_iA_i + A_i^{\mathrm{T}}P_i = -P_{i-1}, \quad i = 1, \cdots, N \tag{3}$$

Then P_N is a CQLF. Moreover, P_N has an analytic expression as

$$\begin{aligned} P_N = \; & \int_0^\infty \mathrm{e}^{A_N^{\mathrm{T}}t_N} \int_0^\infty \mathrm{e}^{A_{N-1}^{\mathrm{T}}t_{N-1}} \cdots \int_0^\infty \mathrm{e}^{A_1^{\mathrm{T}}t_1} \times \\ & P_0\mathrm{e}^{A_1t_1}\mathrm{d}t_1 \cdots \mathrm{e}^{A_{N-1}t_{N-1}}\mathrm{d}t_{N-1}\mathrm{e}^{A_Nt_N}\mathrm{d}t_N \end{aligned} \tag{4}$$

For ease of statement, (4) is called the N-B type CQLF with initial matrix P_0. The N-B structure was firstly extended to constructing common Lyapunov functions for nonlinear switched systems in [2]. Some sufficient conditions based on Lie algebraic structure were proposed in [3]. [4] provided a nice Lie algebraic condition than the solvability of Lie algebra implies CQLF, which generalized the condition of [1]. The most updated and beautiful Lie algebra conditions were proved in [5]. Necessary and sufficient conditions for quadratic stabilization and a geometric description of the set of planar control systems were provided in [6]. A necessary and sufficient condition of CQLF was given in [7]. In fact, the approach in both [7] and [6] is based on constructing CQLF by considering the topological structure of the set of CQLFs. An alternative approach in constructing CQLF is using stable matrices A_i. The purpose of this paper is to investigate how far N-B structure can be extended for non-commutative set of stable matrices. In general the Lie algebra generated by them is not solvable. We intend to combine the Lie algebra structure with N-B structure to enlarge the applicability of N-B structure.

Received September 14, 2005; in revised form September 19, 2006
Supported by National Natural Science Foundation of P. R. China (60274010, 60343001, 60221301, 60334040)
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DOI: 10.1360/aas-007-0202

2 CQLF for Two Matrices

In this section we consider the case of $N = 2$. First consider the involutive case and then non-involutive case.

Definition 1. Let $\mathcal{L} \subset gl(n, R)$ be a sub-algebra, $\dim(\mathcal{L}) = N$, and a linearly independent set of matrices $\mathcal{A} = \{A_1, A_2, \cdots, A_N\} \subset \mathcal{L}$ are given.

1) $\mathcal{A}$ is called a stable basis of $\mathcal{L}$ if all A_i, $i = 1, \cdots, N$ are stable and they form a basis of $\mathcal{L}$.

In this case $\mathcal{A}$ is said to be involutive.

2) $\mathcal{A}$ is called a stable generator of $\mathcal{L}$ if all A_i, $i = 1, \cdots, N$ are stable and $\mathcal{A}_{LA} = \mathcal{L}$.

For $N = 2$, let $\{A_1, A_2\}$ be a stable basis. Our question is: can we use the algorithm (3) to construct a CQLF for A_1 and A_2. Since $\{A_1, A_2\}$ is a basis, we have

$$[A_1, A_2] = \alpha A_1 + \beta A_2 \tag{5}$$

Denote by $\sigma(A)$ the set of eigenvalues of A. Then we can show that in a cone area of $\alpha - \beta$ plane the N-B structure remains available:

Theorem 1. Let $\{A_1, A_2\}$ be a stable basis satisfying (5). Then A_1 and A_2 share an N-B type CQLF if the following two conditions are satisfied:

$$\alpha > -2\min_\sigma |\mathrm{Re}\sigma(A_2)|, \quad \beta < 2\min_\sigma |\mathrm{Re}\sigma(A_1)| \tag{6}$$

Proof. Choosing any positive definite matrix $Q > 0$, we set

$$P_0\left(A_1 + \frac{\beta}{2}I\right) + \left(A_1 + \frac{\beta}{2}I\right)^{\mathrm{T}} P_0 = -Q \tag{7}$$

Then we know that (7) has a unique solution $P_0 > 0$. Using this P_0, we set

$$\begin{cases} P_1A_1 + A_1^{\mathrm{T}}P_1 = -P_0 \\ P_2A_2 + A_2^{\mathrm{T}}P_2 = -P_1 - \mu P_0, \quad \mu > 0 \end{cases} \tag{8}$$

Then we can show that for suitably chosen μ, P_2 (precisely, $x^{\mathrm{T}}P_2x$) is a CQLF.

From the second equation of (8) we have

$$P_1 = -\left(P_2 A_2 + A_2^{\mathrm{T}} P_2 + \mu P_0\right)$$

Plugging it into the first equation of (8) and using (5) and (7) yields

$$\begin{aligned}&\left(P_2 A_1 + A_1^{\mathrm{T}} P_2\right)\left(A_2 - \tfrac{\alpha}{2} I\right) + \left(A_2 - \tfrac{\alpha}{2} I\right)^{\mathrm{T}} \times \\ &\left(P_2 A_1 + A_1^{\mathrm{T}} P_2\right) = -\beta P_1 + P_0 + \mu Q\end{aligned} \tag{9}$$

Now choosing $\mu > 0$ large enough, the right hand side of (9) becomes positive definite. Using (6), we know that $A_2 - \frac{\alpha}{2} I$ is Hurwitz. It follows that

$$P_2 A_1 + A_1^{\mathrm{T}} P_2 < 0$$

That is, P_2 is the CQLF for A_1 and A_2. □

Theorem 1 shows how far the N-B approach can go when we have a stable basis of two dimensional Lie algebra. The advantage is that it provides an easy way to construct a CQLF. But it is conservative. Because we have the following:

Proposition 1. If two stable matrices A_1 and A_2 satisfy (5), then they share a CQLF.

Proof. According to (5) it is easy to verify that $\mathcal{L} = \{A_1, A_2\}_{LA}$ is solvable because $\mathcal{L}^2 = 0$. The conclusion follows from [4]. □

Next, we consider non-solvable case. Roughly speaking, if $[A_1, A_2]$ is sufficiently close to zero, then by continuity this result remains true even though $\{A_1, A_2\}$ is not involutive.

Using the same argument in the proof of Theorem 1, we can prove the following:

Proposition 2. Let A_1, A_2 be two stable matrixes and $[A_1, A_2] = C$. Then A_1, A_2 share a CQLF if there exists a positive definite matrix P_0, such that

$$P_0 + P_2 C + C^{\mathrm{T}} P_2 > 0 \tag{10}$$

where P_2, uniquely determined by P_0 via (3), is a CQLF.

3 CQLF for N Matrices

In this section, we try to extend the Propositions 1 and 2 to N matrix case. Consider the set $\mathcal{A}$ of stable matrices and choose any $A_i \in \mathcal{A}$. For notational brevity, denote it as A_N. Finding a CQLF is equivalent to asking when a QLF of A_N is also the QLF for others. Then we propose the following algorithm: Choose any $P_{N-1} > 0$ and set

$$P_N A_N + A_N^{\mathrm{T}} P_N = -P_{N-1} \tag{11}$$

Following [1], we define

$$P_{i,j} = P_j A_i + A_i^{\mathrm{T}} P_j, \quad 1 \le i \le N;\ j = N-1, N \tag{12}$$

First, we assume $[A_N, \mathrm{Span}\{\mathcal{A}\}] \subset \mathrm{Span}\{\mathcal{A}\}$, which is equivalent to

$$[A_N, A_i] = \sum_{k=1}^{N} c_i^k A_k, \quad i = 1, \cdots, N-1 \tag{13}$$

Then we have the following result:

Theorem 2 Consider the set $\mathcal{A}$ of stable matrices satisfying (13). Then $\mathcal{A}$ share a CQLF P_N, if the following two conditions are satisfied:

$$\begin{aligned}&\max(c_i^i) < 2\min|\mathrm{Re}\sigma(A_N)| \\ &-P_{i,N-1} + c_i^N\ P_{N-1} - \sum_{k=1,k\ne i}^{N-1} c_i^k P_{k,N} > 0\end{aligned} \tag{14}$$

Where $i = 1, \cdots, N-1$.

Proof. Using (14), we have

$$\begin{aligned}&P_{i,N}(A_N + \tfrac{c_i^i}{2} I) + (A_N + \tfrac{c_i^i}{2} I)^{\mathrm{T}} P_{i,N} \\ &= \left(P_N A_i + A_i^{\mathrm{T}} P_N\right) A_N + A_N^{\mathrm{T}} \left(P_N A_i + A_i^{\mathrm{T}} P_N\right) + \\ &c_i^i \left(P_N A_i + A_i^{\mathrm{T}} P_N\right) \\ &= P_N \left(A_N A_i - \sum_{k=1}^{N} c_i^k A_k\right) + A_i^{\mathrm{T}} P_N A_N + A_N^{\mathrm{T}} P_N A_i + \\ &\left(A_i^{\mathrm{T}} A_N^{\mathrm{T}} - \sum_{k=1}^{N} c_i^k A_k^{\mathrm{T}}\right) P_N + c_i^i \left(P_N A_i + A_i^{\mathrm{T}} P_N\right) \\ &= (P_N A_N + A_N^{\mathrm{T}} P_N) A_i + A_i^{\mathrm{T}} (P_N A_N + A_N^{\mathrm{T}} P_N) - \\ &c_i^N (P_N A_N + A_N^{\mathrm{T}} P_N) - \sum_{k=1,k\ne i}^{N-1} c_i^k (P_N A_k + A_k^{\mathrm{T}} P_N) \\ &= -P_{i,N-1} + c_i^N P_{N-1} - \sum_{k=1,k\ne i}^{N-1} c_i^k P_{k,N} > 0\end{aligned}$$

Using the first in equality of (14) we know that $A_N + \frac{c_i^i}{2} I$ is stable, so it follows that

$$P_{i,N} = P_N A_i + A_i^{\mathrm{T}} P_N < 0, \quad i = 1, \cdots, N-1$$

That is, P_N is a CQLF for $\{\mathcal{A}\}$. □

Consider the set $\mathcal{A}$ without the assumption (13), denoted by

$$[A_N, A_j] = C_{N,j}, \quad 1 \le j \le N \tag{15}$$

Using the similar argument in the proof of Theorem 2, we have

Theorem 3. Consider the set $\mathcal{A}$ of stable matrices, $C_{N,j}$, $1 \le j \le N$, defined in (15) and P_{N-1} and P_N defined in (11).

$$P_{i,N-1} + P_N C_{N,i} + C_{N,i}^{\mathrm{T}} P_N < 0, \quad i = 1, \cdots, N-1 \tag{16}$$

Then the P_N constructed in (4) is a CQLF of $\mathcal{A}$.

Example 1. Consider the following three stable matrices

$$A_1 = \begin{pmatrix} -1 & -3 & 2 & 0 \\ -1.8 & 0 & -0.9 & 2 \\ -1 & 6 & -5 & 1 \\ 4.8 & -3.2 & 2.9 & -5.1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1.65 & -4.8 & 2.85 & -0.3 \\ -3. & 0.15 & -1.4 & 3.15 \\ -1.35 & 9.6 & -7.35 & 2.1 \\ 7.5 & -4.65 & 4.3 & -7.65 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -2.2 & -6.5 & 3.8 & -0.5 \\ -4.0 & 0.2 & -1.9 & 4.2 \\ -1.8 & 13. & -9.8 & 3.0 \\ 10 & -6.2 & 5.8 & -10.2 \end{pmatrix}$$

Setting $P_0 = I_4$ and using N-B structure, we can calculate positive definite matrices P_1, P_2 and P_3 sequentially as

$$P_1 = \begin{pmatrix} 0.9582 & -0.6434 & 0.4046 & -0.0615 \\ -0.6434 & 1.5362 & -0.1591 & 0.4611 \\ 0.4046 & -0.1591 & 0.3062 & 0.0271 \\ -0.0615 & 0.4611 & 0.0271 & 0.2842 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0.3931 & -0.2162 & 0.2000 & -0.0279 \\ -0.2162 & 0.7119 & -0.0561 & 0.2954 \\ 0.2000 & -0.0561 & 0.1153 & 0.0106 \\ -0.0279 & 0.2954 & 0.0106 & 0.1442 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0.1147 & -0.0193 & 0.0640 & 0.0094 \\ -0.0193 & 0.1881 & 0.0034 & 0.0908 \\ 0.0640 & 0.0034 & 0.0373 & 0.0122 \\ 0.0094 & 0.0908 & 0.0122 & 0.0476 \end{pmatrix}$$

It is easy to check that

$$P_{1,2} + P_3 C_{3,1} + C_{3,1}^{\mathrm{T}} P_3 < 0$$

$$P_{2,2} + P_3 C_{3,2} + C_{3,2}^{\mathrm{T}} P_3 < 0$$

So (16) follows. Thus P_3 is a CQLF of A_1, A_2, A_3.

4 CQLF for Growing Set

In this section we assume that set $\mathcal{A}$ has a CQLF P, and enlarge the set to $\{\mathcal{A}, B\}$, where $\mathcal{A} := \{A_\lambda |\, \lambda \in \Lambda\}$ with $\Lambda = \{1, \cdots, N\}$ being the set of stable matrices. Then we try to construct a CQLF for the enlarged set. We assume $\mathcal{A}$ has a CQLF M. That is

$$MA_\lambda + A_\lambda^{\mathrm{T}} M := -Q_\lambda, \quad \forall \lambda \in \Lambda \tag{17}$$

where $Q_\lambda > 0$, $\lambda \in \Lambda$, are positive definite matrixes. Construct P by

$$PB + B^{\mathrm{T}} P = -M \tag{18}$$

First, we consider the case where $[B, \mathcal{A}_e] \subset \mathcal{A}_e$, where $\mathcal{A}_e = \mathrm{Span}\{B, \mathcal{A}\}$. That is

$$[A_i, B] = \sum_{j=1}^{N} \alpha_j^i A_j + \beta^i B, \quad i = 1, \cdots, N \tag{19}$$

Theorem 4. Assume (19) and the following two conditions are satisfied

$$\begin{aligned} &\alpha_\lambda^\lambda > -2\min_\sigma |\sigma(B)|, \forall \lambda \in \Lambda \\ &Q_\lambda - \beta^\lambda M + \sum_{j=1,\, j\neq\lambda}^{N} \alpha_j^\lambda \left(PA_j + A_j^{\mathrm{T}} P\right) > 0 \end{aligned} \tag{20}$$

Then $\{B, \mathcal{A}\}$ share a CQLF P, which is defined in (18).

Proof. Applying (18), we have

$$(PB + B^{\mathrm{T}} P)A_\lambda + A_\lambda^{\mathrm{T}} (PB + B^{\mathrm{T}} P) = Q_\lambda \tag{21}$$

Using (19), a straightforward computation shows that (21) can be rewritten as

$$\begin{aligned} &(PA_\lambda + A_\lambda^{\mathrm{T}} P)B + B^{\mathrm{T}} (PA_\lambda + A_\lambda^{\mathrm{T}} P) - \\ &\sum_{j=1}^{N} \alpha_j^\lambda (PA_j + A_j^{\mathrm{T}} P) = Q_\lambda - \beta^\lambda M, \quad \forall \lambda \in \Lambda \end{aligned} \tag{22}$$

Now the second in equality of (20) assures that

$$(PA_\lambda + A_\lambda^{\mathrm{T}} P) < 0, \quad \forall \lambda \in \Lambda \tag{23}$$

□

In general, without assumption (19), we denote

$$[A_\lambda, B] = C_\lambda, \quad \forall \lambda \in \Lambda \tag{24}$$

Using the similar argument, it is easy to prove that

Proposition 3. $\{B, \mathcal{A}\}$ share a CQLF P as defined in (18) if

$$Q_\lambda + \left(PC_\lambda + C_\lambda^{\mathrm{T}} P\right) > 0, \quad \forall \lambda \in \Lambda \tag{25}$$

5 Conclusion

The N-B method for constructing CQLF has been generalized to the case where the set of stable matrices are not commutative. The generalized constructing methods have been proposed. Some conditions have been obtained to assure the resulting matrix being CQLF.

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