A priori prejudice in Weyl's unintended unification of gravitation and electricity

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1 Introduction

It is almost always claimed[1] that Weyl *deliberately* unified gravitation and electricity in the rectification of general relativity he attempted in 1918. In fact the unification, as Bergia [5] and Ryckman [6] have pointed out and a couple of passages[2,3] show, was the unintended outcome of apparently gratuitous *a priori*[4] prejudice.[5] But what prejudice?

The evidence suggests that the theory came straight out of Weyl's sense of mathematical 'justice,' which led him to put the direction and length of a vector on an equal footing. Levi-Civita [9] had discovered that the parallel transport determined by Einstein's covariant derivative was not integrable—while length, far from depending on the path taken, remained unaltered.[6] For Weyl this was unfair:

[1] Certainly by Folland [1], Trautman [2], Perlick [3], Vizgin [4] and others. [2] Weyl [7] pages 148-9: Indem man die erwähnte Inkonsequenz beseitigt, kommt eine Geometrie zustande, die überraschenderweise, auf die Welt angewendet, *nicht nur die Gravitationserscheinungen, sondern auch die des elektromagnetischen Feldes erklärt*. [3] Weyl [8]: Übrigens müssen Sie nicht Glauben, daß ich von der Physik her dazu gekommen bin, neben der quadratische noch die lineare Differentialform in die Geometrie einzuführen; sondern ich wollte wirklich diese "Inkonsequenz," die mir schon immer ein Dorn im Auge gewesen war, endlich einmal beseitigen und bemerkte dann zu meinem eigenen erstaunen: das sieht so aus, als erklärt es die Elektrizität. [4] As opposed to "experimentally founded" or even "empirically justified" (with respect to the past; *a posteriori* justification is of course another matter). *A priori* considerations can be æsthetic or mathematical, for instance. [5] I say "prejudice"—and not "principle" or "assumption," for instance—to emphasize the unexpected, gratuitous, almost unaccountable character of the considerations. [6] For there are three cases: length is left unchanged; or length varies, but integrably; or length varies, in a way that depends on the path taken.

both features deserved the same treatment.[7] He remedied with a connection that made *congruent* transport (of length) just as path-dependent as parallel transport. This 'total' connection restored justice through a *length connection* it included, an inexact one-form Weyl couldn't help identifying with the electromagnetic potential A,[8] whose curl $F = dA$, being closed (for $dF = d^2A$ vanishes everywhere), provides Maxwell's two homogeneous equations. Source-free electromagnetism (up to Hodge duality at any rate) thus came, quite unexpectedly, out of Weyl's surprising sense of mathematical justice.

Admittedly there were also intimations,[9] from the beginning, announcing an 'infinitesimal' agenda of sorts; but it was largely unmotivated back then, and too vague to produce the theory on its own—in fact it may even have been *suggested by* the theory. The programme would take shape over the next years, acquiring justification and grounding; Ryckman has found roots in Husserl, and connected, or even identified it with a 'telescepticism' opposed to distant comparisons. But his compelling reconstruction of the agenda and its philosophical background rests largely on a couple of texts (footnotes 24 and 25) from a subsequent 'context of justification.'

'Mathematical justice' has, in the context of discovery, a more conspicuous (though sometimes, as we shall see in 3.1, thinly disguised) presence than the infinitesimal agenda. It is also logically stronger, being enough—together with a handful of simple and natural operations—to yield all of source-free electromagnetism. So I contend that what was really at work in the spring of 1918, what effectively gave rise to Weyl's theory of gravitation and electricity, was the equal rights of direction and length.

[7] Weyl [7] page 148: [. . .] und es ist dann von vornherein ebensowenig anzunehmen, daß das Problem der Längenübertragung von einem Punkte zu einem endlich entfernten integrabel ist, wie sich das Problem der Richtungsübertragung als integrabel herausgestellt hat. [8] Providing the geometrical objects of which Weyl gives only the components (except in words, as we are about to see) may seem anachronistic. But Weyl—who for instance writes "[. . .] die g_{ik} [. . .] bilden die Komponenten des Gravitationspotentials. [. . .] einem Viererpotential [. . .] dessen Komponenten φ_i [. . .]" ([7] page 148)—undeniably sees the geometry behind the components, and indeed even *refers to* the geometrical objects I denote with single letters, like A (Viererpotential) or g (Gravitationspotential). [9] Weyl [7] page 148, for instance: In der oben charakterisierten Riemannschen Geometrie hat sich nun ein letztes ferngeometrisches Element erhalten [. . .]. Or (same page): Eine Wahrhafte Nahe-Geometrie darf jedoch nur ein Prinzip der Übertragung einer Länge von einem Punkt zu einem unendlich benachbarten kennen [. . .].

2 Background: Einstein, Levi-Civita

We can begin with aspects of Einstein's theory of gravitation, since Weyl's theory grew out of it. What interests us above all is affine structure, given by the Christoffel symbols Γ^a_{bc}. Through the geodesic equation

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{1}$$

and the wordlines satisfying it, the Christoffel symbols provide a notion of (parametrised[10]) straightness, of inertial, unaccelerated motion, of free fall.

The left-hand side of (1) gives the components $\langle dx^a, \nabla_{\dot\sigma}\dot\sigma\rangle$ of the covariant derivative $\nabla_{\dot\sigma}\dot\sigma$ of the vector $\dot\sigma$ with components $dx^a/ds = \langle dx^a, \dot\sigma\rangle$, in the direction $\dot\sigma$ tangent to the worldline

$$\sigma : I \to M : s \mapsto \sigma(s)$$

with coordinates $\sigma^a(s) = x^a(\sigma(s))$, where I is an appropriate interval and M the differential manifold representing the universe; $a = 0, \ldots, 3$. The Christoffel symbols are related to ∇ by

$$\Gamma^a_{bc} = \langle dx^a, \nabla_{\partial_b}\partial_c\rangle,$$

where the basis vectors $\partial_a = \partial_{a(x)} = \partial/\partial x^a$ are tangent to the coordinate lines of the system x^a.

Einstein only appears to have explored the *infinitesimal* behaviour of the parallel transport determined by his covariant derivative. It was Levi-Civita [9] who first understood that if ∇g vanishes, as in Einstein's theory, the direction of the vector $V_s \in T_{\sigma(s)}M$ transported according to $\nabla_{\dot\sigma}V_s = 0$ depends[11] on the path σ taken—whereas the squared length $l_s = g(V_s, V_s)$ remains constant along σ, for

$$\nabla g = 0 = \nabla_{\dot\sigma}V_s$$

means that $dl_s/ds = \nabla_{\dot\sigma}l_s$ vanishes.

[10] For (1) determines an equivalence class $[s]$ of affine parameters, each parameter of which gives the proper time of a regular clock, with its own zero and unit of time. The parameters belonging to $[s]$ are related by affine transformations $s \mapsto \upsilon s + \zeta$, where the constants υ and ζ give the unit and zero. The constant υ is typically chosen so that $g(\partial_0, \partial_0) = 1$, where g is the metric tensor and ∂_0 the timelike basis vector. [11] Page 175: La direzione parallela in un punto generico P ad una direzione (α) uscente da un altro punto qualsiasi P_0 dipende in generale dal cammino secondo cui si passa da P_0 a P. L'indipendenza dal cammino è proprietà esclusiva delle varietà euclidee.

3 The emergence of Weyl's theory

3.1 The equal rights of direction and length

Weyl felt that as parallel transport depended on the path taken, congruent transport ought to as well. In fact his generalisation of Einstein's theory appears to have been *entirely determined* by the intention of putting direction and length on an equal footing. The following table[12]—parts of which may for the time being be more intelligible than others—outlines Weyl's programme.

DIRECTION	**LENGTH**
coordinates (up to gauge)	gauge
parallel transport	congruent transport
gravitation	electricity
Levi-Civita connection Γ^a_{bc}	length connection A
$\delta V^a = -\Gamma^a_{bc} X^b V^c$	$\delta l = -\langle \alpha, X \rangle = -A_b X^b l$
directional curvature R^a_{bcd} (of Γ^a_{bc})	length curvature $F = dA$
geodesic coord. y^a (at P): $\Gamma^a_{bc} = 0$	geod. gauge (at P): $A' = A + d\lambda = 0$
equival. principle: $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c \mapsto \ddot{y}^a$	equiv. principle: $\alpha = -lA \mapsto \alpha' = 0$

A few words about "coordinates (up to gauge)." The parallel between coordinates and gauge, which Weyl draws[13,14,15,16] over and over, can be seen as a parallel between direction and length. Surely Weyl does not mean "coordinates *including gauge*, as opposed to gauge," for that would be redundant. And up to gauge,

[12] Parts of it were inspired by Coleman and Korté [10] pages 204-5 and 211-2. [13] Weyl [7]: Die auftretenden Formeln müssen dementsprechend eine doppelte Invarianzeigenschaft besitzen: 1. sie müssen *invariant* sein *gegenüber beliebigen stetigen Koordinatentransformationen*, 2. sie müssen ungeändert bleiben, *wenn man die g_{ik} durch λg_{ik} ersetzt*, wo λ eine willkürliche stetige Ortsfunktion ist. [14] Weyl [11] page 396: Zum Zwecke der analytischen Darstellung denken wir uns 1. ein bestimmtes Koordinatensystem und 2. den an jeder Stelle willkürlich zu wählenden Proportionalitätsfaktor im skalaren Produkt festgelegt; damit ist ein "*Bezugssystem*"[9] für die analytische Darstellung gewonnen. And footnote 9: Ich unterscheide also zwischen "Koordinatensystem" und "Bezugssystem." [15] Weyl [11] page 398: In alle Größen oder Beziehungen, welche metrische Verhältnisse analytisch darstellen, müssen demnach die Funktionen g_{ik}, φ_i in solcher Weise eingehen, daß Invarianz stattfindet 1. gegenüber einer beliebigen Koordinatentransformation ("Koordinaten-Invarianz") und 2. gegenüber der Ersetzung von (7) durch (8) ("Maßstab-Invarianz"). [16] Weyl [12] page 101: Um den physikalischen Zustand der Welt an einer Weltstelle durch Zahlen charakterisieren zu können, muß 1. die Umgebung dieser Stelle auf *Koordinaten* bezogen sein und müssen 2. gewisse *Maßeinheiten* festgelegt werden. Die bisherige Einsteinsche Relativitätstheorie bezieht sich nur auf den ersten Punkt, die Willkürlichkeit des Koordinatensystems; doch gilt es, eine ebenso prinzipielle Stellungnahme zu dem zweiten Punkt, der Willkürlichkeit der Maßeinheit, zu gewinnen.

coordinates provide no more than direction: The coordinates x^a assign to each event $P \in M$ a basis $\partial_a \in T_P M$, and a dual basis

$$dx^a = g^\flat(\partial_a) = g(\partial_a, \cdot\,) \in T_P^* M$$

providing the components $V^a = \langle dx^a, V \rangle$ of any vector $V \in T_P M$; $a = 0, \ldots, 3$. The gauge transformation $g \mapsto e^{2\lambda} g$ induces a transformation $V \mapsto e^\lambda V$, or $V^a \mapsto e^\lambda V^a$, through

$$e^{2\lambda} g(V, V) = g(e^\lambda V, e^\lambda V) = g(e^\lambda \partial_a, e^\lambda \partial_b) V^a V^b = g(\partial_a, \partial_b) e^\lambda V^a e^\lambda V^b.$$

Direction, given by the ratios

$$e^\lambda V^0 : e^\lambda V^1 : e^\lambda V^2 : e^\lambda V^3 = V^0 : V^1 : V^2 : V^3,$$

remains unaffected.

Weyl clearly distinguishes between a 'stretch' (like a *stretch* of road) and its numerical length, determined by the gauge chosen. Just as a direction $[e^\lambda V]_{(\text{all } \lambda)}$ is 'expanded' with respect to a coordinate system, which provides its numerical representation (the ratios $V^0 : \cdots : V^3$), a stretch gets 'expanded' in a gauge, which likewise gives a numerical representation, the squared length $l = e^{2\lambda} g(V, V)$.

The rest of the table should in due course become clearer. Let us now see how the inexact one-form A, from which so much of electromagnetism can be derived, emerges from the equal rights of direction and length.

3.2 Electromagnetism from equal rights

Weyl calls a manifold M *affinely connected* if the tangent space $T_P M$ at every point $P \in M$ is connected to all the neighbouring tangent spaces $T_{P'} M$ by a mapping

$$\Xi_X : T_P M \to T_{P'} M : V_P \mapsto V_{P'} = \Xi_X V_P$$

linear both in the 'main' argument $V_P \in T_P M$ and in the (short)[17] directional argument $X = P' - P$, where P' (being near P) and hence X are viewed as lying in $T_P M$. Being linear, Ξ_X will be represented by a matrix, in fact by

$$\Xi^a_c = \langle dx^a, \Xi_X \partial_c \rangle = \Xi^a_{bc} X^b = \langle dx^a, \Xi_{\partial_b} \partial_c \rangle \langle dx^b, X \rangle.$$

[17] The necessary shortness of X seems inconsistent with linearity, which would 'connect' P with the entire tangent space $T_P M$ and not just with the small neighbourhood 'covering' M. In this context it may be best to view the linearity in the directional argument as being appropriately restricted (of course the length of X does not matter in differentiation, in which limits are taken).

Weyl specifically refers to the components $\delta V^a = \langle dx^a_{P'}, V_{P'}\rangle - \langle dx^a_P, V_P\rangle$, requiring them to be linear in the components X^b and $V^c_P = \langle dx^c_P, V_P\rangle$. The bilinear function $\Gamma^a(\{X^b\},\{V^c\}) = \delta V^a$ will be a matrix, represented by Γ^a_{bc}; the difference δV^a is therefore $-\Gamma^a_{bc}X^bV^c$.

With respect to the *geodesic* coordinates y^a which make

$$\Gamma^a_c = \Gamma^a_{bc}X^b = \langle dy^a, \nabla_X \partial_{c(y)}\rangle$$

and δV^a vanish, leaving the components V^a unchanged, Ξ^a_c becomes the identity matrix $\delta^a_c = \langle dy^a, \Xi_X \partial_{c(y)}\rangle \leftrightarrow \mathrm{diag}(1,1,1,1)$. Physically this has to do with the equivalence principle, according to which a gravitational field Γ^a_{bc} can always be eliminated or generated at P by an appropriate choice of coordinates.

With equal rights in mind Weyl turns to length, using the very same scheme. To clarify his procedure we can take just a single component of the difference $\{\delta V^0, \ldots, \delta V^3\}$, calling it δl (this will be the 'squared-length-difference scalar').[18] The upper index of Γ^a_{bc} accordingly disappears, leaving $\delta l = \Gamma_{bc}X^bV^c$.[19] If we now take a single component of the main argument $\{V^0, \ldots, V^3\}$, calling it l (this will be the squared length), the second index of Γ_{bc} disappears as well, and we are left with $\delta l = \Gamma_b X^b l$, where $\Gamma_b = \langle A, \partial_b\rangle$ are the components of a one-form,[20] denoted A with electricity in mind.

But this is not really Weyl's argument, which is better conveyed as follows. The object A generating the squared-length-difference scalar δl has to be linear in the squared length l and the direction X. A linear function $A(l, X) = \delta l$ of a scalar l and vector X yielding a scalar δl will be a one-form:[21]

$$\begin{aligned}\delta l &= -\langle \alpha, X\rangle = -\langle \alpha, \partial_b\rangle\langle dx^b, X\rangle = -\alpha_b X^b \\ &= -\langle A, X\rangle l = -\langle A, \partial_b\rangle\langle dx^b, X\rangle l = -A_b X^b l,\end{aligned}$$

[18] Weyl seems to use d and δ interchangeably, and d in a way—see footnote 17—that is unusual not only today, but even then. He does not distinguish between the *scalar* representing the difference in squared length, and the corresponding *one-form* (as we would call it); but the distinction seems useful. [19] One can perhaps think of the hybrid, intermediate connection Γ_{bc} as being something like $\langle A, \nabla_{\partial_b}\partial_c\rangle$. [20] One may wonder how the tensor A can be the counterpart of the connection Γ^a_{bc}, which is not a tensor. The components $A_a = \langle A, \partial_a\rangle = \Gamma_a$ only transform as a tensor with respect to coordinate transformations $A_a \mapsto \bar{A}_b = A_a\langle d\bar{x}^b, \partial_{a(x)}\rangle$, however; with respect to recalibration $A_a \mapsto A'_a = A_a + \partial_a\lambda$ the components A_a do not transform 'tensorially,' and can be locally cancelled, for instance. [21] Weyl in fact writes $dl = -l d\varphi$, whereas I write $\alpha = -lA$. The misleading d's cannot be understood globally—or even locally, in the theory of gravitation and electricity, in which $F = d^2\varphi$ will be the Faraday two-form: where $d\varphi$ is closed, in other words the differential (even on a small neighbourhood) of a function φ, there would be no electromagnetism.

where α is the squared-length-difference one-form. An exact one-form $A = d\mu$ would make congruent transfer integrable, removing the dependence of the recalibration

$$e^{\int_\gamma A} = e^{\int d\mu} = e^{\Delta\mu}$$

on the path $\gamma : [0, 1] \to M$, where $\Delta\mu = \mu_1 - \mu_0$ is the difference between the values $\mu_1 = \mu(P_1)$ and $\mu_0 = \mu(P_0)$ of μ at $P_1 = \gamma(1)$ and $P_0 = \gamma(0)$. Mathematical justice therefore demands that A be inexact, and that the curl $F = dA$ accordingly not vanish everywhere.

Confirmation that A has to be one-form, possibly inexact, is provided by Weyl's requirement that the squared-length-difference one-form $\alpha = -lA$ be eliminable at any point P by recalibration.[22] As l is given (and does not vanish), this amounts to $A + d\lambda = 0$ at P, where the gauge λ is *geodesic*.[23] Since $d\lambda$ is a one-form, A must be one too. Though $d\lambda$ is exact, Weyl only asks that it cancel A *at* P—so A needn't even be *closed*, or locally exact.

With $F = dA$ and its consequence $dF = 0$ before him Weyl couldn't help seeing the electromagnetic four-potential A, the Faraday two-form $F = dA$ (which vanishes wherever A is closed) and Maxwell's two homogeneous equations,[24] expressed by $dF = 0$—not to mention an electromagnetic 'equivalence principle' according to which the squared-length-difference scalar δl and one-form α, as well as the electromagnetic four-potential A, can be eliminated or generated at a point by the differential $d\lambda$ of an appropriate gauge λ.

In coordinates

$$F_{ab} = F(\partial_a, \partial_b) = \partial_a A_b - \partial_b A_a \leftrightarrow \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

where E_x, E_y, E_z are the components of the electric field and B_x, B_y, B_z those of the magnetic field. Or $F = F_{ab}\, dx^a \wedge dx^b/2$. And the vanishing three-form

$$dF = \frac{1}{2} dF_{bc} \wedge dx^b \wedge dx^c = \frac{1}{6} \partial_a F_{bc}\, dx^a \wedge dx^b \wedge dx^c$$

[22] Weyl [13] page 122: Ein Punkt P hängt also mit seiner Umgebung metrisch zusammen, wenn von jeder Strecke in P feststeht, welche Strecke aus ihr durch kongruente Verpflanzung von P nach dem beliebigen zu P unendlich benachbarten Punkte P' hervorgeht. Die einzige Forderung, welche wir an diesen Begriff stellen (zugleich die weitgehendste, die überhaupt möglich ist), ist diese: Die Umgebung von P läßt sich so eichen, daß die Maßzahl einer jeden Strecke in P durch kongruente Verpflanzung nach den unendlich benachbarten Punkten keine Änderung erleidet.
[23] By analogy one might even call it 'inertial' or 'unaccelerated.'
[24] In full, $\nabla \cdot B = 0$ and $\nabla \times E + \partial B/\partial t = 0$.

has components $dF(\partial_a, \partial_b, \partial_c) = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab}$.

Maxwell's other two equations are obtained, in 'source-free' form, by setting $d{}^*F$ equal to zero, where ${}^*F$ is the Hodge dual of the Faraday two-form, with coordinates

$$({}^*F)_{ab} = ({}^*F)(\partial_a, \partial_b) \leftrightarrow \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}.$$

Electromagnetism thus emerged, altogether unexpectedly, from the equal rights of direction and length.

3.3 The illegitimacy of distant comparisons

Weyl has another *a priori* prejudice, rooted, as Ryckman [6] has cogently argued, in Husserl. It is expressed in two similar passages,[25,26] which roughly say: As the curvature $R(P)$ is subtle and hard to perceive directly, a "cognizing ego" at the "ego center" $P \in M$ takes itself to be immersed in the 'psychologically privileged' tangent space $T_P M$. The universe M resembles $T_P M$ in the immediate vicinity $\mathfrak{U}$ of P, where they practically coincide, and 'cover' one another. Beyond $\mathfrak{U}$ the relation between M and the 'intuitive' space $T_P M$ grows looser, as the universe goes its own way, bending as the energy-momentum tensor T varies.

Ryckman writes (page 148) that

[25] Weyl [14] page 173: Erkennt man neben dem physischen einen *Anschauungsraum* an und behauptet von ihm, daß seine Maßstruktur aus Wesensgründen die euklidischen Gesetze erfülle, so steht dies mit der Physik nicht in Widerspruch, sofern sie an der euklidischen Beschaffenheit der *unendlich kleinen Umgebung* eines Punktes O (in dem sich das Ich momentan befindet) festhält [...]. Aber man muß dann zugeben, daß die Beziehung des Anschauungsraumes auf den physischen um so vager wird, je weiter man sich vom Ichzentrum entfernt. Er ist einer Tangentenebene zu vergleichen, die im Punkte O an eine krumme Fläche, dem physischen Raum, gelegt ist: in der unmittelbaren Umgebung von O decken sich beide, aber je weiter man sich von O entfernt, um so willkürlicher wird die Fortsetzung dieser Deckbeziehung zu einer eindeutigen Korrespondenz zwischen Ebene und Fläche. [26] Weyl [15] page 52: Die Philosophen mögen recht haben, daß unser Anschauungsraum, gleichgültig, was die physikalische Erfahrung sagt, euklidische Struktur trägt. Nur bestehe ich allerdings dann darauf, daß zu diesem Anschauungsraum das Ich-Zentrum gehört und daß die Koinzidenz, die Beziehung des Anschauungsraumes auf den physischen um so vager wird, je weiter man sich vom Ich-Zentrum entfernt. In der theoretischen Konstruktion spiegelt sich das wider in dem Verhältnis zwischen der krümmen Fläche und ihrer Tangentenebene im Punkte P: beide decken sich in der unmittelbaren Umgebung des Zentrums P, aber je weiter man sich von P entfernt, um so willkürlicher wird die Fortsetzung dieser Deckbeziehung zu einer eindeutigen Korrespondenz zwischen Fläche und Ebene.

> Weyl restricted the homogeneous space of phenomenological intuition, the locus of phenomenological *Evidenz*, to what is given at, or neighboring, the cognizing ego [...]. But in any case, by delimiting what Husserl termed "the sharply illuminated circle of perfect givenness," the domain of "eidetic vision," to the infinitely small homogeneous space of intuition surrounding the "ego-center" [...]

This restriction or delimitation can be understood in two ways: directly, in terms of the limitations of our senses, and of an accordingly circumscribed domain of sensory access; or more mathematically, as follows: The conscience attaches a kind of intuitive 'certainty' to all of $T_P M$, which, being flat and homogeneous,[27] can be captured or 'understood' in its entirety once any little piece is. The universe shares that certainty as long as it resembles $T_P M$, and hence only in $\mathfrak{U}$, outside of which it is subject to all sorts of unforeseeable variations. Integrable congruent propagation had to be rejected as allowing the certain comparison of lengths well beyond $\mathfrak{U}$, indeed at any distance, without the welcome ambiguities related to the path followed. Returning to Ryckman (page 149):

> Guided by the phenomenological methods of "eidetic insight" and "eidetic analysis", the epistemologically privileged purely infinitesimal comparison relations of *parallel transport* of a vector, and the *congruent displacement* of vector magnitude, will be the foundation stones of Weyl's reconstruction. The task of comprehending "the sense and justification" of the mathematical structures of classical field theory is accordingly to be addressed through a construction or *constitution* of the latter within a world geometry entirely built up from these basic geometrical relations immediately evident within a purely infinitesimal space of intuition. A wholly *epistemological* project, it nonetheless coincides with the explicitly *metaphysical* aspirations of Leibniz and Riemann to "understand the world from its behaviour in the infinitesimally small."

3.4 The two prejudices

Removed from the context of Weyl's theory, the two prejudices are entirely distinct. While one is markedly infinitesimal, 'mathematical justice' has nothing (necessarily) infinitesimal about it: in a spirit of equal rights one could require,

[27] Curvature (which vanishes everywhere) and the metric are constant.

for instance, both the directions and lengths of the vectors in some set to have the same kind of distribution—uniform, say, or Gaussian—around a given vector. Nothing infinitesimal about that.

An abundant insistence in the early going on the equal rights of direction and length, together with the absence, back then, of any developed, articulated expression of the telescepticism of footnotes 24 and 25, suggests the following account. First, then, there was mathematical justice, which, far from being at odds with Weyl's nascent telescepticism, supported and perhaps even suggested it. In due course Weyl's 'purely infinitesimal geometry' acquired more explicit philosophical grounding—expressed in footnotes 24 and 25 and rooted in the thought of Husserl—which can in hindsight be viewed as justifying and motivating the surprising, apparently gratuitous early insistence on equal rights.

4 Compensating transformations

We have seen how Weyl's theory, building on general relativity, came out of the inexact one-form A—whose transformations

$$A \mapsto A' = A + d\mu \tag{2}$$

are counterbalanced in the theory by

$$g \mapsto g' = e^{\mu} g, \tag{3}$$

leaving length unaltered. Such compensation is fundamental enough to be worth looking at briefly.

Freedom to transform A according to (2) is left by the length curvature $F = dA$, which is indifferent to an exact term $d\mu$, as

$$F = dA' = dA + d^2\mu = dA.$$

But (2) does change length. Transporting the vector X_0 from point P_0 with value $\mu_0 = \mu(P_0)$ to point P_1 with value $\mu_1 = \mu(P_1)$, the final squared length $g_1(X_1, X_1)$ acquires the additional (integrable) factor $e^{\Delta\mu}$, where $\Delta\mu = \mu_1 - \mu_0$. For μ recalibrates, along a curve γ, according to

$$e^{\int_\gamma A} \mapsto e^{\int_\gamma A'} = e^{\int_\gamma (A + d\mu)} = e^{\int_\gamma A} e^{\Delta\mu} = e^{\int_\gamma A} e^{\mu_1} e^{-\mu_0} \neq e^{\int_\gamma A},$$

and therefore

$$g_1(X_1, X_1) = e^{\int_\gamma A} g_0(X_0, X_0) \neq e^{\int_\gamma A'} g_0(X_0, X_0).$$

But the conformal transformation (3) compensates, leaving length unchanged:

$$g_1'(X_1, X_1) = e^{\mu_1} g_1(X_1, X_1) = e^{\int_\gamma A'} g_0'(X_0, X_0) = e^{\int_\gamma A} e^{\Delta\mu} e^{\mu_0} g_0(X_0, X_0)$$

The exponents cancel, yielding the original dilation

$$g_1(X_1, X_1) = e^{\int_\gamma A} g_0(X_0, X_0).$$

The metric g is *compatible* with the covariant derivative ∇ if ∇g vanishes, in which case the straightest worldlines (satisfying $\nabla_{\dot\sigma}\dot\sigma = 0$) will also be stationary, satisfying

$$\delta \int \sqrt{g(\dot\sigma, \dot\sigma)}\, ds = \delta \int ds = 0$$

too. The covariant derivative of the recalibrated metric g' only vanishes if μ is a constant (for then $d\mu$ vanishes); otherwise

$$\nabla g' = d\mu \otimes g',$$

which combines (2) and (3), to express the weaker *Weyl compatibility*.

5 Einstein's objection

Out of a sense of mathematical justice, then, Weyl made congruent displacement just as path-dependent as parallel transport. But experience, objected Einstein, is unfair, showing congruent dispacement to be integrable. In a letter to Weyl dated April 15th (1918) he argued[28] that clocks running at the same rate at one point will *continue* to run at the same rate at another point, however they get there—whatever the requirements of mathematical justice. Four days later he reformu-

[28] So schön Ihre Gedanke ist, muss ich doch offen sagen, dass es nach meiner Ansicht ausgeschlossen ist, dass die Theorie die Natur entspricht. Das ds selbst hat nämlich reale Bedeutung. Denken Sie sich zwei Uhren, die relativ zueinander ruhend neben einander gleich rasch gehen. Werden sie voneinander getrennt, in beliebiger Weise bewegt und dann wieder zusammen gebracht, so werden sie wieder gleich (rasch) gehen, d. h. ihr relativer Gang hängt nicht von der Vorgeschichte ab. Denke ich mir zwei Punkte P_1 & P_2 die durch eine Zeitartige Linie verbunden werden können. Die an P_1 & P_2 anliegenden zeitartigen Elemente ds_1 und ds_2 können dann durch mehrere zeitartigen Linien verbunden werden, auf denen sie liegen. Auf diesen laufende Uhren werden ein Verhältnis $ds_1 : ds_2$ liefern, welches von der Wahl der verbindenden Kurven unabhängig ist.—Lässt man den Zusammenhang des ds mit Massstab- und Uhr-Messungen fallen, so verliert die Rel. Theorie überhaupt ihre empirische Basis.

lated[29] the objection in terms of the 'proper frequencies' of atoms (rather than genuine macroscopic clocks) "of the same sort": if such frequencies depended on the path followed, and hence on the (electromagnetic) vicissitudes of the atoms, the chemical elements the atoms would make up if brought together would not have the clean spectral lines one sees.

But even if experience shows congruent displacement to be integrable, it would be wrong to conclude that the equal rights of direction and length led nowhere; for the structure that came out of Weyl's ostensibly groundless sense of mathematical justice would survive in our standard gauge theories, whose accuracy is much less questionable.

6 Final remarks

There are various levels of 'experience,' ranging from the most concrete to the most abstract: from the most obvious experimental level, having to do with the results of particular experiments, to principles, perhaps even instincts, distilled from a lifetime of experience. One such principle could be Einstein's "I am convinced that God does not play dice," to which, having—we may conjecture—noticed that the causal regularities behind apparent randomness eventually tend to emerge, he may ultimately have been led by experience: by his own direct experience, together with his general knowledge of science and the world. One would nonetheless hesitate to view so general and abstract a principle as being *a posteriori*, empirical. It is clearly not *a posteriori* with respect to any particular experiment; only, if at all, with respect to a very loose, general and subjective kind of ongoing experience, capable of being interpreted in very different ways.

An unexpected empirical fertility of apparently *a priori* and unempirical prejudice can sometimes be accounted for in terms of a derivation, however indirect, from experience: by attributing remote empirical roots to considerations which at first seem to have nothing at all to do with experience. Fair enough, the world can be experienced in very different ways, some much less obvious and straightforward than others. But here we have a prejudice which—however subtle and

[29] [...] wenn die Länge eines Einheitsmassstabes (bezw. die Gang-Geschwindigkeit einer Einheitsuhr) von der Vorgeschichte abhingen. Wäre dies in der Natur wirklich so, dann könnte es nicht chemische Elemente mit Spektrallinien von bestimmter Frequenz geben, sondern es müsste die relative Frequenz zweier (räumlich benachbarter) Atome der gleichen Art im Allgemeinen verschieden sein. Da dies nicht der Fall ist, scheint mir die Grundhypothese der Theorie leider nicht annehmbar, deren Tiefe und Kühnheit aber jeden Leser mit Bewunderung erfüllen muss.

developed one's faculties for interpreting experience—seems to be completely unempirical. Perhaps the empirical shortcomings of the theory are best blamed, then, on the totally unempirical nature of the prejudice from which it was derived.

Or is it so completely unempirical? The principle of sufficient reason comes to mind: if there is a difference, an imbalance, there had better be a reason for it—failing which, balance, or rather justice should prevail. Even Einstein's dice may come to mind: If a situation of apparent balance, such as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |\beta\rangle) \tag{4}$$

gives rise to an imbalance (as it must, if a measurement is made), such as the eigenvalue $+1$ of the operator $A = |\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|$, there had better be a reason: a circumstance unrepresented in (4) which favours $|\alpha\rangle$—for God does not play dice. But the 'balance' before the disruption is not always so easily seen; what tells us in general which objects or entities are to be put on an equal footing, for imbalances to become apparent? *Judgment*, surely; a judgment somehow *founded in experience*, which assesses the relevant peculiarities of the context and determines accordingly.

And what about the great empirical success of the progeny, of the gauge theories that would follow? Fluke? Or are the descendants 'illegitimate,' and not so direct after all? Is the connection between today's gauge theories and the equal rights of direction and length too tenuous to be worth speaking of? The scheme of compensation outlined in Section 4 survives in today's gauge theories, and is central to their success ... but any attempt to answer these questions would take us too far from our subject.

Whatever the relationship between mathematical justice and experience, we have a surprising example of how directly an elaborate theory can emerge from simple *a priori* prejudice. The prejudice here seems gratuitous and arbitrary in the context of discovery, and only acquires justification and grounding years later, by association with an articulated 'telescepticism,' which provides epistemology and motivation.

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