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Invariant Subspaces of Sobolev Disk Algebra

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Abstract: In this paper, we study the invariant subspaces of the operator M_z on the Sobolev disk algebra $R(\mathbf{D})$ and characterize the invariant subspace with finite codimension.

Key words: Invariant subspace; Sobolev disk algebra; multiplication operator.
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1. Introduction

Let Ω be an analytic Cauchy domain in the complex plane and let $W^{2,2}(\Omega)$ denote the Sobolev space: $W^{2,2}(\Omega) = \{f \in L^2(\Omega, \mathrm{d}m)$: the distributional partial derivatives of first and second order of f belong to $L^2(\Omega, \mathrm{d}m)\}$, where $\mathrm{d}m$ denotes the planar Lebesgue measure. For $f, g \in W^{2,2}(\Omega)$, define $\langle f, g\rangle = \sum_{|\alpha|\le 2} \int D^\alpha f \overline{D^\alpha g} \mathrm{d}m$, then $W^{2,2}(\Omega)$ is a Hilbert space and a Banach algebra with identity under an equivalent norm. Let $\mathbf{D} = \{\mathbf{z} \in \mathbf{C} : |\mathbf{z}| < \mathbf{1}\}$ be the unit disk and $R(\mathbf{D})$ be the closure in the Sobolev space $W^{2,2}(\mathbf{D})$ of all rational functions with poles outside $\overline{\mathbf{D}}$. For $f \in R(\mathbf{D})$, the multiplication operator M_f on $R(\mathbf{D})$ is defined as follows:

$$M_f g = fg,\ g \in R(\mathbf{D}).$$

We have the following properties of the space $R(\mathbf{D})$ and the multiplication operators on it[7,8,11]:

Proposition 1[11,Proposition1.3] *(i) Hilbert space $R(\mathbf{D})$ has an orthonormal basis $\{e_n\}_{n=0}^\infty$, where*

$$e_n = \beta_n z^n, \beta_n = [\frac{n+1}{(3n^4 - n^2 + 2n + 1)\pi}]^{\frac{1}{2}}, n = 0, 1, \cdots;$$

(ii) If $f = \sum_{n=0}^\infty f_n z^n$ is analytic in $\mathbf{D}$, then $f \in R(\mathbf{D})$ if and only if $\sum_{n=0}^\infty \frac{|f_n|^2}{\beta_n^2} < +\infty$.

For a separable Hilbert space H, let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on H. If $T \in \mathcal{L}(H)$, then denote $\sigma_e(T)$ the essential spectrum of T, $\rho_F(T) = \mathbf{C}\backslash\sigma_\mathbf{e}(\mathbf{T})$ the Fredholm domain of T. Denote $\mathrm{ind}(T-\lambda)$ the index of T at λ, where $\mathrm{ind}(T-\lambda) = \mathrm{nul}(T-\lambda) - \mathrm{nul}(T-\lambda)^*$, $\lambda \in \rho_F(T)$.

Proposition 1.2[11],Proposition 1.1] *(i) M_z is an essentially normal weighted shift: $M_z e_n = w_n e_{n+1}, w_n = \frac{\beta_n}{\beta_{n+1}}$ $(n = 0, 1, \cdots)$;*

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(ii) $\mathcal{A}'(M_z) = \{M_f : f \in R(\mathbf{D})\}$, *where* $\mathcal{A}'(M_z)$ *is the commutant algebra of* M_z.

Proposition 1.3[11,Proposition1.5] *If* $f \in \mathbf{D}$, *then* $\sigma(M_f) = f(\overline{\mathbf{D}})$, $\sigma_e(M_f) = \sigma_{lre}(M_f) = f(\partial\mathbf{D})$. *If* $z_0 \in \mathbf{D}$, $f(z_0) \notin f(\partial\mathbf{D})$, *then*

$$\mathrm{ind}(M_f - f(z_0)) = -\mathrm{nul}(M_f - f(z_0))^* = -n,$$

where n *is the number of zeros of* $f(z) - f(z_0)$ *in* $\mathbf{D}$ *(including multiplicity).*

Note that $f \in W^{2,2}(\Omega)$ implies that $f \in C(\overline{\Omega})$. Therefore, $R(\mathbf{D})$ is a subalgebra of the disk algebra $A(\mathbf{D})$, hence a subalgebra of H^∞. Because of the special definition of the inner product and the complex behavior of the boundary value, the structure of the space $R(\mathbf{D})$ is much different from H^∞ or H^2.

2. Invariant subspace of M_z

A classical result on invariant subspace is the Beurling's theorem on Hardy space H^2. Beurling's theorem classifies the invariant subspace of H^2 by virtue of inner functions, and it plays an important role in operator theory and function theory. However, it is only recently that progress has been made in proving analogues for the other space of analytic function in $\mathbf{D}$[2]. Since a function in $R(\mathbf{D})$ must be continuous on $\overline{\mathbf{D}}$, it is not all inner functions that are in $R(\mathbf{D})$. But Blaschke products of finite factors are still important in characterizing the structure of invariant subspaces.

Proposition 2.1 *Let* $f \in R(\mathbf{D})$, $B_a(z) = \frac{z-a}{1-\overline{a}z}$, $a \in \mathbf{D}$, *then* $M_{f\circ B_a}$ *is similar to* M_f.

Proof It is easy to see that $f \circ B_a \in R(\mathbf{D})$. For this a, define an operator S_a as follows:

$$S_a f = f \circ B_a, \text{ for all } f \in R(\mathbf{D}).$$

Computations show that there are positive numbers M_1, M_2, M_3 and M_4 such that

$$\int |f \circ B_a|^2 \mathrm{d}m \le M_1 \int |f|^2 \mathrm{d}m,$$

$$\int |(f \circ B_a)'|^2 \mathrm{d}m \le M_2 \int |f'|^2 \mathrm{d}m,$$

and

$$\int |(f \circ B_a)''|^2 \mathrm{d}m \le M_3 \int |f'|^2 \mathrm{d}m + M_4 \int |f''|^2 \mathrm{d}m.$$

We can see $\|f \circ B_a\| \le M\|f\|$ for some $M > 0$, i.e., S_a is bounded.

For all $g \in R(\mathbf{D})$,

$$S_a M_f g = S_a(f \cdot g) = f(B_a)g(B_a).$$

But

$$M_{f\circ B_a} S_a g = M_{f\circ B_a} g(B_a) = f(B_a)g(B_a).$$

Thus $S_a M_f = M_{f\circ B_a} S_a$.

Since B_a is an invertible analytic function, S_a is one-to-one and onto. Therefore, S_a is invertible and $M_{f\circ B_a} = S_a M_f S_a^{-1}$.

Proposition 2.2 *If $\varphi \in R(\mathbf{D})$, then M_φ is similar to M_z if and only if*

$$\varphi(z) = \lambda \frac{z-a}{1-\overline{a}z}, \quad \textit{for some } \lambda \textit{ and } a,\ |\lambda| = 1, a \in \mathbf{D}.$$

Proof If $\varphi(z) = \lambda\frac{z-a}{1-\overline{a}z}$, then it is an immediate corollary of Proposition 2.1 that $M_\varphi \sim M_z$.

On the other hand, if $M_f \sim M_z$, it follows from Proposition 1.3 that

$$f(\overline{\mathbf{D}}) = \sigma(M_f) = \sigma(M_z) = \overline{\mathbf{D}}.$$

Thus $f(\mathbf{D}) \subset \overline{\mathbf{D}}$. Since f is continuous on $\overline{\mathbf{D}}$, by Maximum Modulus Theorem, f cannot take its maximum modulus 1 in $\mathbf{D}$. Therefore, $f(\mathbf{D}) \subset \mathbf{D}$. But for $\lambda \in \mathbf{D}$,

$$\lambda \in \sigma(M_z) \setminus \sigma_e(M_z) = f(\overline{\mathbf{D}}) \setminus f(\partial\mathbf{D}).$$

Then $\lambda \in f(\mathbf{D})$, that is, $\mathbf{D} \subset \mathbf{f}(\mathbf{D})$. Consequently, f is a map from $\mathbf{D}$ onto $\mathbf{D}$.

Notice that

$$\mathrm{nul}\ (\lambda - \mathrm{M_f^*}) = \mathrm{nul}\ (\lambda - \mathrm{M_z^*}) = 1, \quad \text{for all } \ \lambda \in \mathbf{D},$$

which indicates that $f(z) - \lambda$ has a unique zero in $\mathbf{D}$ for all $\lambda \in \mathbf{D}$. Therefore, f is a one-to-one map from $\mathbf{D}$ onto $\mathbf{D}$ and must be a Möbius transformation with a coefficient modulus one.

Recall that the class of Cowen-Douglas operator $B_n(\Omega)$ of index n is the set of all bounded linear operators $B \in \mathcal{L}(\mathcal{H})$ which satisfy:

(i) $\sigma(B) \supset \Omega$;

(ii) $\mathrm{ran}\ (\mathrm{B} - \lambda) = \mathrm{H}$ for all $\lambda \in \Omega$;

(iii) $\mathrm{nul}\ (\mathrm{B} - \lambda) = \mathrm{n}$ for all $\lambda \in \Omega$;

(iv) $\bigvee\{\ker\ (\mathrm{B} - \lambda) : \lambda \in \Omega\} = \mathrm{H}$.

Lemma 2.3[11,Proposition1.6] *Let $f \in R(\mathbf{D})$ and $f(z_0) \notin f(\partial\mathbf{D}), \mathbf{z_0} \in \mathbf{D}$. Denote the component of the Fredholm domain $\rho_F(M_f)$ containing $f(z_0)$ as Ω, then $M_f^* \in B_n(\Omega)$, where n is the number of zeros of $f(z) - f(z_0)$ in $\mathbf{D}$.*

Lemma 2.4 *If $f_1, f_2, \cdots, f_n$ are functions in $R(\mathbf{D})$ with no common zeros in the closed disk $\overline{\mathbf{D}}$, then there exist $g_1, g_2, \cdots, g_n \in R(\mathbf{D})$ such that $f_1g_1 + f_2g_2 + \cdots + f_ng_n = 1$.*

Proof Let

$$J = \{g_1f_1 + g_2f_2 + \cdots + g_nf_n : g_1, \cdots, g_n \in R(\mathbf{D})\}.$$

It is easy to see that J is an ideal of $R(\mathbf{D})$. If J is a proper ideal, then J is contained in a maximal ideal. This is impossible since the maximal ideal space of $R(\mathbf{D})$ is $\overline{\mathbf{D}}$ but $f_1, f_2, \cdots, f_n$ have no common zeros in $\overline{\mathbf{D}}$. Thus J contains the constant 1 and so there exist $g_1, g_2, \cdots, g_n \in R(\mathbf{D})$ such that $f_1g_1 + f_2g_2 + \cdots + f_ng_n = 1$.

Lemma 2.5 *Let M be an invariant subspace of M_z with common zeros $z_1, z_2, \cdots, z_n$ in $\mathbf{D}$ (including multiplicity), then $N := \{g \in R(\mathbf{D}) : (z - z_1) \cdots (z - z_n)g \in M\}$ is a closed subspace of $R(\mathbf{D})$.*

Proof Obviously, N is a subspace of $R(\mathbf{D})$. Suppose that $g_k \in N$, $g_k \to g_0$ in $R(\mathbf{D})$ $(k \to \infty)$, let

$$h_k = (z - z_1) \cdots (z - z_n)g_k$$

and

$$h_0 = (z - z_1) \cdots (z - z_n)g_0.$$

Then $h_k \to h_0$ in $R(\mathbf{D})$. It follows from $h_k \in M$ that $h_0 \in M$, which implies that $g_0 \in N$. Therefore, N is closed.

Lemma 2.6[11,Proposition1.2] *The set of all polynomials is dense in $R(\mathbf{D})$.*

Proposition 2.7 *Let M be a subspace of $R(\mathbf{D})$.*

(i) M is an invariant subspace of M_z with common zeros $z_1, z_2, \cdots, z_n$ in $\mathbf{D}$ (including multiplicity) if and only if $M = (z - z_1)(z - z_2) \cdots (z - z_n)R(\mathbf{D})$;

(ii) If M is an invariant subspace in (i), then the projection from $R(\mathbf{D})$ onto M is

$$P_M = M_\chi (M_\chi^* M_\chi)^{-1} M_\chi^*,$$

where

$$\chi = \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}.$$

Proof (i) Clearly, if $M = (z - z_1)(z - z_2) \cdots (z - z_n)R(\mathbf{D})$, then M is an invariant subspace of M_z.

On the other hand, as defined in Lemma 2.5, let

$$N = \{g \in R(\mathbf{D}) : (z - z_1) \cdots (z - z_n)g \in M\},$$

then N is a closed subspace of M_z. For each $w \in \overline{\mathbf{D}}$, since $z_1, z_2, \cdots, z_n \in \mathbf{D}$, there exists a function $f_w \in N$ such that $f_w(w) \neq 0$. Observing f_w is continuous on $\overline{\mathbf{D}}$, we can choose a neighborhood $U(w, \varepsilon)$ such that f_w does not vanish in $U(w, \varepsilon)$. Consequently, there exists a finite covering $U(w_1, \varepsilon_1), \cdots, U(w_k, \varepsilon_k)$ of $\overline{\mathbf{D}}$ and functions $f_{w_1}, \cdots, f_{w_k} \in R(\mathbf{D})$ such that f_{w_i} does not vanish in $U(w_i, \varepsilon_i)$, $i = 1, \cdots, k$. Note that $f_{w_1}, \cdots, f_{w_k}$ have no common zeros in $\overline{\mathbf{D}}$, otherwise, if $f_{w_i}(w_0) = 0$ for all $1 \leq i \leq k$, then w_0 must be in certain $U(w_j, \varepsilon_j)$ $(1 \leq j \leq k)$, this contradicts to the fact $f_{w_j}(z) \neq 0$ in $U(w_j, \varepsilon_j)$. Now it follows from Lemma 2.4 that there exist functions $g_1, g_2, \cdots, g_k \in R(\mathbf{D})$ such that $f_{w_1} g_1 + f_{w_2} g_2 + \cdots + f_{w_k} g_k = 1$, so that $1 \in N$. Thus $(z - z_1)(z - z_2) \cdots (z - z_n) \in M$. Since M is invariant under M_z, by Lemma 2.6, $(z - z_1)(z - z_2) \cdots (z - z_n)R(\mathbf{D}) \subset M$.

Conversely, it is obvious that,

$$M \subset (z - z_1)(z - z_2) \cdots (z - z_n)R(\mathbf{D}).$$

Therefore,

$$M=(z-z_1)(z-z_2)\cdots(z-z_n)R(\mathbf{D}).$$

(ii) Since M_χ^* is a Fredholm operator and $\ker \mathrm{M}_\chi=\{0\}$, $M_\chi^* M_\chi$ is invertible. It is obvious that $M_\chi(M_\chi^* M_\chi)^{-1}M_\chi^*$ is an idempotent and self-adjoint operator, thus is a projection.

By Lemma 2.3, $M_\chi^*\in B_n(\mathbf{D})$, ran $M_\chi^*=R(\mathbf{D})$, we see that $P_M=\mathrm{ran}M_\chi=\chi R(\mathbf{D})$. But $M=\chi R(\mathbf{D})$, thus $\mathrm{ran}P_M=M$ and so P_M is the projection onto M.

It is well known that for each invariant subspace M of M_z in H^2, M_z is similar to its restriction on M. The following proposition shows that in Sobolev disk algebra, this property holds if and only if M has finite common zeros in $\mathbf{D}$.

Proposition 2.8 *Let M be an invariant subspace of M_z, then M_z is similar to its restriction on M if and only if*

$$M=(z-z_1)(z-z_2)\cdots(z-z_n)R(\mathbf{D}),$$

where $\{z_i\}_{i=1}^n$ are numbers in $\mathbf{D}$.

Proof If $M=(z-z_1)(z-z_2)\cdots(z-z_n)R(\mathbf{D})$, let

$$p=(z-z_1)(z-z_2)\cdots(z-z_n).$$

Define $T_p: R(\mathbf{D})\to M$, $T_pf=pf$ for $f\in R(\mathbf{D})$. It is easy to see that T_p is a one-to-one map from $R(\mathbf{D})$ onto M. Since $T_p^{-1}M_z|_M T_p f=zf=M_zf$ for all $f\in R(\mathbf{D})$, $M_z=T_p^{-1}M_z|_M T_p$.

On the other hand, if there exists $W: R(\mathbf{D})\to M$ such that $M_z=W^{-1}M_z|_M W$, then $WM_z=M_z|_M W$. Denote $h=W1$, then computations show that $Wz^n=z^n h$, and so $Wp_n=p_n h=M_h p_n$ for all polynomials. By Lemma 2.6, $Wf=M_h f$ for all $f\in R(\mathbf{D})$, which implies $W=M_h$ and the range of M_h is closed. It follows from Proposition 1.3 that $h(\partial D)\neq 0$. Since h is analytic, it must have only finite zeros in $\mathbf{D}$. The proof is completed by Proposition 2.7.

Given a set M, let $[M]$ be the smallest invariant subspace of M_z generated by M. If $M\neq\{0\}$ is an invariant subspace of H^2, it follows from Beurling's theorem that

$$\dim[M\ominus zM]=1.$$

It also has been known that the invariant subspace of Bergman Space L_a^2 is very complicated. In [3], C. Apostol, H. Bercovici, C. Foias and C. Pearcy showed that if n is any positive integer or ∞, then there is an invariant subspace M of L_a^2 such that $\dim[M\ominus zM]=n$.

In [10], S. Richter studied a class of Banach spaces of analytic functions $\mathcal{B}$ such that:

(i) The functional of evaluation at λ is continuous for all $\lambda\in\mathbf{D}$.

(ii) If $f\in\mathcal{B}$, then $zf\in\mathcal{B}$.

(iii) If $f\in\mathcal{B}$ and $f(\lambda)=0$, then $f=(z-\lambda)g$ for some $g\in\mathcal{B}$.

Lemma 2.9[10,Theorem5.3] *Let $\mathcal{B}$ be a Hilbert space satisfies the above conditions.*

(i) If $\mathcal{B}$ is an algebra and I is a closed ideal of $\mathcal{B}$, then $\dim[I\ominus zI]=1$.

(ii) Suppose $\sigma(M_z)=\overline{\mathbf{D}}$ and $\|M_zf\|\geq\|f\|$ for all $f\in\mathcal{B}$. If M is an invariant subspace of M_z and $\dim[M\ominus zM]=1$, *then $M\subset H^2f_0\cap\mathcal{B}$, where $f_0\in[M\ominus zM]$.*

Corollary 2.10 *Let $M \neq \{0\}$ be an invariant subspace of M_z in $R(\mathbf{D})$, then*

$$\dim[M \ominus zM] = 1,$$

and for each $f \in M$, $f = \varphi f_0$, where $\varphi \in H^2$ and $f_0 \in [M \ominus zM]$.

Proof Since an invariant subspace of M_z in $R(\mathbf{D})$ is also a closed ideal, we have $\dim[M \ominus zM] = 1$. To show that $f = \varphi f_0$, we need only to verify that $\|M_z f\| \geq \|f\|$ for all $f \in R(\mathbf{D})$. In fact, if $f = \sum_{n=0}^{\infty} f_n z^n$, then

$$\|M_z f\|^2 = \sum_{n=0}^{\infty} \frac{|f_n|^2}{\beta_{n+1}^2}.$$

Thus

$$\|M_z f\|^2 - \|f\|^2 = \sum_{n=0}^{\infty} \frac{9n^4 + 30n^3 + 29n^2 + 12n + 3}{(n+2)(n+1)} \pi |f_n|^2 > 0.$$

Remark As we know that in both the Hardy space and Bergman space, an invariant subspace M is just equal to the subspace generated by $M \ominus zM$, that is, $M = [M \ominus zM]$. We may expect that this property holds in $R(\mathbf{D})$, while this problem is still open.

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Sobolev **圆盘代数的不变子空间**

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摘要: 本文研究了 Sobolev 圆盘代数 $R(D)$ 上的乘法算子 M_z 的不变子空间，完全刻画了余维有限的不变子空间的结构.

关键词: 不变子空间； Sobolev 圆盘代数；乘法算子.