Further Result on Robust Stabilization for Uncertain Nonlinear Time-delay Systems

JIAO Xiao-Hong[1] SHEN Tie-Long[2] SUN Yuan-Zhang[3]

Abstract The systematic recursive design method of the robust stabilizing controller for general uncertain nonlinear time-delay systems is investigated in this paper. A delay-independent state feedback control law can be obtained by recursively constructing Lyapunov-Razumikhin function. It is shown that by some design techniques the obstacle that is intrinsic to the application of the Razumikhin condition can be removed such that the design of the robust stabilizing control law is free of any restriction for the systems.

Key words Nonlinear time-delay systems, robust stabilization, Lyapunov-Razumikhin function, recursive design

1 Introduction

Motivated by the systematic design method[1] for nonlinear nondelay systems stabilization based on Lyapunov function, how to extend nonlinear time-delay systems is naturally regarded as an interesting and challenging research topic. In recent years, many research efforts in this area have been made. However, most of the results were based on the linear matrix inequalities (LMI) method, that is to say, in these results the considered nonlinear systems were essentially handled as linear systems with nonlinear uncertainties satisfying the linear matching conditions. Several researches such as [2,3] were based on the essence of nonlinear time-delay systems. A functional based version of recursive approach was first presented in [2], but the stabilizing controller proposed could not be obtained constructively[4]. Based on the control Lyapunov-Razumikhin function, another version of recursive method was provided in [3]. However, the control law can only be constructed through checking the existence of the domination function. Thus, as pointed out by [3] itself, it is a difficult task for higher dimensional systems. Apparently, for time-delay systems described by the functional differential equations[5], it is not a trivial extension of the recursive design of nonlinear nondelay systems. In [6∼9] some attempts have been made to solve this issue. A recursive design based on Lyapunov-Razumikhin function was given in [6] for a class of time-delay systems with restriction, where the bounding functions of the uncertain related-delay functions were required to be only related to x_{1t}. In [8], with the help of the proposed LaSalle-Yoshizawa-like theorem, this result was further extended to the adaptive stabilization for a class of nonlinear time-delay systems. The restriction was relaxed in [9] by the requirement for the related-delay functions to satisfy the linear growth condition. However, how to recursively design a stabilizing controller for general nonlinear time-delay systems without any restriction is still an open nontrivial problem.

This paper addresses the methodology and makes a discussion on the results of [6∼9] in order to further develop novel results on general nonlinear time-delay systems. A Lyapunov-Razumikhin function based version of a similar backstepping approach is developed for the general nonlinear time-delay systems. It is shown that a robust stabilizing controller can be explicitly constructively obtained by properly using the Razumikhin condition without additional conditions. And the constructed controller is independent of the state-delay. Thus, the value of the delay is allowed to be unknown. We call it a similar backstepping approach to mean that there exists some distinctive difference from the conventional backstepping method. In comparison to the backstepping design for the nonlinear nondelay systems, one more difficulty arises in the new problem setting for time-delay systems. The difficulty is caused by two aspects. One is that due to the use of the Razumikhin condition the triangular structure form of the system is changed. The other is that the coordinate transformation has effect on the Razumikhin condition. Thus, the key to recursive design for time-delay systems is how to overcome this obstacle to explicitly obtain a robust stabilizing controller.

2 Problem statement

The nonlinear time-delay systems considered are described by

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{x}_t) + \boldsymbol{G}(\boldsymbol{x})u, \quad \boldsymbol{x}_0(\tau),\ \tau \in [-r, 0] \tag{1}$$

where $\boldsymbol{x} \in R^n$ represents the state, $\boldsymbol{x}_t := \boldsymbol{x}(t+\tau)$ the delayed state, and $\tau \in [-r, 0]$, $r > 0$ is a constant representing the largest value of delay. $u \in R$ is control input. $\boldsymbol{F}(\cdot)$ and $\boldsymbol{G}(\cdot)$ are smooth functions with appropriate dimensions with $\boldsymbol{F}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{G}(\boldsymbol{x}) \neq \boldsymbol{0}, \forall \boldsymbol{x}$. For $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{x}_t)$, the following decomposition is reasonable

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{x}_t) = \boldsymbol{f}(\boldsymbol{x}) + \tilde{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{x}_t)\boldsymbol{x}_t$$

with $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{0})$, and $\tilde{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{x}_t)$ can be found analytically[10]. Thus, (1) can be rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{e}(\boldsymbol{x}, \boldsymbol{x}_t) + \boldsymbol{G}(\boldsymbol{x})u \tag{2}$$

where $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{x}_t) := \tilde{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{x}_t)\boldsymbol{x}_t$ and $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{e}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0}$. With the help of a certain geometric condition[1], (2) can be changed into the following form

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1) + e_1(x_1, x_{1t}) \\ \dot{x}_2 = x_3 + f_2(\tilde{x}_2) + e_2(\tilde{x}_2, \tilde{x}_{2t}) \\ \quad \vdots \\ \dot{x}_n = u + f_n(\tilde{x}_n) + e_n(\tilde{x}_n, \tilde{x}_{nt}) \end{cases} \tag{3}$$

where $\tilde{x}_i = [x_1 x_2 \cdots x_i]^{\mathrm{T}}$, $\tilde{x}_{it} = [x_{1t} \cdots x_{it}]^{\mathrm{T}}$ $(i = 1, \cdots, n)$. $f_i(\cdot)$ are smooth functions, $f_i(0) = 0$, and $e_i(\cdot)$ reasonably satisfy the following conditions with known functions

Received September 7, 2005; in revised form September 21, 2006
Supported by National Natural Science Foundation of P. R. China (50595413), China Postdoctoral Science Foundation (2004036064)
1. Institute of Electrical Engineering, Yanshan University, Qinhuangdao 066004, P. R. China 2. Department of Mechanical Engineering, Sophia University, Tokyo 102-8554, Japan 3. Department of Electrical Engineering, Tsinghua University, Beijing 100080, P. R. China
DOI: 10.1360/aas-007-0164

$b_{ij}(\cdot) > 0$ and class $\mathcal{K}$ functions $\mu_{ij}(\cdot)$ $(i = 1, \cdots, n,\ j = 1, \cdots, i)$:

$$|e_i(\tilde{x}_i, \tilde{x}_{it})| \le \sum_{j=1}^{i} b_{ij}(\tilde{x}_i)\mu_{ij}(|x_{jt}|) \quad (4)$$

It should be noted that $e_i(\cdot)$ may represent the uncertainties if only it satisfies (4).

The robust stabilization problem addressed in this paper is, for the general system (3), how to find a smooth feedback controller $u = c(\boldsymbol{x})$, which is independent of the delayed state, such that the closed loop system is globally asymptotically stable at $\boldsymbol{x} = \boldsymbol{0}$.

To this end, the following technical lemma will serve as a basis for the explicit construction of the robust stabilizing controller.

Lemma 1. Consider nonlinear time-delay systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}_t),\ \ \boldsymbol{x}_0(\tau),\ \ \tau \in [-r, 0] \quad (5)$$

If there exist a continuous function $V(\boldsymbol{x})$ and $\mathcal{K}_\infty$ functions $\kappa_1(\cdot)$, $\kappa_2(\cdot)$ and $\kappa_3(\cdot)$ such that

$$\kappa_1(\|\boldsymbol{x}\|) \le V(\boldsymbol{x}) \le \kappa_2(\|\boldsymbol{x}\|) \quad (6)$$

$$\dot{V}(\boldsymbol{x}) \le -\kappa_3(\|\boldsymbol{x}\|),\ \text{if} \max_{-r\le\tau\le 0} V(\boldsymbol{x}_t(\tau)) < pV(\boldsymbol{x}_t(0)) \quad (7)$$

the solution $\boldsymbol{x}(t) = \boldsymbol{0}$ of (5) is globally asymptotically stable, where $p > 1$ is a given constant, $\max_{-r\le\tau\le 0} V(\boldsymbol{x}_t(\tau)) < pV(\boldsymbol{x}_t(0))$ is called Razumikhin condition.

This lemma is a case of the Razumikhin stability theorem in [5] since a linear function ps with a constant $p > 1$ is used to replace the function $p(s)$ $(p(s) > s,\ \forall s > 0)$.

3 Main result

For simplicity of presenting the basic idea, the case with scalar x of (2) is first considered.

Theorem 1. For the scalar case of (2), a delay independent state feedback controller is given by

$$u = -\frac{1}{g(x)}\left\{f(x) + \frac{1}{2}xb^2(x) + \frac{1}{2}q^2x\tilde{\mu}^2(q|x|) + x\right\} \quad (8)$$

The resulting closed-loop system is globally asymptotically stable at $x = 0$, where $q > 1$ is a given constant, and the function $\tilde{\mu}(\cdot)$ satisfies the function decomposition $\mu(s) = s\tilde{\mu}(s)$.

Proof. Choose a candidate for Lyapunov-Razumikhin function as follows.

$$V(x) = \frac{1}{2}x^2 \quad (9)$$

Since $|e(x, x_t)| \le b(x)\mu(|x_t|)$, the time derivative of V along any trajectories of the system satisfies

$$\dot{V}(x) \le x[f(x) + g(x)u] + |x|b(x)\mu(|x_t|) \quad (10)$$

When the Razumikhin condition $|x_t| < q|x|$ holds, the time derivative of V becomes

$$\dot{V}(x) \le x\left\{g(x)u + f(x) + \frac{1}{2}xb^2(x) + \frac{1}{2}q^2x\tilde{\mu}(q|x|)\right\} \quad (11)$$

Therefore, a feedback law defined by (8) gives

$$\dot{V}(x) \le -x^2,\ \ \text{if } \|x_t(\tau)\| < q\|x_t(0)\|,\ \ \tau \in [-r, 0] \quad (12)$$

Thus, by Lemma 1, the asymptotical stability follows from (9) and (12). □

Now the design method presented in Theorem 1 is extended to the higher order systems (3). To demonstrate the idea of recursive design, the result on the two-dimensional system of (3) is presented.

Theorem 2. For the two-dimensional system of (3), a delay independent stabilizing controller

$$u = -z_1 - f_2 + \frac{\partial\alpha_1}{\partial x_1}[x_2 + f_1] - \frac{1}{2}z_2b_{21}^2 - \frac{1}{2}z_2\left|\frac{\partial\alpha_1}{\partial x_1}\right|^2 b_{11}^2 - \frac{1}{2}z_2b_{22}^2\left[\sum_{l=1}^{2}\tilde{\nu}_{221}^2(nq|z_l|) + 1\right] - 2q^2z_2 - z_2\sum_{i=1}^{2}\sum_{j=1}^{i}4(3-i)j^2q^2\tilde{\mu}_{ij}^2(j2q|z_2|) - z_2 \quad (13)$$

can be recursively obtained, where $q > 1$, $\alpha_1(x_1)$ is a smooth function determined in the design procedure.

Proof. First, note that in the recursive design, Lyapunov-Razumikhin function of the whole system will be in a quadratic form on the new coordinate z under the change of the coordinate

$$z_1 = x_1,\ \ \ z_2 = x_2 - \alpha_1(x_1) \quad (14)$$

with $\alpha_1(0) = 0$. Then, the Razumikhin condition becomes

$$\max_{-r\le\tau\le 0} V(\boldsymbol{z}_t(\tau)) < pV(\boldsymbol{z}_t(0)) \quad (15)$$

It is equivalent to the following condition with a given constant $q = \sqrt{p} > 1$

$$\|\boldsymbol{z}_t(\tau)\| < q\|\boldsymbol{z}_t(0)\|,\ \ \ \tau \in [-r, 0] \quad (16)$$

First Step. For the x_1-subsystem with x_2 viewed as a virtual control signal, we define a positive definite function $V_1(x_1)$ as

$$V_1(z_1) = \frac{1}{2}z_1^2 \quad (17)$$

then, we obtain the derivative of V_1 as follows.

$$\dot{V}_1 \le z_1\{x_2 + f_1(x_1)\} + \frac{1}{2}z_1^2b_{11}^2(\tilde{x}_1) + \frac{1}{2}\mu_{11}^2(|x_{1t}|)$$

When the Razumikhin condition (16) holds, $|x_{1t}| = |z_{1t}| \le \|\boldsymbol{z}_t\| < q\|\boldsymbol{z}\|$ holds, thus, we get

$$\dot{V}_1 \le z_1\left\{x_2 + f_1(x_1) + \frac{1}{2}z_1b_{11}^2(\tilde{x}_1)\right\} + \frac{1}{2}\mu_{11}^2(q\|\boldsymbol{z}\|) \quad (18)$$

It is clear that the last term cannot be cancelled with the virtual control law $\alpha_1(x_1)$. But, by the virtual control law $\alpha_1(x_1)$, additional function terms on z_1 must be contained in order to ensure the derivative of the Lyapunov-Razumikhin function in the final step to be negative. Thus, the virtual control law is chosen as

$$\alpha_1 = -f_1 - \frac{1}{2}x_1b_{11}^2 - 2q^2z_1 - z_1\sum_{i=1}^{2}\sum_{j=1}^{i}4(3-i)j^2q^2\tilde{\mu}_{ij}^2(j2q|z_1|) - z_1 \quad (19)$$

where $\tilde{\mu}_{ij}(\cdot)$ satisfies the decompositions $\mu_{ij}(s) = s\tilde{\mu}_{ij}(s)$, which makes the derivative of V_1 satisfy

$$\dot{V}_1 \le z_1z_2 + \frac{1}{2}\mu_{11}^2(q\|\boldsymbol{z}\|) - 2q^2z_1^2 - \sum_{i=1}^{2}\sum_{j=1}^{i}(3-i)\mu_{ij}^2(j2q|z_1|) - z_1^2 \quad (20)$$

whenever the Razumikhin condition holds.

Second Step. For the whole system, define

$$V(z_1, z_2) = V_1(z_1) + \frac{1}{2}z_2^2 \tag{21}$$

Then, along the trajectories of the system (x_1, z_2), the time derivative of V is

$$\dot{V} \leq z_2\left\{u + f_2 - \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1]\right\} + |z_2|\left|\frac{\partial \alpha_1}{\partial x_1}\right| b_{11}\mu_{11}(|x_{1t}|) + |z_2|[b_{21}\mu_{21}(|x_{1t}|) + b_{22}\mu_{22}(|x_{2t}|)] + \dot{V}_1 \tag{22}$$

A problem arises, *i.e.* how to use the Razumikhin condition in $\mu_{22}(|x_{2t}|)$. Let

$$M_2 := |z_2| b_{22}(\tilde{x}_2)\mu_{22}(|x_{2t}|)$$

and note that $x_{2t} = z_{2t} + \alpha_1(x_{1t})$, where $\alpha_1(\cdot)$ has been determined in the former step. Then one can find a class of $\mathcal{K}$ functions $c_{11}(\cdot)$ such that $|\alpha_1(x_{1t})| \leq c_{11}(|x_{1t}|)$. Thus, we have

$$M_2 \leq \frac{1}{2}z_2^2 b_{22}^2(\tilde{x}_2) + \frac{1}{2}\mu_{22}^2(2|z_{2t}|) + |z_2| b_{22}(\tilde{x}_2)\nu_{221}(|x_{1t}|) \tag{23}$$

where $\nu_{221}(s) := \mu_{22}(2c_{11}(s))$, $s \geq 0$. Substituting (23) into (22) and considering $|x_{1t}| < q\|\boldsymbol{z}\|$ and $|z_{2t}| < q\|\boldsymbol{z}\|$ when the Razumikhin condition holds, we obtain

$$\dot{V} \leq z_2\left\{u + f_2 - \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1] + \frac{1}{2}z_2(b_{21}^2 + b_{22}^2) + \frac{1}{2}z_2\left|\frac{\partial \alpha_1}{\partial x_1}\right|^2 b_{11}^2\right\} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{i}\mu_{ij}^2(jq\|\boldsymbol{z}\|) + |z_2| b_{22}\nu_{221}(q\|\boldsymbol{z}\|) + \dot{V}_1 \tag{24}$$

Obviously, another problem arises: how to deal with $N_2 := |z_2| b_{22}(\tilde{x}_2)\nu_{221}(q\|\boldsymbol{z}\|)$. The difficulty lies in the fact that the function $\nu_{221}(\cdot)$ is not dealt with in the same way as the functions $\mu_{ij}(\cdot)$ since $\nu_{221}(\cdot)$ is closely related to $\alpha_1(\cdot)$ designed in the former step. Thus, to overcome this difficulty, we handle N_2 as follows.

$$\begin{aligned} N_2 &\leq |z_2| b_{22}(\tilde{x}_2)\sum_{l=1}^{2}\nu_{221}(2q|z_l|) \leq \\ &\frac{1}{2}z_2^2 b_{22}^2(\tilde{x}_2)\sum_{l=1}^{2}\tilde{\nu}_{221}^2(2q|z_l|) + \sum_{l=1}^{2}2q^2 z_l^2 \end{aligned} \tag{25}$$

where $\tilde{\nu}_{221}(\cdot)$ satisfies the decomposition $\nu_{221}(s) = s\tilde{\nu}_{221}(s)$. The first term in the last inequality of (25) can be cancelled with the virtual control law in this step. In the second term, the quadratic form of z_2 can be cancelled in this step and the quadratic form of z_1 can be dominated by the predesigned additional term in $\alpha_1(\cdot)$. These features are just the distinctive difference from the backstepping design of nonlinear non-delay systems. Substituting (25) and (20) into (24) yields

$$\begin{aligned} \dot{V} \leq\; & z_2\left\{u + z_1 + f_2 - \frac{\partial \alpha_1}{\partial x_1}[x_2 + f_1] + \frac{1}{2}z_2 b_{21}^2 + \frac{1}{2}z_2\left|\frac{\partial \alpha_1}{\partial x_1}\right|^2 b_{11}^2\right\} + \\ & \frac{1}{2}z_2^2 b_{22}^2\left[\sum_{l=1}^{2}\tilde{\nu}_{221}^2(2q|z_l|) + 1\right] + 2q^2 z_2^2 - z_1^2 + \\ & \sum_{i=1}^{2}\sum_{j=1}^{i}\frac{3-i}{2}\mu_{ij}^2(jq\|\boldsymbol{z}\|) - \sum_{i=1}^{2}\sum_{j=1}^{i}(3-i)\mu_{ij}^2(j2q|z_1|) \end{aligned} \tag{26}$$

whenever the Razumikhin condition holds. Therefore, a feedback law defined by (13) renders

$$\dot{V}(z_1, z_2) \leq -z_1^2 - z_2^2 \;\text{ if } \|\boldsymbol{z}_t(\tau)\| < q\|\boldsymbol{z}_t(0)\|,\; \tau \in [-r, 0] \tag{27}$$

Thus, by Lemma 1, the asymptotical stability follows from (21) and (27). □

From the design presented by Theorem 2, it can be seen that the key of the recursive design is how to deal with the system without triangular structure due to the use of the Razumikhin condition and the effect of the coordinate transformation on the Razumikhin condition, so that the derivative of the Lyapunov-Razumikhin function for the whole system along the closed-loop system trajectories satisfying the Razumikhin condition is negative. Recursive application of the proposed design step described above leads to backstepping method for system (3).

Theorem 3. Consider system (3) with (4). A stabilizing controller $u = c(x_1, \cdots, x_n)$, which is independent of delay, can be recursively obtained.

See Appendix A for the proof.

4 Numerical example

To illustrate the proposed recursive method, we determine a robust asymptotically stabilizing feedback control for the two-dimensional system

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1) + e_1(x_1, x_{1t}) \\ \dot{x}_2 = u + f_2(x_1, x_2) + e_2(x_1, x_{1t}, x_2, x_{2t}) \end{cases} \tag{28}$$

where $f_1(x_1) = x_1^2 + 2x_1$, $f_2(x_1, x_2) = x_1x_2 + x_1^2 + x_2^2$, $e_1(\cdot)$ and $e_2(\cdot)$ are unknown functions satisfying

$$\begin{aligned} &|e_1(x_1, x_{1t})| \leq \frac{1}{2}|x_{1t}| \\ &|e_2(x, x_t)| \leq \frac{1}{2}(1 + x_1^2)|x_{1t}| + \frac{1}{2}|x_{2t}| \end{aligned} \tag{29}$$

i.e. $e_1(\cdot)$ and $e_2(\cdot)$ are bounded by (4) with the bounding functions

$$\begin{aligned} &b_{11}(x_1) = 1,\; b_{21}(x) = 1 + x_1^2,\; b_{22}(x) = 1 \\ &\mu_{ij}(s) = \tfrac{1}{2}s \;(i, j = 1, 2) \end{aligned} \tag{30}$$

Hence, by applying Theorem 2 to the system, we obtain a robust stabilizing controller ($q = 1.005$):

$$\begin{aligned} u = & -x_1 - f_2 + \frac{\partial \alpha_1}{\partial x_1}(x_2 + f_1) - \frac{1}{2}z_2(1 + x_1^2)^2 - \frac{1}{2}z_2\left(\frac{\partial \alpha_1}{\partial x_1}\right)^2 - \\ & 16.16z_2(z_1^2 + z_2^2) - 302.43z_2 \end{aligned} \tag{31}$$

where $\alpha_1(x_1) = -x_1^2 - 12.09x_1$, $z_2 = x_2 - \alpha_1(x_1)$.

When the initial conditions are chosen as

$$\phi_1(\tau) = 0.1e^{\tau},\; \phi_2(\tau) = -0.8\sin(\tau + \frac{\pi}{2}),\; \tau \in [-0.2, 0] \tag{32}$$

and the uncertainties satisfying the bounding condition (29) with the bounding functions (30) are described by

$$e_1(x_1, x_{1t}) = \frac{1}{2}x_{1t}, \quad e_2(x, x_t) = x_1x_{1t} + \frac{1}{2}x_{2t} \tag{33}$$

The simulation of the closed loop system consisting of (28) and the robust feedback controller (31) is shown in Fig. 1. This simulation demonstrates that the system with the delay-related uncertainty can be stabilized with a satisfactory dynamic performance by the robust feedback controller constructed recursively.

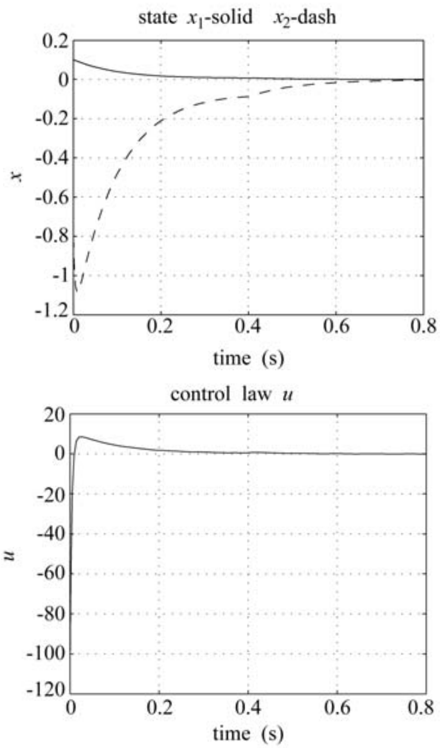


Fig. 1 The response curse of the closed-loop system

5 Conclusion

The robust stabilization problem for general nonlinear time-delay systems is investigated. A Lyapunov-Razumikhin function based version of similar backstepping approach is developed. The key feature of this approach is that in the recursive design, the subsystems forced by the virtual control laws at each step are not necessarily stable but contain the additional signals dominating the delay-related uncertainties such that in the final step the derivative of the Lyapunov-Razumikhin function is negative whenever the Razumikhin condition holds.

References

1 Isidori A. *Nonlinear Control Systems*. New York: Springer-Verlag, 1995.

2 Nguang S K. Robust stabilization of a class of time-delay nonlinear systems, *IEEE Transactions on Automatic Control*, 2000, **45**(4): 756∼762.

3 Jankovic M. Control Lyapunov-Razumikhin functions and robust stabilization of timedelay systems, *IEEE Transactions on Automatic Control*, 2001, **46**(7): 1048∼1060.

4 Zhou S S, Feng G and Nguang S K. Comments on "Robust stabilization of a class time-delay nonlinear system", *IEEE Transactions on Automatic Control*, 2002, **47**(9): 1586∼1586

5 Hale J K and Lunel S M V. *Introduction to functional differential equations*. New York: Springer-Verlag, 1993.

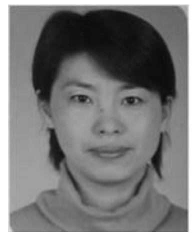
6 Jiao X H, Shen T. Domination design approach to robust stabilization of nonlinear systems with time-delay *via* Lyapunov-Razumikhin Function, *Transaction of Society of Instrument and Control Engineers of Japan*, 2004, **40**(9): 890∼897.

7 Jiao X H, Shen T, Sun Y Z and Tamura K. Krasovskii functional, Razumikhin function and backstepping. In: Proceedings of the 8th International Conference on Control, Automation, Robotics and Vision (ICARCV), Beijing: Tsinghua University Press 2004, **2**: 1200∼1205

8 Jiao X H, Shen T. Adaptive feedback control of nonlinear time-delay systems the LaSalle-Razumikhin-based approach, *IEEE Transactions on Automatic Control*, 2005, **50**(11): 1909∼1913.

9 Jiao X H, Sun Y Z and Shen T. Backstepping design for robust stabilizing control of nonlinear systems with time-delay, In: Proceedings of the 16th IFAC World Congress, Prague, July 3, 2005,

10 Lin W and Shen T. Robust passivity and feedback design for minimum-phase nonlinear systems with structural uncertainty, *Automatica*, 1999, **35**(1): 35∼47.



JIAO Xiao-Hong Received her bachelor and master degrees in Automatic Control from Northeast Heavy Machinery Institute, China in 1988 and 1991, respectively, and the Ph. D. degree in Mechanical Engineering from Sophia University, Tokyo, Japan, 2004. She is a professor at Institute of Electrical Engineering, Yanshan University, China. Her research interests include robust adaptive control of nonlinear systems and time-delay systems. Corresponding author of this paper. Email: jiaoxh@tsinghua.edu.cn

SHEN Tie-Long Received his bachelor and master degrees in Automatic Control from Northeast Heavy Machinery Institute, China in 1982 and 1986, respectively, and Ph. D. degree in Mechanical Engineering from Sophia University, Tokyo, Japan, 1992. He is an associate professor at Department of Mechanical Engineering, Sophia University of Japan. His research interests include robust and adaptive control theory of nonlinear systems and its applications in mechanical systems, power systems, robotics and mobile dynamical system. Email: tetu-sin@sophia.ac.jp

SUN Yuan-Zhang Received bachelor degree from Wuhan University of Hydro and Electrical Engineering, China in 1978, the master degree from Electric Power Research Institute, China in 1982, and Ph. D. degree from Tsinghua University, Beijing in 1988. He is a professor at Tsinghua University and the head of Dynamic Simulation Power System Lab, and he has been a "ChangJiang Scholar" since 1999. His research interests include robust and adaptive control of nonlinear systems and its applications in power systems. Email: yzsun@tsinghua.edu.cn

Appendix A

Proof of Theorem 3. First, note that in the recursive design, Lyapunov-Razumikhin function of the whole system will be of a quadratic form on the new coordinate $\boldsymbol{z}$ under the change of the coordinate

$$z_i = x_i - \alpha_{i-1}(\tilde{x}_{i-1}), \quad i = 1, \cdots, n$$

with $\alpha_0 = 0$ and $\alpha_{i-1}(0) = 0$. Then, the Razumikhin condition becomes

$$\max_{-r \le \tau \le 0} V(\boldsymbol{z}_t(\tau)) < pV(\boldsymbol{z}_t(0))$$

It is equivalent to the following condition with a given constant $q = \sqrt{p} > 1$

$$\|\boldsymbol{z}_t(\tau)\| < q\|\boldsymbol{z}_t(0)\|, \quad \tau \in [-r, 0] \qquad (A1)$$

First Step. For x_1-subsystem, similar to the proof of Theorem 2, the derivative of V_1 defined as (17) can be obtained as (18) when Razumikhin condition (A1) holds. Noticing $\boldsymbol{z}\in R^n$, we choose

$$\alpha_1(\tilde{x}_1)=-f_1-\frac{1}{2}x_1b_{11}^2-\sum_{s=2}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_1-z_1-z_1\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{(n+1-i)}{2}j^2n^3q^2\tilde{\mu}_{ij}^2(jnq|z_1|) \quad (A2)$$

to make the derivative of V_1 satisfy the following form whenever the Razumikhin condition holds.

$$\dot{V}_1\le z_1z_2+\frac{1}{2}\mu_{11}^2(q\|\boldsymbol{z}\|)-\sum_{s=2}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{j-1}{2}n^2q^2z_1^2-\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{n(n+1-i)}{2}\mu_{ij}^2(jnq|z_1|)-z_1^2 \quad (A3)$$

Second Step. Similarly, for the z_2-subsystem with $x_3=z_3+\alpha_2(\tilde{x}_2)$, the time derivative of V_2 defined as (21) can be derived as the form (24) when the Razumikhin condition holds. But, since $\boldsymbol{z}\in R^n$, it is slightly different from Theorem 2 in dealing with N_2:

$$\begin{aligned}N_2 &\le |z_2|b_{22}(\tilde{x}_2)\left\{\sum_{l=1}^{2}\nu_{221}(nq|z_l|)+\sum_{l=3}^{n}\nu_{221}(nq|z_l|)\right\}\\ &\le\frac{1}{2}z_2^2b_{22}^2(\tilde{x}_2)\sum_{l=1}^{2}\tilde{\nu}_{221}^2(nq|z_l|)+\frac{n-2}{2}z_2^2b_{22}^2(\tilde{x}_2)+\\ &\quad\frac{1}{2}\sum_{l=1}^{2}n^2q^2z_l^2+\frac{1}{2}\sum_{l=3}^{n}\nu_{221}^2(nq|z_l|) \quad (A4)\end{aligned}$$

where $\nu_{221}(s)=s\tilde{\nu}_{221}(s)$. The last inequality in (A4) consists of three parts. The treatment of the first two parts, namely the first three terms, is the same as that in Theorem 2. The third part that is the fourth term will be dealt with in the later steps. Substituting (A4) and (A3) into (24), we obtain

$$\begin{aligned}\dot{V}_2\le&-z_1^2+z_2\left\{z_3+\alpha_2+z_1+f_2-\frac{\partial\alpha_1}{\partial x_1}[x_2+f_1]+\frac{1}{2}z_2\left|\frac{\partial\alpha_1}{\partial x_1}\right|^2b_{11}^2\right\}+\\ &\frac{1}{2}z_2^2b_{21}^2+\frac{1}{2}z_2^2b_{22}^2\left[\sum_{l=1}^{2}\tilde{\nu}_{221}^2(nq|z_l|)+n-1\right]+\sum_{l=1}^{2}\frac{n^2q^2}{2}z_l^2-\\ &\sum_{s=2}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_1^2+\sum_{i=1}^{2}\sum_{j=1}^{i}\frac{3-i}{2}\mu_{ij}^2(jq\|\boldsymbol{z}\|)+\\ &\frac{1}{2}\sum_{l=3}^{n}\nu_{221}^2(nq|z_l|)-\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{n(n+1-i)}{2}\mu_{ij}^2(jnq|z_1|) \quad (A5)\end{aligned}$$

whenever the Razumikhin condition holds. Therefore, a virtual feedback law defined by

$$\begin{aligned}\alpha_2(\tilde{x}_2)=&-z_1-f_2+\frac{\partial\alpha_1}{\partial x_1}[x_2+f_1]-\frac{1}{2}z_2b_{21}^2-\frac{1}{2}z_2\left|\frac{\partial\alpha_1}{\partial x_1}\right|^2b_{11}^2-\\ &\frac{1}{2}z_2b_{22}^2\left[\sum_{l=1}^{2}\tilde{\nu}_{221}^2(nq|z_l|)+n-1\right]-\sum_{s=2}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_2-\\ &z_2\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{j^2n^3q^2(n+1-i)}{2}\tilde{\mu}_{ij}^2(jnq|z_2|)-z_2 \quad (A6)\end{aligned}$$

is such that

$$\begin{aligned}\dot{V}_2\le& z_2z_3+\frac{1}{2}\sum_{l=3}^{n}\nu_{221}^2(nq|z_l|)+\sum_{i=1}^{2}\sum_{j=1}^{i}\frac{3-i}{2}\mu_{ij}^2(jq\|\boldsymbol{z}\|)-\\ &\sum_{l=1}^{2}z_l^2-\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{l=1}^{2}\frac{n(n+1-i)}{2}\mu_{ij}^2(jnq|z_l|)-\\ &\sum_{s=3}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}\sum_{l=1}^{2}z_l^2 \quad (A7)\end{aligned}$$

whenever the Razumikhin condition holds.

Induction Step. Suppose at the $(k-1)$-th step $(3\le k\le n)$, there are a set of virtual control laws $\alpha_i(\tilde{x}_i)$, $(i=1,\cdots,k-1)$ and a positive definite function $V_{k-1}(\tilde{z}_{k-1})$ such that

$$\begin{aligned}\dot{V}_{k-1}(\tilde{z}_{k-1})\le& z_{k-1}z_k+\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\sum_{l=k}^{n}\frac{k-i}{2}\nu_{ijs}^2(nq|z_l|)+\\ &\sum_{i=1}^{k-1}\sum_{j=1}^{i}\frac{k-i}{2}\mu_{ij}^2(jq\|\boldsymbol{z}\|)-\sum_{s=k}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}\sum_{l=1}^{k-1}z_l^2-\\ &\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{l=1}^{k-1}\frac{n(n+1-i)}{2}\mu_{ij}^2(jnq|z_l|)-\sum_{l=1}^{k-1}z_l^2 \quad (A8)\end{aligned}$$

whenever the Razumikhin condition holds, where $\nu_{ijs}(\cdot):=\mu_{ij}(j\eta_{(j-1)s}(\cdot))$ and $\eta_{(j-1)s}(\cdot)$ is a class $\mathcal{K}$ functions satisfying $|\alpha_{j-1}(\tilde{x}_{j-1})|\le\sum_{s=1}^{j-1}c_{(j-1)s}(|x_s|)=\sum_{s=1}^{j-1}\eta_{(j-1)s}(|z_s|)$ with the class $\mathcal{K}$ function $c_{(j-1)s}(\cdot)$.

Thus, in the following we will show that for the k-th subsystem of (3) the time derivative of V_k also satisfies the inequality form as (A8) if the positive definite function V_k is defined as

$$V_k(\tilde{z}_k)=V_{k-1}(\tilde{z}_{k-1})+\frac{1}{2}z_k^2 \quad (A9)$$

The time derivative of V_k along the trajectories of (3) can be calculated as

$$\begin{aligned}\dot{V}_k\le&\dot{V}_{k-1}+z_k\left\{x_{k+1}+f_k-\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}[x_{i+1}+f_i]\right\}+\frac{1}{2}z_k^2b_{k1}^2+\\ &\frac{1}{2}z_k^2\sum_{i=1}^{k-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2b_{i1}^2+\frac{1}{2}\sum_{i=1}^{k}\mu_{i1}^2(|x_{1t}|)+M_k \quad (A10)\end{aligned}$$

where

$$M_k=|z_k|\sum_{i=2}^{k-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|\sum_{j=2}^{i}b_{ij}\mu_{ij}(|x_{jt}|)+|z_k|\sum_{j=2}^{k}b_{kj}\mu_{kj}(|x_{jt}|)$$

Notice

$$z_i=x_i-\alpha_{i-1},\ \ |\alpha_{j-1}(\tilde{x}_{(j-1)t})|\le\sum_{s=1}^{j-1}\eta_{(j-1)s}(|z_{st}|)$$

with class $\mathcal{K}$ functions $\eta_{(j-1)s}(\cdot)$. Then,

$$M_k \le \frac{1}{2}z_k^2\sum_{j=2}^{k} b_{kj}^2+\frac{1}{2}\sum_{j=2}^{k}\mu_{kj}^2(j|z_{jt}|)+\frac{1}{2}\sum_{i=2}^{k-1}\sum_{j=2}^{i}\mu_{ij}^2(j|z_{jt}|)+ \frac{1}{2}z_k^2\sum_{i=2}^{k-1}\sum_{j=2}^{i}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2 b_{ij}^2+|z_k|\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}\nu_{kjs}(|z_{st}|)+ |z_k|\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right| b_{ij}\nu_{ijs}(|z_{st}|) \qquad (A11)$$

Substituting (A11) into (A10), we have

$$\dot V_k\le\dot V_{k-1}+z_k\left\{x_{k+1}+f_k-\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}(x_{i+1}+f_i)\right\}+\frac{1}{2}z_k^2\sum_{j=1}^{k}b_{kj}^2+ \frac{1}{2}z_k^2\sum_{i=1}^{k-1}\sum_{j=1}^{i}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2 b_{ij}^2+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{i}\mu_{ij}^2(jq\|\boldsymbol{z}\|)+N_k \quad (A12)$$

where

$$N_k=|z_k|\left[\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}\nu_{kjs}(q\|\boldsymbol{z}\|)+\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right| b_{ij}\nu_{ijs}(q\|\boldsymbol{z}\|)\right]$$

From the property of the class $\mathcal{K}$ function, Young$'$s Inequality and the function decomposition, it follows that

$$N_k\le\sum_{i=2}^{k}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}\sum_{l=1}^{k}z_l^2+\frac{1}{2}\sum_{i=2}^{k}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\sum_{l=k+1}^{n}\nu_{ijs}^2(nq|z_l|)+ \frac{1}{2}z_k^2\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2 b_{ij}^2\left[\sum_{l=1}^{k}\tilde\nu_{ijs}^2(nq|z_l|)+n-k\right]+ \frac{1}{2}z_k^2\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}^2\left[\sum_{l=1}^{k}\tilde\nu_{kjs}^2(nq|z_l|)+n-k\right] \qquad (A13)$$

Consider (A12), (A13), and (A8). According to a virtual feedback law defined by

$$\alpha_k(\tilde x_k)=-z_{k-1}-f_k+\sum_{i=1}^{k-1}\frac{\partial\alpha_{k-1}}{\partial x_i}[x_{i+1}+f_i]-\frac{1}{2}z_k\sum_{j=1}^{k}b_{kj}^2- \frac{1}{2}z_k\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2 b_{ij}^2\left[\sum_{l=1}^{k}\tilde\nu_{ijs}^2(nq|z_l|)+n-k\right]- \frac{1}{2}z_k\sum_{j=2}^{k}\sum_{s=1}^{j-1}b_{kj}^2\left[\sum_{l=1}^{k}\tilde\nu_{kjs}^2(nq|z_l|)+n-k\right]- z_k\sum_{i=2}^{k-1}\sum_{j=2}^{i}\sum_{s=1}^{j-1}\frac{(k-i)n^2q^2}{2}\tilde\nu_{ijs}^2(nq|z_k|)-z_k- \frac{1}{2}z_k\sum_{i=1}^{k-1}\sum_{j=1}^{i}\left|\frac{\partial\alpha_{k-1}}{\partial x_i}\right|^2 b_{ij}^2-\sum_{s=k}^{n}\sum_{i=2}^{s}\sum_{j=2}^{i}\frac{(j-1)n^2q^2}{2}z_k- z_k\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{j^2n^3q^2(n+1-i)}{2}\tilde\mu_{ij}^2(jnq|z_k|) \qquad (A14)$$

renders the derivative of V_k satisfy (A8) ($k{-}1\to k$) whenever the Razumikhin condition holds.

Obviously, this recursive procedure will terminate at the n-th step, where $V(\tilde z_n)=\frac{1}{2}\sum_{j=1}^{n}z_j^2$. According to the virtual control law $\alpha_n(\tilde x_n)$ (let $k{=}n$ in (A14)) the derivative of V satisfies

$$\dot V\le z_n(u-\alpha_n(\tilde x_n))+\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{n+1-i}{2}\mu_{ij}^2(jq\|\boldsymbol{z}\|)-\sum_{l=1}^{n}z_l^2- \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{l=1}^{n}\frac{n(n+1-i)}{2}\mu_{ij}^2(jnq|z_l|)$$

whenever the Razumikhin condition holds. Hence, by choosing the feedback control law $u=\alpha_n(\tilde x_n)$ and by using

$$\|\boldsymbol{z}\|\le\sum_{l=1}^{n}|z_l|,\ \ \mu_{ij}(jq\sum_{l=1}^{n}|z_l|)\le\sum_{l=1}^{n}\mu_{ij}(jnq|z_l|)$$

we obtain

$$\dot V\le-\|\boldsymbol{z}\|^2\quad \text{if } \|\boldsymbol{z}_t(\tau)\|<q\|\boldsymbol{z}_t(0)\|,\ \ \tau\in[-r,0] \qquad (A15)$$

Thus, the asymptotical stability follows from Lemma 1. □