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数据缺失场合三参数 Weibull 分布的参数估计

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摘　要：给出了缺失数据场合三参数 Weibull 分布参数的近似极大似然估计，通过 Monte-Carlo 模拟例子及实例证实了所给方法的可行性.

关键词：缺失数据；三参数 Weibull 分布；近似极大似然估计

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0　引　言

在可靠性统计分析中，三参数 Weibull 分布是最为广泛使用的寿命分布之一. 在产品的寿命分析上，如轴承的疲劳寿命[8]；在生物医学上，如实验动物的肿瘤出现时间[7]；在气候分析上，如年降水量和月降水量[14]，风速的概率分布[9]，河的最大流量[3]等. 关于三参数 Weibull 分布的统计分析已有许多文献作了研究，可参阅文献[1]～[13]. 现考虑如下情况，即由于某种原因，使得试验中的某一些样本寿命数据丢失，而只能知道其前后的寿命数据. 这种情况在现实中是经常发生的，称其为数据丢失或数据缺失. 关于 Weibull 分布数据缺失场合下的统计分析，可参阅文献[15]～[20]，这些文献仅针对两参数 Weibull 分布，而三参数 Weibull 分布数据缺失场合下的统计分析，至今未见有这方面的文献. 本文给出了参数的近似极大似然估计(AMLE)，当然，这种方法可适用于三参数对数正态分布.

1　参数的近似极大似然估计(AMLE)

设产品的寿命 t 服从三参数 Weibull 分布，其分布函数和密度函数分别为：

$$F(t;m,\eta,\theta) = 1 - e^{-(\frac{t-\theta}{\eta})^m},\ f(t;m,\eta,\theta) = \frac{m}{\eta}\left(\frac{t-\theta}{\eta}\right)^{m-1} e^{(\frac{t-\theta}{\eta})^m}, \qquad (1)$$

其中 $t>\theta$，$m>0$ 为形状参数，$\eta>0$ 为刻度参数，θ 为位置参数. 现假定有 n 个产品进行寿命试验，到有 r 个产品失效时停止试验，其次序失效数据为：$t_{(1)} \leqslant t_{(2)} \leqslant \cdots \leqslant t_{(r)}$. 现考虑如下情形：上述 r 个失效数据由于某些原因而使得有若干个数据缺失，设剩下 k 个数据，剩下的失效数据为：$0 \triangleq t_{(r_0)} \leqslant t_{(r_1)} \leqslant t_{(r_2)} \leqslant \cdots \leqslant t_{(r_k)}$，$r_0=0$.

令：$Y=\ln(t-\theta)$，则 Y 服从位置参数为 $\mu=\ln\eta$，刻度参数 $=1/m$ 的极小值分布，其分布函数和密度函数分别为：

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$$F(y;\mu,\sigma)=1-e^{-e^{\frac{y-\mu}{\sigma}}},\ f(y;\mu,\sigma)=\frac{1}{\sigma}e^{\frac{y-\mu}{\sigma}}e^{-e^{\frac{y-\mu}{\sigma}}},\tag{2}$$

令：$Z=\dfrac{Y-\mu}{\sigma}$，则 Z 服从标准极小值分布，其分布函数和密度函数分别为：

$$F(z)=1-e^{-e^{z}},\ f(z)=e^{z}e^{-e^{z}},\tag{3}$$

而 $Y_{(i)}=\ln(t_{(i)-\theta})$，$Z_{(i)}=\dfrac{Y_{(i)}-\mu}{\sigma}$，则 $Y_{(1)}\leqslant Y_{(2)}\leqslant\cdots\leqslant Y_{(r)}$ 为来自位置参数为 μ、刻度参数为 σ 的极小值分布的样本容量为 n 的前 r 个次序统计量，$Z_{(1)}\leqslant Z_{(2)}\leqslant\cdots\leqslant Z_{(r)}$ 为来自标准极小值分布的样本容量为 n 的前 r 个次序统计量，考虑到数据有缺失场合，则有：$0\triangleq Y_{(r_0)}\leqslant Y_{(r_1)}\leqslant Y_{(r_2)}\leqslant\cdots\leqslant Y_{(r_k)}$ 及 $0\triangleq Z_{(r_0)}\leqslant Z_{(r_1)}\leqslant Z_{(r_2)}\leqslant\cdots\leqslant Z_{(r_k)}$.

易知似然函数 $L(\mu,\sigma,\theta)$ 为(其中 C 为正常数)：

$$L(\mu,\sigma,\theta)=C\sigma^{-k}\left[F(Z_{(r_1)})\right]^{r_1-1}\prod_{i=1}^{k-1}\left[F(Z_{(r_{i+1})})-F(Z_{(r_i)})\right]^{r_{i+1}-r_i-1}\left[1-F(Z_{(r_k)})\right]^{n-r_k}\prod_{i=1}^{k}f(Z_{(r_i)}).\tag{4}$$

考虑到 $\dfrac{\partial Z}{\partial\mu}=-\dfrac{1}{\sigma}$，$\dfrac{\partial Z}{\partial\sigma}=-\dfrac{Z}{\sigma}$，$\dfrac{\partial Z}{\partial\theta}=-\dfrac{1}{\sigma}\dfrac{1}{t-\theta}=-\dfrac{1}{\sigma}e^{-(\sigma Z+\mu)}$，$\dfrac{\partial f(Z)}{\partial\mu}=-\dfrac{1}{\sigma}f'(Z)$，$\dfrac{\partial f(Z)}{\partial\sigma}=-\dfrac{Z}{\sigma}f'(Z)$，$\dfrac{\partial f(Z)}{\partial\theta}=-\dfrac{1}{\sigma}e^{-(\sigma Z+\mu)}f'(Z)$，$f'(Z)=e^{Z}e^{-e^{Z}}-e^{2Z}e^{-e^{Z}}$，

$$\frac{\partial\ln L(\mu,\sigma,\theta)}{\partial\mu}=-\frac{1}{\sigma}\left\{(r_1-1)\frac{f(Z_{(r_1)})}{F(Z_{(r_1)})}+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\frac{f(Z_{(r_{i+1})})-f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}-\right.$$
$$\left.(n-r_k)\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})}+\sum_{i=1}^{k}\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}\right\},\tag{5}$$

$$\frac{\partial\ln L(\mu,\sigma,\theta)}{\partial\mu}=-\frac{1}{\sigma}\left\{(r_1-1)\frac{f(Z_{(r_1)})}{F(Z_{(r_1)})}+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\frac{Z_{(r_i+1)}f(Z_{(r_{i+1})})-Z_{(r_i)}f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}-\right.$$
$$\left.(n-r_k)Z_{(r_k)}\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})}+\sum_{i=1}^{k}Z_{(r_i)}\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}+k\right\},\tag{6}$$

$$\frac{\partial\ln L(\mu,\sigma,\theta)}{\partial\mu}=-\frac{1}{\sigma}\left\{(r_1-1)e^{-(\sigma Z_{(r_1)}+\mu)}\frac{f(Z_{(r_1)})}{F(Z_{(r_1)})}+\right.$$
$$\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\frac{e^{-(\sigma Z_{(r_{i+1})}+\mu)}f(Z_{(r_{i+1})})-e^{-(\sigma Z_{(r_i)}+\mu)}f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}-$$
$$\left.(n-r_k)e^{-(\sigma Z_{(r_k)}+\mu)}\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})}+\sum_{i=1}^{k}e^{-(\sigma Z_{(r_i)}+\mu)}\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}\right\},\tag{7}$$

令 $\dfrac{\partial\ln L(\mu,\sigma,\theta)}{\partial\mu}=0$，$\dfrac{\partial\ln L(\mu,\sigma,\theta)}{\partial\sigma}=0$，$\dfrac{\partial\ln L(\mu,\sigma,\theta)}{\partial\theta}=0$，得如下三个方程：

$$(r_1-1)\frac{f(Z_{(r_1)})}{F(Z_{(r_1)})}+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\frac{f(Z_{(r_{i+1})})-f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}-(n-r_k)\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})}+\sum_{i=1}^{k}\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}=0,\tag{8}$$

$$k+(r_1-1)\frac{f(Z_{(r_1)})}{F(Z_{(r_1)})}+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\frac{Z_{(r_i+1)}f(Z_{(r_{i+1})})-Z_{(r_i)}f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}-$$
$$(n-r_k)Z_{(r_k)}\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})}+\sum_{i=1}^{k}Z_{(r_i)}\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}=0,\tag{9}$$

$$(r_1-1)e^{-(\sigma Z_{(r_1)}+\mu)}\frac{f(Z_{(r_1)})}{F(Z_{(r_1)})}+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\frac{e^{-(\sigma Z_{(r_{i+1})}+\mu)}f(Z_{(r_{i+1})})-e^{-(\sigma Z_{(r_i)}+\mu)}f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}-$$
$$(n-r_k)e^{-(\sigma Z_{(r_k)}+\mu)}\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})}+\sum_{i=1}^{k}e^{-(\sigma Z_{(r_i)}+\mu)}\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}=0,\tag{10}$$

令：$p_{r_i}=\frac{r_i}{n+1}, q_{r_i}=1-p_{r_i}, F(\zeta_{r_i})=p_{r_i}, \zeta_{r_i}=\ln(-\ln q_{r_i})$. 将函数$\frac{f(Z_{(r_i)})}{F(Z_{(r_i)})}$在点$\zeta_{r_i}$处泰勒展开得：

$$\frac{f(Z_{(r_i)})}{F(Z_{(r_i)})} \approx \alpha_0 - \beta_0 Z_{(r_i)}, \tag{11}$$

其中：
$$\frac{f(Z_{(r_i)})}{F(Z_{(r_i)})} \approx \frac{f(\zeta_{r_i})}{F(\zeta_{r_i})} + (Z_{(r_i)}-\zeta_{r_i})\frac{f'(\zeta_{r_i})F(\zeta_{r_i})-f^2(\zeta_{r_i})}{F^2(\zeta_{r_i})} =$$
$$\left\{\frac{f(\zeta_{r_i})}{F(\zeta_{r_i})} - \zeta_{r_i}\frac{f'(\zeta_{r_i})F(\zeta_{r_i})-f^2(\zeta_{r_i})}{F^2(\zeta_{r_i})}\right\} - \frac{f^2(\zeta_{r_i})-f'(\zeta_{r_i})F(\zeta_{r_i})}{F^2(\zeta_{r_i})}Z_{(r_i)},$$

$$\alpha_0 = \frac{f(\zeta_{r_i})}{F(\zeta_{r_i})} - \zeta_{r_i}\frac{f'(\zeta_{r_i})F(\zeta_{r_i}) - f^2(\zeta_{r_i})}{F^2(\zeta_{r_i})}, \tag{12}$$

$$\beta_0 = \frac{f^2(\zeta_{r_i}) - f'(\zeta_{r_i})F(\zeta_{r_i})}{F^2(\zeta_{r_i})}. \tag{13}$$

易见 $\beta_0>0$，事实上

$$f^2(\zeta_{r_1}) - f'(\zeta_{r_1})F(\zeta_{r_1}) = e^{\zeta_{r_1}}e^{-e^{\zeta_{r_1}}} - [e^{\zeta_{r_1}}e^{-e^{\zeta_{r_1}}} - e^{2\zeta_{r_1}}e^{-e^{\zeta_{r_1}}}][1 - e^{-e^{\zeta_{r_1}}}] =$$
$$e^{\zeta_{r_1}}e^{-e^{\zeta_{r_1}}}\left[1 - (1 - e^{\zeta_{r_1}})(1 - e^{-e^{\zeta_{r_1}}})\right] =$$
$$e^{\zeta_{r_1}}e^{-e^{\zeta_{r_1}}}\left[e^{\zeta_{r_1}} + e^{-e^{\zeta_{r_1}}} - e^{\zeta_{r_1}}e^{-e^{\zeta_{r_1}}}\right] =$$
$$e^{\zeta_{r_1}}e^{-e^{\zeta_{r_1}}}\left[e^{\zeta_{r_1}}(1 - e^{-e^{\zeta_{r_1}}}) + e^{-e^{\zeta_{r_1}}}\right] > 0$$

将函数$\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})}$在点$\zeta_{r_k}$处泰勒展开得：

$$\frac{f(Z_{(r_k)})}{1 - F(Z_{(r_k)})} \approx \alpha_k + \beta_k Z_{(r_k)}, \tag{14}$$

其中：
$$\frac{f(Z_{(r_k)})}{1-F(Z_{(r_k)})} \approx \frac{f(\zeta_{r_k})}{1-F(\zeta_{r_k})} + (Z_{(r_k)}-\zeta_{r_k})\frac{f'(\zeta_{r_k})[1-F(\zeta_{r_k})]+f^2(\zeta_{r_k})}{[1-F(\zeta_{r_k})]^2},$$

$$\alpha_k = \frac{f(\zeta_{r_k})}{1 - F(\zeta_{r_k})} - \zeta_{r_k}\frac{f'(\zeta_{r_k})[1 - F(\zeta_{r_k})] + f^2(\zeta_{r_k})}{[1 - F(\zeta_{r_k})]^2}, \tag{15}$$

$$\beta_k = \frac{f'(\zeta_{r_k})[1 - F(\zeta_{r_k})] + f^2(\zeta_{r_k})}{[1 - F(\zeta_{r_k})]^2}. \tag{16}$$

易见 $\beta_k>0$，事实上

$$f'(\zeta_{r_k})[1 - F(\zeta_{r_k})] + f^2(\zeta_{r_k}) = (e^{\zeta_{r_k}}e^{-e^{\zeta_{r_k}}} - e^{2\zeta_{r_k}}e^{-e^{\zeta_{r_k}}})e^{-e^{\zeta_{r_k}}} + e^{2\zeta_{r_k}}e^{-2e^{\zeta_{r_k}}} =$$
$$e^{\zeta_{r_k}}e^{-2e^{\zeta_{r_k}}}[e^{\zeta_{r_k}} + 1 - e^{\zeta_{r_k}}] = e^{\zeta_{r_k}}e^{-2e^{\zeta_{r_k}}} > 0$$

将二元函数$\frac{f(Z_{(r_{i+1})})-f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}$在点$(\zeta_{r_{i+1}}, \zeta_{r_i})$处泰勒展开得：

$$f(\frac{Z_{(r_{i+1})}) - f(Z_{(r_i)})}{F(Z_{(r_{i+1})}) - F(Z_{(r_i)})} \approx \varepsilon_i + \omega_i Z_{(r_i)} - \delta_i Z_{(r_{i+1})}, \tag{17}$$

其中：
$$\frac{f(Z_{(r_{i+1})})-f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})} \approx \frac{f(\zeta_{r_{i+1}})-f(\zeta_{r_i})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})} +$$
$$(Z_{(r_{i+1})}-\zeta_{r_{i+1}})\frac{f'(\zeta_{r_{i+1}})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-[f(\zeta_{r_{i+1}})-f(\zeta_{r_i})]f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2} +$$
$$(Z_{(r_i)}-\zeta_{r_i})\frac{-f'(\zeta_{r_i})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]+[f(\zeta_{r_{i+1}})-f(\zeta_{r_i})]f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}$$

$$\varepsilon_i = \frac{f(\zeta_{r_{i+1}})-f(\zeta_{r_i})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})} - \zeta_{r_{i+1}}\frac{f'(\zeta_{r_{i+1}})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-[f(\zeta_{r_{i+1}})-f(\zeta_{r_i})]f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2} -$$

$$\zeta_{r_i}\frac{-f'(\zeta_{r_i})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]+[f(\zeta_{r_{i+1}})-f(\zeta_{r_i})]f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2},\tag{18}$$

$$\omega_i=\frac{-f'(\zeta_{r_i})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]+[f(\zeta_{r_{i+1}})-f(\zeta_{r_i})]f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2},\tag{19}$$

$$\delta_i=-\frac{f'(\zeta_{r_{i+1}})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-[f(\zeta_{r_{i+1}})-f(\zeta_{r_i})]f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}.\tag{20}$$

将二元函数$\frac{Z_{(r_{i+1})}f(Z_{(r_{i+1})})-Z_{(r_i)}f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}$在点$(\zeta_{r_{i+1}},\zeta_{r_i})$处泰勒展开得:

$$\frac{Z_{(r_{i+1})}f(Z_{(r_{i+1})})-Z_{(r_i)}f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}\approx a_i+b_iZ_{(r_i)}-c_iZ_{(r_{i+1})},\tag{21}$$

其中: $$\frac{Z_{(r_{i+1})}f(Z_{(r_{i+1})})-Z_{(r_i)}f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}\approx\frac{\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})}+$$

$$(Z_{(r_{i+1})}-\zeta_{r_{i+1}})\frac{[f(\zeta_{r_{i+1}})+\zeta_{r_{i+1}}f'(\zeta_{r_{i+1}})][F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-[\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})]f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}+$$

$$(Z_{(r_i)}-\zeta_{r_i})\frac{[-f(\zeta_{r_i})-\zeta_{r_i}f'(\zeta_{r_i})][F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]+[\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})]f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2},$$

$$\begin{aligned}a_i=&\frac{\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})}-\\&\zeta_{r_{i+1}}\frac{[f(\zeta_{r_{i+1}})+\zeta_{r_{i+1}}f'(\zeta_{r_{i+1}})][F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-[\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})]f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}-\\&\zeta_{r_i}\frac{[-f(\zeta_{r_i})-\zeta_{r_i}f'(\zeta_{r_i})][F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]+[\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})]f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2},\end{aligned}\tag{22}$$

$$b_i=\frac{[-f(\zeta_{r_i})-\zeta_{r_i}f'(\zeta_{r_i})][F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]+[\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})]f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2},\tag{23}$$

$$c_i=-\frac{[f(\zeta_{r_{i+1}})+\zeta_{r_{i+1}}f'(\zeta_{r_{i+1}})][F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-[\zeta_{r_{i+1}}f(\zeta_{r_{i+1}})-\zeta_{r_i}f(\zeta_{r_i})]f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}.\tag{24}$$

将函数$\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}$在点$\zeta_{r_i}$处泰勒展开得:

$$\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}\approx d_i-e_iZ_{(r_i)},\tag{25}$$

其中: $$\frac{f'(Z_{(r_i)})}{f(Z_{(r_i)})}\approx\frac{f'(\zeta_{r_i})}{f(\zeta_{r_i})}+(Z_{(r_i)}-\zeta_{r_i})\frac{f''(\zeta_{r_i})f(\zeta_{r_i})-[f'(\zeta_{r_i})]^2}{f^2(\zeta_{r_i})},$$

$$d_i=\frac{f'(\zeta_{r_i})}{f(\zeta_{r_i})}-\zeta_{r_i}\frac{f''(\zeta_{r_i})f(\zeta_{r_i})-[f'(\zeta_{r_i})]^2}{f^2(\zeta_{r_i})},\tag{26}$$

$$e_i=-\frac{f''(\zeta_{r_i})f(\zeta_{r_i})-[f'(\zeta_{r_i})]^2}{f^2(\zeta_{r_i})},\tag{27}$$

将二元函数$\frac{f(Z_{(r_{i+1})})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}$在点$(\zeta_{r_{i+1}},\zeta_{r_i})$处泰勒展开得:

$$\frac{f(Z_{(r_{i+1})})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}\approx g_i+h_iZ_{(r_i)}-l_iZ_{(r_{i+1})},\tag{28}$$

其中: $$\frac{f(Z_{(r_{i+1})})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}\approx\frac{f(\zeta_{r_{i+1}})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})}+(Z_{(r_i)}-\zeta_{r_i})\frac{f(\zeta_{r_{i+1}})f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}+$$

$$(Z_{(r_{i+1})}-\zeta_{r_{i+1}})\frac{f'(\zeta_{r_{i+1}})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-f^2(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2},$$

$$g_i=\frac{f(\zeta_{r_{i+1}})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})}-\zeta_{r_{i+1}}\frac{f'(\zeta_{r_{i+1}})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-f^2(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}-\zeta_{r_i}\frac{f(\zeta_{r_{i+1}})f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2} \quad (29)$$

$$h_i=\frac{f(\zeta_{r_{i+1}})f(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}, \quad (30)$$

$$l_i=-\frac{f'(\zeta_{r_{i+1}})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-f^2(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}. \quad (31)$$

将二元函数$\frac{f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}$在点$(\zeta_{r_{i+1}},\zeta_{r_i})$处泰勒展开得：

$$\frac{f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}\approx s_i+u_iZ_{(r_i)}-v_iZ_{(r_{i+1})}, \quad (32)$$

其中
$$\frac{f(Z_{(r_i)})}{F(Z_{(r_{i+1})})-F(Z_{(r_i)})}\approx\frac{f(\zeta_{r_i})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})}+(Z_{(r_{i+1})}-\zeta_{r_{i+1}})\frac{-f(\zeta_{r_i})f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}+$$
$$(Z_{(r_i)}-\zeta_{r_i})\frac{f'(\zeta_{r_i})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-f^2(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2},$$

$$s_i=\frac{f(\zeta_{r_i})}{F(\zeta_{r_{i+1}})-F(\zeta_{r_i})}+\zeta_{r_{i+1}}\frac{f(\zeta_{r_i})f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}-\zeta_{r_i}\frac{f'(\zeta_{r_i})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-f^2(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}, \quad (33)$$

$$u_i=\frac{f'(\zeta_{r_i})[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]-f^2(\zeta_{r_i})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}, \quad (34)$$

$$v_i=\frac{f(\zeta_{r_i})f(\zeta_{r_{i+1}})}{[F(\zeta_{r_{i+1}})-F(\zeta_{r_i})]^2}. \quad (35)$$

将(11),(14),(17),(25)代入方程(8)化简得：

$$\Big[(r_1-1)\alpha_0+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\varepsilon_i-(n-r_k)\alpha_k+\sum_{i=1}^{k}d_i\Big]\sigma+\Big[-(r_1-1)\beta_0Y_{(r_1)}+$$
$$\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)(\omega_iY_{(r_i)}-\delta_iY_{(r_{i+1})})+(n-r_k)\beta_kY_{(r_k)})-\sum_{i=1}^{k}e_iY_{(r_i)})\Big]+$$
$$\Big[(r_1-1)\beta_0-\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)(\omega_i-\delta_i)+(n-r_k)\beta_k+\sum_{i=1}^{k}e_i\Big]\mu=0, \quad (36)$$

$$\hat{\mu}=B(\theta)-\hat{\sigma}C, \quad (37)$$

其中：

$$M=(r_1-1)\beta_0-\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)(\omega_i-\delta_i)+(n-r_k)\beta_k+\sum_{i=1}^{k}e_i, \quad (38)$$

$$B(\theta)=\frac{1}{M}\Big[(r_1-1)\beta_0\ln(t_{(r_1)}-\theta)-\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\big(\omega_i\ln(t_{(r_i)}-\theta)-\delta_i\ln(t_{(r_{i+1})}-\theta)+$$
$$(n-r_k)\beta_k\ln(t_{(r_k)}-\theta)+\sum_{i=1}^{k}e_i\ln(t_{(r_i)}-\theta)\Big], \quad (39)$$

$$C=\frac{1}{M}\Big[(r_1-1)\alpha_0+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)\varepsilon_i-(n-r_k)\alpha_k+\sum_{i=1}^{k}d_i\Big]. \quad (40)$$

将(11),(14),(21),(25)代入方程(9)化简得：

$$\Big\{k+(r_1-1)C(\alpha_0-\beta_0C)+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)(a_i+b_iC-c_iC)-(n-r_k)C(\alpha_k+\beta_kC)+$$
$$\sum_{i=1}^{k}C(d_i-e_iC)\Big\}\sigma^2+\Big\{(r_1-1)(Y_{(r_1)}-B(\theta))(\alpha_0-\beta_0C)-(r_1-1)C\beta_0(Y_{(r_1)}-B(\theta))+$$

$$\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)[b_i(Y_{(r_i)}-B(\theta))-c_i(Y_{(r_{i+1})}-B(\theta))]-(n-r_k)C\beta_k(Y_{(r_k)}-B(\theta))-$$

$$(n-r_k)(\alpha_k+\beta_kC)(Y_{(r_k)}-B(\theta))-\sum_{i=1}^{k}Ce_i(Y_{(r_i)}-B(\theta))+\sum_{i=1}^{k}(Y_{(r_i)}-B(\theta))(d_i-e_iC)\Big\}\sigma-$$

$$\Big\{(r_1-1)\beta_0(Y_{(r_1)}-B(\theta))^2+(n-r_k)\beta_k(Y_{(r_k)}-B(\theta))^2+\sum_{i=1}^{k}e_i(Y_{(r_i)}-B(\theta))^2\Big\}=0. \quad (41)$$

即：

$$A\sigma^2+D(\theta)\sigma-E(\theta)=0, \quad (42)$$

其中：

$$A=k+(r_1-1)C(\alpha_0-\beta_0C)+\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)(a_i+b_iC-c_iC)-$$
$$(n-r_k)C(\alpha_k+\beta_kC)+\sum_{i=1}^{k}C(d_i-e_iC), \quad (43)$$

$$D(\theta)=(r_1-1)(\ln(t_{(r_1)}-\theta)-B(\theta))(\alpha_0-\beta_0C)-(r_1-1)C\beta_0(\ln(t_{(r_1)}-\theta)-B(\theta))+$$
$$\sum_{i=1}^{k-1}(r_{i+1}-r_i-1)[b_i(\ln(t_{(r_i)}-\theta)-B(\theta))-c_i(\ln(t_{(r_{i+1})}-\theta)-B(\theta))]-$$
$$(n-r_k)C\beta_k(\ln(t_{(r_k)}-\theta)-B(\theta))-(n-r_k)(\alpha_k+\beta_kC)(\ln(t_{(r_k)}-\theta)-B(\theta))-$$
$$\sum_{i=1}^{k}Ce_i(\ln(t_{(r_i)}-\theta)-B(\theta))\sum_{i=1}^{k}(\ln(t_{(r_i)}-\theta)-B(\theta))(d_i-e_iC). \quad (44)$$

$$E(\theta)=(r_1-1)\beta_0(\ln(t_{(r_1)}-\theta)-B(\theta))^2+(n-r_k)\beta_k(\ln(t_{(r_k)}-\theta)-B(\theta))^2+\sum_{i=1}^{k}e_i(\ln(t_{(r_i)}-\theta)-B(\theta))^2. \quad (45)$$

由此得：

$$\hat{\sigma}=\frac{-D(\theta)+\sqrt{D^2(\theta)+4AE(\theta)}}{2A}, \quad (46)$$

代入(35)得：

$$\hat{\mu}(\theta)=B(\theta)-\hat{\sigma}(\theta)C. \quad (47)$$

将(11)，(14)，(25)，(28)，(32)代入方程(10)，同时注意到：$e^{-(\sigma z+\mu)}=e^{-(\sigma\frac{\ln(t-\theta)-\mu}{\sigma}+\mu)}=\frac{1}{t-\theta}$，化简方程(10)得：

$$\frac{r_1-1}{t_{(r_1)}-\theta}\left[\alpha_0-\beta_0\frac{\ln(t_{(r_1)}-\theta)-\mu}{\sigma}\right]+\sum_{i=1}^{k-1}\frac{r_{i+1}-r_i-1}{t_{(r_{i+1})}-\theta}\left[g_i+h_i\frac{\ln(t_{(r_i)}-\theta)-\mu}{\sigma}-l_i\frac{\ln(t_{(r_{i+1})}-\theta)-\mu}{\sigma}\right]-$$
$$\sum_{i=1}^{k-1}\frac{r_{i+1}-r_i-1}{t_{(r_i)}-\theta}\left[s_i+u_i\frac{\ln(t_{(r_i)}-\theta)-\mu}{\sigma}-v_i\frac{\ln(t_{(r_{i+1})}-\theta)-\mu}{\sigma}\right]-\frac{n-r_k}{t_{(r_k)}-\theta}\left[\alpha_k+\beta_k\frac{\ln(t_{(r_k)}-\theta)-\mu}{\sigma}\right]+$$
$$\sum_{i=1}^{k}\frac{1}{t_{(r_i)}-\theta}\left[d_i-e_i\frac{\ln(t_{(r_i)}-\theta)-\mu}{\sigma}\right]=0. \quad (48)$$

将(46)，(47)代入(48)得：

$$\frac{r_1-1}{t_{(r_1)}-\theta}\left[\alpha_0-\beta_0\frac{(\ln t_{(r_1)}-\theta)-\mu(\theta)}{\sigma(\theta)}\right]+\sum_{i=1}^{k-1}\frac{r_{i+1}-r_i-1}{t_{(r_{i-1})}-\theta}\left[g_i+h_i\frac{(\ln t_{(r_i)}-\theta)-\mu(\theta)}{\sigma(\theta)}\right.$$
$$\left.-l_i\frac{\ln(t_{(r_{i+1})}-\theta)-\mu(\theta)}{\sigma(\theta)}\right]-\sum_{i=1}^{k-1}\frac{r_{i+1}-r_i-1}{t_{(r_i)}-\theta}\left[s_i+u_i\frac{\ln(t_{(r_i)}-\theta)-\mu(\theta)}{\sigma(\theta)}-v_i\frac{\ln(t_{(r_{i+1})}-\theta)-\mu(\theta)}{\sigma(\theta)}\right]-$$
$$\frac{n-r_k}{t_{(r_k)}-\theta}\left[\alpha_k+\beta_k\frac{\ln(t_{(r_k)}-\theta)-\mu(\theta)}{\sigma(\theta)}\right]+\sum_{i=1}^{k}\frac{1}{t_{(r_i)}-\theta}\left[d_i-e_i\frac{\ln(t_{(r_i)}-\theta)-\mu(\theta)}{\sigma(\theta)}\right]=0. \quad (49)$$

方程(49)仅含有位置参数 θ，故可解超越方程(49)得参数 θ 的近似极大似然估计(AMLE)$\hat{\theta}$. 代入(46)，(47)得参数 σ,μ 的近似极大似然估计(AMLE)分别为 $\hat{\sigma},\hat{\mu}$：

$$\hat{\sigma}=\frac{-D(\hat{\theta})+\sqrt{D^2(\hat{\theta})+4AE(\hat{\theta})}}{2A},\ \hat{\mu}=B(\hat{\theta})-\hat{\sigma}C. \quad (50)$$

由此形状参数 m 和刻度参数 η 的近似极大似然估计(AMLE)分别为 $\hat{m}$ 和 $\hat{\eta}$：

$$\hat{m} = \frac{1}{\hat{\sigma}}, \hat{\eta} = e^{\hat{\mu}}. \tag{51}$$

3 Monte-Carlo 模拟及实例

例1:我们对 $n=10$,参数真值取为:$\theta=10,\eta=10,m=0.5(0.5)1.5$,通过 Monte-Carlo 模拟的方法产生10个服从三参数 Weibull 分布的随机数共三组. 利用本文方法可得参数 μ,m,η 的近似极大似然估计列于下表.

表 1 Monte-Carlo 模拟表

参数真值	模拟数据	缺失情况	近似极大似然估计(AMLE)$\hat{\theta},\hat{m},\hat{\eta}$
$\theta=10$	10.0425	$k=10$ 无缺失	9.4500,0.5121,10.1060
$m=0.5$	10.0456		
$\eta=10$	10.1148		
	10.7127	$k=8$ $t_{(1)},t_{(2)}$缺失	9.1500,0.4783,9.9413
	10.9414		
	11.1614	$t_{(1)},t_{(6)}$缺失	9.6500,0.5022,10.0111
	18.3008		
	58.2557	$k=5$ $t_{(1)},t_{(2)}$,	9.0500,0.3768,11.6902
	83.1450	$t_{(3)},t_{(9)},t_{(10)}$缺失	
	98.7374		
$\theta=10$	10.5041	$k=10$ 无缺失	10.3500,1.0903,10.0522
$m=1.0$	12.2163		
$\eta=10$	13.5298		
	15.1095	$k=8$ $t_{(1)},t_{(2)}$缺失	10.3500,1.0655,9.8622
	18.0447		
	19.8137	$t_{(3)},t_{(4)}$缺失	11.8500,1.3367,7.5730
	20.5013		
	21.8150	$k=5$ $t_{(1)},t_{(2)}$,	10.1500,1.0262,9.9205
	23.2187	$t_{(3)},t_{(9)},t_{(10)}$缺失	
	50.7382		
$\theta=10$	11.4449	$k=10$ 无缺失	9.7500,1.5432,10.0603
$m=1.5$	11.7751		
$\eta=10$	14.9218		
	15.5109	$k=8$ $t_{(1)},t_{(2)}$缺失	10.1500,1.6730,10.0491
	16.7013		
	18.6616	$t_{(1)},t_{(10)}$缺失	9.5500,1.5130,10.0288
	20.9133		
	22.0284	$k=5$ $t_{(1)},t_{(2)}$,	9.9500,1.8239,9.5879
	25.6172	$t_{(3)},t_{(9)},t_{(10)}$缺失	
	31.9577		

例2[1]:从1890年到1969年间,宾夕法尼亚州 Harrisburrg 的 Susquehanna 河20个每四年的最大流量($10^6\ ft^3/s$)为:0.654,0.613,0.315,0.449,0.297,0.402,0.379,0.423,0.379,0.3235,0.269,0.740,0.418,0.412,0.494,0.416,0.338,0.392,0.414,0.265这些数据起先由 Dumonceaux 和 Autle(1973)用作说明极限分布的例子的,利用本文方法可得 θ,m,η 的近似极大似然估计,同时列出这一问题的一些参数估计见下表.

表 2 Susquehanna 河20个每年最大流量的参数估计

METHOD	$\hat{\theta}$	$\hat{m}$	$\hat{\eta}$
MME	0.2409	1.4790	0.2015
ME	0.2302	1.5750	0.2149
Zanakis	0.2650	1.3280	0.2020
WBE	0.2465	1.3960	0.1386
AMLE($k=20$) 无缺失	0.2530	1.3846	0.1786
AMLE($k=17$) $t_{(2)},t_{(8)},t_{(19)}$缺失	0.2475	1.1184	0.1746

例3[3]:本例由 MCCOSL(1974)提供的,数据包括同种型号的10个轴承的依小时计的疲劳寿命,依小到大排序为:152.7,172.0,172.5,173.3,193.0,204.7,216.5,234.9,262.6,422.6,利用本文方法可得 θ,m,η 的近似极大似然估计,同时列出这一问题的一些参数估计见下表.

表 3 轴承疲劳寿命的参数估计

METHOD	$\hat{\theta}$	$\hat{m}$	$\hat{\eta}$
MME	146.14	0.949	72.59
ME	129.43	1.165	96.02
Zanakis	151.09	1.170	65.41
WBE	142.48	1.112	56.64
AMLE(k=10) 无缺失	147.7500	1.0908	71.7147
AMLE(k=8) ,缺失	152.3000	1.1039	65.8200

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Statistical Inference for Incomplete Data Three-parameter WEIBULL Distribution

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Abstract: We derive an approximate maximum likelihood estimation of the parameters of the three-parameter Weibull distribution under a TYPE-Ⅱ censored sample. In addition, we show the feasibility of this method by using two Monte-Carlo simulated samples.

Key words: TYPE-Ⅱ censored data;three-parameter Weibull distribution;approximate maximum likelihood