双边截尾场合下BS疲劳寿命分布的参数估计*

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提 要 在双边定数截尾场合下,给出了Birnbaum-Saunders疲劳寿命分布的统计分析,给出了参数的拟最小二乘估计和近似极大似然估计,并用随机模拟方法比较极大似然估计、近似极大似然估计和拟最小二乘估计的偏性和均方误差.

关键词 Birnbaum-Saunders疲劳寿命分布;双边定数截尾寿命试验;拟最小二乘估计;极大似然估计;近似极大似然估计

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0 引 言

在可靠性寿命试验中,产品失效有各种原因,其中有一种是由于在某种周期应力作用下,产品产生裂缝,如果裂缝长度达到或超过某一值时,则产品失效,文[1]在一定条件下对产品这种失效原因导出了一个两参数疲劳寿命分布,其分布的分布函数、密度函数分别为:

$$F(t,\alpha,\beta)=\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t}{\beta}\right)\right),\ t\geq 0,\alpha>0,\beta>0, \tag{1}$$

$$f(t,\alpha,\beta)=\frac{1}{2\alpha t}\left(\sqrt{\frac{t}{\beta}}+\sqrt{\frac{\beta}{t}}\right)\varphi\left(\frac{1}{\alpha}\xi\left(\frac{t}{\beta}\right)\right),\ t\geq 0,\alpha>0,\beta>0. \tag{2}$$

式中 $\xi(x)=\sqrt{x}-\sqrt{\frac{1}{x}}$, $\Phi(x)$, $\varphi(x)$ 分别为标准正态分布的分布函数和密度函数. 参数 α 和 β 分别称为形状参数和刻度参数,记此分布为BS(α,β). 文[2],[3]说明了在较文[1]更弱的条件下产品的寿命分布仍服从BS(α,β),因此用BS(α,β)分布拟合这类产品的寿命分布比常用的Weibull、对数正态分布更合适. 在完全样本情形,文[4]讨论了BS(α,β)分布参数的点估计,导出了参数的极大似然估计. 文[5]利用模拟方法和极大似然估计的渐近正态性讨论了参数的区间估计和假设检验. 文[6]研究了BS(α,β)分布的对数线性模型,导出了参数的极大似然估计和最小二乘估计,利用MLE的渐近正态性给出了参数的近似置信区

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间. 文[7]讨论了可靠度函数的区间估计方法. 最近, 文[8]讨论了BS(α, β)分布在截尾试验情形下的统计分析, 给出了参数的拟最小二乘估计和极大似然估计, 另外还给出了刻度参数 β 的区间估计.

本文讨论了Birnbaum-Saunders 疲劳寿命分布在双边定数截尾场合下的统计分析, 给出了参数的拟最小二乘估计(QLSE)、近似极大似然估计(AMLE).

1 双边定数截尾场合下参数的拟最小二乘估计(QLSE)

设产品的寿命分布服从两参数BS(α, β)分布, 设 $t_{(r_1)} \leq t_{(r_1+1)} \leq \cdots\cdots \leq t_{(r_2)}$ 是来自BS(α, β)分布样本容量为 n 的第 r_1 个到第 r_2 个次序统计量, 共有 $k = r_2 - r_1 + 1$ 个寿命数据, $r_1 \geq 1$. 令: $Z = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)$, 由文献[8]知: Z 服从标准正态分布 $N(0, 1)$, $Z_{(i)} = \frac{1}{\alpha}\left(\sqrt{\frac{t_{(i)}}{\beta}} - \sqrt{\frac{\beta}{t_{(i)}}}\right)$, $Z_{(r_1)} \leq Z_{(r_1+1)} \leq \cdots\cdots \leq Z_{(r_2)}$ 是来自样本容量为 n 的标准正态分布 $N(0, 1)$ 的第 r_1 个到第 r_2 个次序统计量.

完全可以类似文[8]的方法得到参数 α, β 的拟最小二乘估计 $\hat{\alpha}_1$、$\hat{\beta}_1$:

$$\hat{\beta}_1 = \left[\frac{c\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} V^{ij}\sqrt{t_{(i)}t_{(j)}} - \left(\sum_{i=r_1}^{r_2} E(n, r_1, r_2, i)\sqrt{t_{(i)}}\right)^2}{c\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} \frac{V^{ij}}{\sqrt{t_{(i)}t_{(j)}}} - \left(\sum_{i=r_1}^{r_2} \frac{E(n, r_1, r_2, i)}{\sqrt{t_{(i)}}}\right)^2}\right]^{\frac{1}{2}}, \tag{3}$$

$$\hat{\alpha}_1 = \frac{1}{A_{r_1, r_2, n}}\sum_{i=r_1}^{r_2} E(n, r_1, r_2, i)\left(\sqrt{\frac{t_{(i)}}{\hat{\beta}_1}} - \sqrt{\frac{\hat{\beta}_1}{t_{(i)}}}\right). \tag{4}$$

其中: $a_i = E(Z_{(i)})$ 为标准正态分布 $N(0, 1)$ 样本容量为 n 的第 i 个次序统计量的数学期望.

$\sum = (V_{ij})_{k\times k}$ 为标准正态分布 $N(0, 1)$ 样本容量为 n 的 $(Z_{(r_1)}, Z_{(r_1+1)}, \cdots\cdots, Z_{(r_2)})$ 的协方差阵, V_{ij} 的值可查文献[10]. $\sum^{-1} = (V^{ij})_{k\times k}$ 为 $\sum$ 的逆矩阵.

$$\triangle = \left(\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} V^{ij}\right)\left(\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} V^{ij}a_i a_j\right) - \left(\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} V^{ij}a_j\right)^2,$$

$$A_{r_1, r_2, n} = \frac{1}{\triangle}\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} V^{ij}a_i a_j, B_{r_1, r_2, n} = -\frac{1}{\triangle}\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} V^{ij}a_j, l_{r_1, r_2, n} = \frac{1}{\triangle}\sum_{i=r_1}^{r_2}\sum_{j=r_1}^{r_2} V^{ij},$$

$$C(n, r_1, r_2, i) = l_{r_1, r_2, n}\sum_{j=r_1}^{r_2} V^{ij}a_j + B_{r_1, r_2, n}\sum_{j=r_1}^{r_2} V^{ij},$$

$$D(n, r_1, r_2, i) = A_{r_1, r_2, n}\sum_{j=r_1}^{r_2} V^{ij}a_j + B_{r_1, r_2, n}\sum_{j=r_1}^{r_2} V^{ij}a_j, c = A_{r_1, r_2, n}\left(l_{r_1, r_2, n}A_{r_1, r_2, n} - B^2_{r_1, r_2, n}\right),$$

$$E(n, r_1, r_2, i) = A_{r_1, r_2, n} C(n, r_1, r_2, i) - B_{r_1, r_2, n} D(n, r_1, r_2, i).$$

2　双边定数截尾场合下参数的近似极大似然估计(AMLE)

似然函数为 $L(\alpha, \beta)$ (其中 C 为正常数): $L(\alpha, \beta) =$

$$\frac{C}{2^k}\alpha^{-k}\left[\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_1)}}{\beta}\right)\right)\right]^{r_1-1}\left[1-\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_2)}}{\beta}\right)\right)\right]^{n-r_2}\prod_{i=r_1}^{r_2}\left[\frac{1}{t_{(i)}}\left(\sqrt{\frac{t_{(i)}}{\beta}}+\sqrt{\frac{\beta}{t_{(i)}}}\right)\varphi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(i)}}{\beta}\right)\right)\right]. \quad (5)$$

令 $\frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = 0$, $\frac{\partial \ln L(\alpha, \beta)}{\partial \beta} = 0$, 得如下两个方程:

$$k\alpha + (r_1-1)\frac{\varphi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_1)}}{\beta}\right)\right)}{\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_1)}}{\beta}\right)\right)}\xi\left(\frac{t_{(r_1)}}{\beta}\right) -$$

$$(n-r_2)\frac{\varphi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_2)}}{\beta}\right)\right)}{1-\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_2)}}{\beta}\right)\right)}\xi\left(\frac{t_{(r_2)}}{\beta}\right) - \frac{1}{\alpha}\sum_{i=r_1}^{r_2}\xi^2\left(\frac{t_{(i)}}{\beta}\right) = 0, \quad (6)$$

$$(r_1-1)\frac{\varphi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_1)}}{\beta}\right)\right)}{\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_1)}}{\beta}\right)\right)}\left(\sqrt{\frac{t_{(r_1)}}{\beta}}+\sqrt{\frac{\beta}{t_{(r_1)}}}\right) - (n-r_2)\frac{\varphi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_2)}}{\beta}\right)\right)}{1-\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(r_2)}}{\beta}\right)\right)}\left(\sqrt{\frac{t_{(r_2)}}{\beta}}+\sqrt{\frac{\beta}{t_{(r_2)}}}\right) +$$

$$\alpha\sum_{i=r_1}^{r_2}\frac{\sqrt{\frac{t_{(i)}}{\beta}}-\sqrt{\frac{\beta}{t_{(i)}}}}{\sqrt{\frac{t_{(i)}}{\beta}}+\sqrt{\frac{\beta}{t_{(i)}}}} - \frac{1}{\alpha}\sum_{i=r_1}^{r_2}\xi\left(\frac{t_{(i)}}{\beta}\right)\left(\sqrt{\frac{t_{(i)}}{\beta}}+\sqrt{\frac{\beta}{t_{(i)}}}\right) = 0. \quad (7)$$

注: 文中 $\varphi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(i)}}{\beta}\right)\right) = \varphi(x)\Big|_{x=\frac{1}{\alpha}\xi\left(\frac{t_{(i)}}{\beta}\right)}$, $\Phi\left(\frac{1}{\alpha}\xi\left(\frac{t_{(i)}}{\beta}\right)\right) = \Phi(x)\Big|_{x=\frac{1}{\alpha}\xi\left(\frac{t_{(i)}}{\beta}\right)}$, $i=\overline{r_1, r_2}$.

要求解上述两个方程是很不容易的. 由此可以根据文[9]给出参数的近似MLE (AMLE), 似然函数可以写为: $L(\alpha, \beta) =$

$$C\left[\Phi(Z_{(r_1)})\right]^{r_1-1}\left[1-\Phi(Z_{(r_2)})\right]^{n-r_2}\prod_{i=r_1}^{r_2}\left[\varphi(Z_{(i)})\frac{1}{\alpha\beta}\frac{\sqrt{\alpha^2 Z_{(i)}^2+4}}{\alpha^2 Z_{(i)}^2+\alpha Z_{(i)}\sqrt{\alpha^2 Z_{(i)}^2+4}+2}\right], \quad (8)$$

$$\frac{\partial Z_{(i)}}{\partial \alpha} = -\frac{1}{\alpha^2}\left(\sqrt{\frac{t_{(i)}}{\beta}}-\sqrt{\frac{\beta}{t_{(i)}}}\right) = -\frac{Z_{(i)}}{\alpha}, \quad \frac{\partial Z_{(i)}}{\partial \beta} = -\frac{1}{2\alpha\beta}\left(\sqrt{\frac{t_{(i)}}{\beta}}+\sqrt{\frac{\beta}{t_{(i)}}}\right) = -\frac{\sqrt{\alpha^2 Z_{(i)}^2+4}}{2\alpha\beta}.$$

令 $\frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = 0$, $\frac{\partial \ln L(\alpha, \beta)}{\partial \beta} = 0$, 得如下两个方程:

$$\sum_{i=r_1}^{r_2} Z_{(i)}^2 + (n-r_2)Z_{(r_2)}\frac{\varphi(Z_{(r_2)})}{1-\Phi(Z_{(r_2)})} - (r_1-1)Z_{(r_1)}\frac{\varphi(Z_{(r_1)})}{\Phi(Z_{(r_1)})} - k = 0, \quad (9)$$

$$\sum_{i=r_1}^{r_2} Z_{(i)}\sqrt{\alpha^2 Z_{(i)}^2+4}+(n-r_2)\sqrt{\alpha^2 Z_{(r_2)}^2+4}\,\frac{\varphi(Z_{(r_2)})}{1-\Phi(Z_{(r_2)})}-$$

$$(r_1-1)\sqrt{\alpha^2 Z_{(r_1)}^2+4}\,\frac{\varphi(Z_{(r_1)})}{\Phi(Z_{(r_1)})}-\alpha^2\sum_{i=r_1}^{r_2}\frac{Z_{(i)}}{\sqrt{\alpha^2 Z_{(i)}^2+4}}=0. \qquad (10)$$

似然方程(9), (10)**没有明显的解析表达式, 可以给出近似的似然方程. 记**: $p_{r_j}=\frac{r_j}{n+1}$, $q_{r_j}=1-p_{r_j}$, $\xi_{r_j}=\Psi(p_{r_j})=\Phi^{-1}(p_{r_j})$, $j=1,2$.

将函数$\frac{\varphi(Z_{(r_1)})}{\Phi(Z_{(r_1)})}$**在点** ξ_{r_1} **处, 函数**$\frac{\varphi(Z_{(r_2)})}{1-\Phi(Z_{(r_2)})}$**在点** ξ_{r_2} **处一阶泰勒展开, 由文**[9]**可知**:

$$\frac{\varphi(Z_{(r_1)})}{\Phi(Z_{(r_1)})}\approx a_1-b_1 Z_{(r_1)},\quad \frac{\varphi(Z_{(r_2)})}{1-\Phi(Z_{(r_2)})}\approx a_2+b_2 Z_{(r_2)}.$$

其中

$$a_1=\frac{\varphi(\xi_{r_1})}{p_{r_1}}\left[1+\xi_{r_1}^2+\frac{\xi_{r_1}\varphi(\xi_{r_1})}{p_{r_1}}\right],\quad b_1=\frac{\varphi(\xi_{r_1})}{p_{r_1}^2}\left[\varphi(\xi_{r_1})+p_{r_1}\xi_{r_1}\right]>0,$$

$$a_2=\frac{\varphi(\xi_{r_2})}{q_{r_2}}\left[1+\xi_{r_2}^2-\frac{\xi_{r_2}\varphi(\xi_{r_2})}{q_{r_2}}\right],\quad b_2=\frac{\varphi(\xi_{r_2})}{q_{r_2}}\left[\varphi(\xi_{r_2})-q_{r_2}\xi_{r_2}\right]>0.$$

将(9)**式化简为如下近似的似然方程**(11), (12):

$$k\alpha^2-\left[(n-r_2)a_2\left(\sqrt{\frac{t_{(r_2)}}{\beta}}-\sqrt{\frac{\beta}{t_{(r_2)}}}\right)-(r_1-1)a_1\left(\sqrt{\frac{t_{(r_1)}}{\beta}}-\sqrt{\frac{\beta}{t_{(r_1)}}}\right)\right]\alpha-\left[\sum_{i=r_1}^{r_2}\left(\frac{t_{(i)}}{\beta}+\frac{\beta}{t_{(i)}}-2\right)+\right.$$

$$\left.(n-r_2)b_2\left(\frac{t_{(r_2)}}{\beta}+\frac{\beta}{t_{(r_2)}}-2\right)+(r_1-1)b_1\left(\frac{t_{(r_1)}}{\beta}+\frac{\beta}{t_{(r_1)}}-2\right)\right]=0. \qquad (11)$$

令 $$A_1(\beta)=(n-r_2)a_2\left(\sqrt{\frac{t_{(r_2)}}{\beta}}-\sqrt{\frac{\beta}{t_{(r_2)}}}\right)-(r_1-1)a_1\left(\sqrt{\frac{t_{(r_1)}}{\beta}}-\sqrt{\frac{\beta}{t_{(r_1)}}}\right),$$

$$B(\beta)=\sum_{i=r_1}^{r_2}\left(\frac{t_{(i)}}{\beta}+\frac{\beta}{t_{(i)}}-2\right)+(n-r_2)b_2\left(\frac{t_{(r_2)}}{\beta}+\frac{\beta}{t_{(r_2)}}-2\right)+(r_1-1)b_1\left(\frac{t_{(r_1)}}{\beta}+\frac{\beta}{t_{(r_1)}}-2\right).$$

进而得

$$k\alpha^2-A_1(\beta)\alpha-B(\beta)=0. \qquad (12)$$

将(10)**式化简为如下近似的似然方程**(13), (14):

$$\sum_{i=r_1}^{r_2}\frac{\sqrt{\frac{t_{(i)}}{\beta}}-\sqrt{\frac{\beta}{t_{(i)}}}}{\sqrt{\frac{t_{(i)}}{\beta}}+\sqrt{\frac{\beta}{t_{(i)}}}}\alpha^2-\left[(n-r_2)a_2\left(\sqrt{\frac{t_{(r_2)}}{\beta}}+\sqrt{\frac{\beta}{t_{(r_2)}}}\right)-(r_1-1)a_1\left(\sqrt{\frac{t_{(r_1)}}{\beta}}+\sqrt{\frac{\beta}{t_{(r_1)}}}\right)\right]\alpha-$$

$$\left[\sum_{i=r_1}^{r_2}\left(\frac{t_{(i)}}{\beta}-\frac{\beta}{t_{(i)}}\right)+(n-r_2)b_2\left(\frac{t_{(r_2)}}{\beta}-\frac{\beta}{t_{(r_2)}}\right)+(r_1-1)b_1\left(\frac{t_{(r_1)}}{\beta}-\frac{\beta}{t_{(r_1)}}\right)\right]=0. \qquad (13)$$

令 $$A_2(\beta)=(n-r_2)a_2\left(\sqrt{\frac{t_{(r_2)}}{\beta}}+\sqrt{\frac{\beta}{t_{(r_2)}}}\right)-(r_1-1)a_1\left(\sqrt{\frac{t_{(r_1)}}{\beta}}+\sqrt{\frac{\beta}{t_{(r_1)}}}\right),$$

令
$$C(\beta)=\sum_{i=r_1}^{r_2}\frac{\sqrt{\frac{t_{(i)}}{\beta}}-\sqrt{\frac{\beta}{t_{(i)}}}}{\sqrt{\frac{t_{(i)}}{\beta}}+\sqrt{\frac{\beta}{t_{(i)}}}}=\sum_{i=r_1}^{r_2}\frac{t_{(i)}-\beta}{t_{(i)}+\beta},$$

$$D(\beta)=\sum_{i=r_1}^{r_2}\left(\frac{t_{(i)}}{\beta}-\frac{\beta}{t_{(i)}}\right)+(n-r_2)b_2\left(\frac{t_{(r_2)}}{\beta}-\frac{\beta}{t_{(r_2)}}\right)+(r_1-1)b_1\left(\frac{t_{(r_1)}}{\beta}-\frac{\beta}{t_{(r_1)}}\right),$$

进而得
$$C(\beta)\alpha^2-A_2(\beta)\alpha-D(\beta)=0,\tag{14}$$

$$\alpha=\frac{kD(\beta)-B(\beta)C(\beta)}{A_1(\beta)C(\beta)-kA_2(\beta)},\tag{15}$$

$$k[kD(\beta)-B(\beta)C(\beta)]^2-A_1(\beta)[kD(\beta)-B(\beta)C(\beta)][A_1(\beta)C(\beta)-kA_2(\beta)]-B(\beta)\left[A_1(\beta)C(\beta)-kA_2(\beta)\right]=0.\tag{16}$$

于是可以从方程(16)中解得参数 β 的近似极大似然估计 $\hat{\beta}_2$, 进而可得参数 α 的近似极大似然估计 $\hat{\alpha}_2$:

$$\hat{\alpha}_2=\frac{kD(\hat{\beta}_2)-B(\hat{\beta}_2)C(\hat{\beta}_2)}{A_1(\hat{\beta}_2)C(\hat{\beta}_2)-kA_2(\hat{\beta}_2)}.\tag{17}$$

易知 $E\left(-\frac{\partial^2\ln L(\alpha,\beta)}{\partial\alpha^2}\right)\approx\frac{M}{\alpha^2}$, $E\left(-\frac{\partial^2\ln L(\alpha,\beta)}{\partial\alpha\partial\beta}\right)\approx\frac{MV_1}{\alpha^2}$, $E\left(-\frac{\partial^2\ln L(\alpha,\beta)}{\partial\beta^2}\right)\approx\frac{MV_2}{\alpha^2}$,

$$M=-k-2(r_1-1)a_1E\left(Z_{(r_1)}\right)+3(r_1-1)b_1E\left(Z^2_{(r_1)}\right)+2(n-r_2)a_2E\left(Z_{(r_2)}\right)+3(n-r_2)b_2E\left(Z^2_{(r_2)}\right)+3\sum_{i=r_1}^{r_2}E\left(Z^2_{(i)}\right),$$

$$V_1=\frac{1}{2\beta M}\left\{-(r_1-1)a_1E\left(\sqrt{\alpha^2Z^2_{(r_1)}+4}\right)+2(r_1-1)b_1E\left(Z_{(r_1)}\sqrt{\alpha^2Z^2_{(r_1)}+4}\right)+(n-r_2)a_2E\left(\sqrt{\alpha^2Z^2_{(r_2)}+4}\right)+2(n-r_2)b_2E\left(Z_{(r_2)}\sqrt{\alpha^2Z^2_{(r_2)}+4}\right)+2\sum_{i=r_1}^{r_2}E\left(Z_{(i)}\sqrt{\alpha^2Z^2_{(i)}+4}\right)\right\},$$

$$V_2=\frac{1}{4\beta^2M}\left\{2\alpha\sum_{i=r_1}^{r_2}E\left[Z_{(i)}\sqrt{\alpha^2Z^2_{(i)}+4}\right]+2\alpha(n-r_2)E\left[\sqrt{\alpha^2Z^2_{(r_2)}+4}\left(a_2+b_2Z_{(r_2)}\right)\right]-2\alpha(r_1-1)E\left[\sqrt{\alpha^2Z^2_{(r_1)}+4}\left(a_1-b_1Z_{(r_1)}\right)\right]-2\alpha^3\sum_{i=r_1}^{r_2}E\left[\frac{Z_{(i)}}{\sqrt{\alpha^2Z^2_{(i)}+4}}\right]+2\alpha^2\sum_{i=r_1}^{r_2}E\left(Z^2_{(i)}\right)+4k+(n-r_2)\alpha^2E\left(Z_{(r_2)}\right)+2(n-r_2)b_2\alpha^2E\left(Z^2_{(r_2)}\right)+4(n-r_2)b_2-(r_1-1)a_1\alpha^2E\left(Z_{(r_1)}\right)+2(r_1-1)b_1\alpha^2E\left(Z^2_{(r_1)}\right)+4(r_1-1)b_1-\alpha^2\sum_{i=r_1}^{r_2}E\left[\frac{4}{\alpha^2Z^2_{(i)}+4}\right]\right\}.$$

由此
$$\mathrm{Var}(\hat{\alpha}_2)\approx\frac{\alpha^2}{M}\frac{V_2}{V_2-V_1^2},\ \mathrm{Var}(\hat{\beta}_2)\approx\frac{\alpha^2}{M}\frac{1}{V_2-V_1^2},\ \mathrm{Cov}(\hat{\alpha}_2,\hat{\beta}_2)\approx-\frac{\alpha^2}{M}\frac{V_1}{V_2-V_1^2}.$$

特别地　当 $r_1=1$ 时, 即为一般定数截尾寿命试验场合, 令 $r=r_2$, 方程(9), (10)变为:

$$\sum_{i=1}^{r}Z^2_{(i)}+(n-r)Z_{(r)}\frac{\varphi(Z_{(r)})}{1-\Phi(Z_{(r)})}-r=0,\tag{18}$$

$$\sum_{i=1}^{r} Z_{(i)}\sqrt{\alpha^2 Z_{(i)}^2+4}+(n-r)\sqrt{\alpha^2 Z_{(r)}^2+4}\,\frac{\varphi(Z_{(r)})}{1-\Phi(Z_{(r)})}-\alpha^2\sum_{i=1}^{r}\frac{Z_{(i)}}{\sqrt{\alpha^2 Z_{(i)}^2+4}}=0. \tag{19}$$

化简(18), (19) **式得**

$$\alpha^2=\frac{G(\beta)}{E(\beta)}. \tag{20}$$

$$G(\beta)=\sum_{i=1}^{r}\left(\frac{t_{(i)}}{\beta}-\frac{\beta}{t_{(i)}}\right)-\frac{t_{(r)}+\beta}{t_{(r)}-\beta}\sum_{i=1}^{r}\left(\frac{t_{(i)}}{\beta}+\frac{\beta}{t_{(i)}}-2\right),\ E(\beta)=\sum_{i=1}^{r}\frac{t_{(i)}-\beta}{t_{(i)}+\beta}-r\,\frac{t_{(r)}+\beta}{t_{(r)}-\beta}.$$

又由(12) **式知**

$$r\alpha^2-H(\beta)\alpha-Q(\beta)=0. \tag{21}$$

$$H(\beta)=(n-r)a_2\left(\sqrt{\frac{t_{(r)}}{\beta}}-\sqrt{\frac{\beta}{t_{(r)}}}\right),$$

$$Q(\beta)=\sum_{i=1}^{r}\left(\frac{t_{(i)}}{\beta}+\frac{\beta}{t_{(i)}}-2\right)+(n-r)b_2\left(\frac{t_{(r)}}{\beta}+\frac{\beta}{t_{(r)}}-2\right).$$

由(20), (21) **式可得**

$$\alpha=\frac{rG(\beta)-E(\beta)Q(\beta)}{E(\beta)H(\beta)}. \tag{22}$$

由此得

$$G(\beta)E(\beta)H^2(\beta)-[E(\beta)Q(\beta)-rG(\beta)]^2=0. \tag{23}$$

从方程(23) **便可解得参数** β **的近似**MLE $\hat{\beta}_2^*$, **进而得参数** α **的近似**MLE $\hat{\alpha}_2^*$

$$\hat{\alpha}_2^*=\frac{rG(\hat{\beta}_2^*)-E(\hat{\beta}_2^*)Q(\hat{\beta}_2^*)}{E(\hat{\beta}_2^*)H(\hat{\beta}_2^*)}. \tag{24}$$

易知 $E\left(-\frac{\partial^2\ln L(\alpha,\beta)}{\partial\alpha^2}\right)\approx\frac{M}{\alpha^2}$, $E\left(-\frac{\partial^2\ln L(\alpha,\beta)}{\partial\alpha\partial\beta}\right)\approx\frac{MV_1}{\alpha^2}$, $E\left(-\frac{\partial^2\ln L(\alpha,\beta)}{\partial\beta^2}\right)\approx\frac{MV_2}{\alpha^2}$.

$$M=-r+2(n-r)a_2E\left(Z_{(r)}\right)+3(n-r)b_2E\left(Z_{(r)}^2\right)+3\sum_{i=1}^{r}E\left(Z_{(i)}^2\right),$$

$$V_1=\frac{1}{2\beta M}\left\{(n-r)a_2E\left(\sqrt{\alpha^2Z_{(r)}^2+4}\right)+2(n-r)b_2E\left(Z_{(r)}\sqrt{\alpha^2Z_{(r)}^2+4}\right)+2\sum_{i=1}^{r}E\left(Z_{(i)}\sqrt{\alpha^2Z_{(i)}^2+4}\right)\right\},$$

$$V_2=\frac{1}{4\beta^2M}\left\{2\alpha\sum_{i=1}^{r}E\left[Z_{(i)}\sqrt{\alpha^2Z_{(i)}^2+4}\right]+2\alpha(n-r)E\left[\sqrt{\alpha^2Z_{(r)}^2+4}\left(a_2+b_2Z_{(r)}\right)\right]-2\alpha^3\sum_{i=1}^{r}E\left[\frac{Z_{(i)}}{\sqrt{\alpha^2Z_{(i)}^2+4}}\right]+2\alpha^2\sum_{i=1}^{r}E\left(Z_{(i)}^2\right)+4r+(n-r)\alpha^2E\left(Z_{(r)}\right)+2(n-r)b_2\alpha^2E\left(Z_{(r)}^2\right)+4(n-r)b_2-\alpha^2\sum_{i=1}^{r}E\left[\frac{4}{\alpha^2Z_{(i)}^2+4}\right]\right\}.$$

由此 $\mathrm{Var}(\hat{\alpha}_2)\approx\frac{\alpha^2}{M}\frac{V_2}{V_2-V_1^2}$, $\mathrm{Var}(\hat{\beta}_2)\approx\frac{\alpha^2}{M}\frac{1}{V_2-V_1^2}$, $\mathrm{Cov}(\hat{\alpha}_2,\hat{\beta}_2)\approx-\frac{\alpha^2}{M}\frac{V_1}{V_2-V_1^2}$

为了作比较, 取 $n=20$, $r=10$, **对参数** α, β **的**QLSE、MLE **及**AMLE **进行了** 1000 **次** Monte-carlo **模拟, 参数** α **的真值分别为** 0.5, 1, 1.5, 2, **而** β **的真值为** 500, **模拟结果见表** 1, **从表中可以看出**, AMLE **与**MLE **相差不大, 精度上**AMLE **略低于**MLE.

表 1　$n= 20, r= 10$ 时模拟结果

α	参　数	偏　　差			均方误差		
		QLSE	MLE	AMLE	QLSE	MLE	AMLE
0.5	α	-0.0059	-0.0459	-0.0553	0.0157	0.0157	0.0210
	β	-0.0089	-0.0191	-0.0230	0.0177	0.0179	0.0193
1	α	-0.0539	-0.0968	-0.0991	0.0610	0.0704	0.0819
	β	-0.0638	-0.0222	-0.0272	0.0618	0.0702	0.0830
1.5	α	-0.1652	-0.1567	-0.1701	0.1490	0.1841	0.2004
	β	-0.1555	-0.0167	-0.0188	0.1210	0.1551	0.1753
2	α	-0.3329	-0.2294	-0.3004	0.3185	0.3813	0.3942
	β	-0.2521	-0.0104	-0.2301	0.1852	0.2649	0.2813

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Statistical Analysis of Birnbaum-Saunders Fatigue Life Distribution

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Abstract Offers point estimators under Type Ⅱ bilateral censored life tests for Birnbaum-Saunders fatigue life distribution. The distribution mentioned here appeared in literature to deal with failures caused by cracks when a periodic stress is imposed on a tested product.

Key words Birnbaum-Saunders fatigue life distribution; Type Ⅱ bilateral censored life test; quasi-least-square estimato; maximum likelihood estimator; approximate maximum likelihood estimator