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Bifurcation analysis of reaction-diffusion equations in developmental biology

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Abstract: Using the Liapunov-Schmidt reduction, we investigate the bifurcation of a class of nonlinear reaction-diffusion equations in developmental biology. Near the bifurcation point we obtain nontrivial solution branches bifurcated from the trivial solution. Approximate analytical expressions of the nontrivial solutions are given to compare with the numerical solutions of the nonlinear problem.

Key words: reaction-diffusion equations; bifurcation; Liapunov-Schmidit reduction

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1 Introduction

In the developmental biology[4], the pattern formation processes in a variety of morphogenetic situations can usually be described by the following class of nonlinear reaction-diffusion equations:

$$\begin{cases} u_t = \gamma f(u, v) + u_{xx} \\ v_t = \gamma g(u, v) + dv_{xx} \end{cases} \tag{1}$$

with boundary condition

$$u_x(t, 0) = u_x(t, \pi) = v_x(t, 0) = v_x(t, \pi) = 0, \tag{2}$$

where

$$f(u,v) = a - bu + \frac{u^2}{v}, \ g(u,v) = u^2 - v, \tag{3}$$

a, b and γ are constants and d is a parameter.

The rest of the paper is organized as follows. In Sec. 2, we apply the Liapunov-Schmidt reduction process[1,2] to the above nonlinear problem (1), (2) and (3)at the bifurcation point. Sec. 3 is devoted to bifurcation analysis to get the bifurcation equation and the approximate analytical expressions of the nontrivial solutions of the nonlinear problem. In the last section we take an example to show the effectiveness of our analysis by comparison of the approximate nontrivial solutions with the numerical solutions.

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2 Liapunov-Schmidt reduction

It is easy to know that (1), (2) and (3) have the stationary solution

$$u_0 = (a+1)/b\ ,\ v_0 = ((a+1)/b)^2. \tag{4}$$

Let $p = u - u_0$, $q = v - v_0$. Substituting them into (1) and (2) we have

$$\begin{cases} p_t = -\gamma bp - \gamma + \gamma \dfrac{(bp+a+1)^2}{b^2 q + (a+1)^2} + p_{xx} \\ q_t = 2\gamma \dfrac{a+1}{b} p + \gamma p^2 - \gamma q + dq_{xx} \end{cases} \tag{5}$$

and

$$p_x(t,0) = p_x(t,\pi) = q_x(t,0) = q_x(t,\pi) = 0. \tag{6}$$

Separating the linear item from Eqs. (5) we get $y_t = L(d)y + h(y)$ where $y = (p,q)^T$,

$$L(d) = \begin{bmatrix} \dfrac{1-a}{1+a}\gamma b + \dfrac{d^2}{dx^2} & -\gamma(\dfrac{b}{a+1})^2 \\ 2\dfrac{a+1}{b}\gamma & -\gamma + d\dfrac{d^2}{dx^2} \end{bmatrix},\ \text{and } h(y) = \begin{bmatrix} h_1(y) \\ h_2(y) \end{bmatrix} = \begin{bmatrix} \dfrac{b^2\gamma}{b^2 q + (a+1)^2}(p - \dfrac{b}{a+1}q)^2 \\ \gamma p^2 \end{bmatrix}.$$

As $y = \hat{y} = (p_n \cos nx, q_n \cos nx)^T$, $L(d)y = \hat{L}_n \hat{y}$ where $\hat{L}_n = \begin{bmatrix} \dfrac{1-a}{1+a}\gamma b - n^2 & -\gamma(\dfrac{b}{a+1})^2 \\ 2\dfrac{a+1}{b}\gamma & -\gamma - dn^2 \end{bmatrix}$.

The character equation of $\hat{L}_n$ is $|\lambda I - \hat{L}_n| = 0$, that is $\lambda^2 + H_n\lambda + G_n = 0$, in which

$$H_n = (d+1)n^2 + \gamma\frac{a+1+(a-1)b}{a+1},\ G_n = dn^4 + b\gamma^2 + \gamma n^2\frac{a+1+(a-1)bd}{a+1}.$$

Hence the eigenvalues of $\hat{L}_n$ are $\lambda_n^{\pm} = \dfrac{-H_n \pm \sqrt{(H_n)^2 - 4G_n}}{2}$.

Solving $G_n = 0$ (i. e. $\lambda_n^+ = 0$) we obtain

$$d_n = \frac{(a+1)(b\gamma^2 + n^2\gamma)}{(1-a)bn^2\gamma - (a+1)n^4}, \tag{7}$$

which is a bifurcation point. And

$$\ker L_n = \text{span}\{\varphi_n(x)\} = \text{span}\{(\cos nx,\ M_n \cos nx)^T\},$$

where $L_n = L(d_n)$, $M_n = \dfrac{(1-a^2)b\gamma - n^2(a+1)^2}{b^2\gamma}$.

Similarly

$$\ker L_n^* = \text{span}\{\psi_n(x)\} = \text{span}\{(\cos nx,\ N_n \cos nx)^T\},$$

where L_n^* is the conjugate operator of L_n and $N_n = \dfrac{(a-1)b^2}{2(a+1)^2} + \dfrac{bn^2}{2(a+1)\gamma}$.

L_n is a Fredholm operator of index zero.

Let $Y = \{(p,q)^T \mid p,q \in C^2[0,\pi],\ p_x(0) = p_x(\pi) = q_x(0) = q_x(\pi) = 0\}$,

$$Z = \{(z_1,z_2)^T \mid z_1,z_2 \in C^0[0,\pi]\}$$

and

$$F(y,d) = L(d)y + h(y) = 0. \tag{8}$$

Clearly $F(y,d)$ is a map from $Y \times R$ onto Z.

Split these spaces into

$$Y = \ker L_n \oplus M\ ,\ \ Z = \text{range}\ L_n \oplus N\ , \tag{9}$$

where $M = (\ker L_n)^{\perp} \cap Y$ and $N = (\text{range}\ L_n)^{\perp}$. According to the Fredholm alternative, $(\text{range}\ L_n)^{\perp} = \ker L_n^*$. Define the orthogonal projector P_n from Z onto range L_n as

$$P_n z = z - \frac{\langle z, \psi_n \rangle}{\langle \psi_n, \psi_n \rangle} \psi_n\ \ z \in Z\ , \tag{10}$$

where the inner product $\langle u, v \rangle = \int_0^{\pi} u(x)^T v(x) \mathrm{d}x$. Let $\lambda = d - d_n$ and $y = \tau \varphi_n + \omega$. Then

$$F(y, \lambda) = L_n \omega + h(\tau \varphi_n + \omega) + \lambda\ (0, \frac{\mathrm{d}^2 \omega_2}{\mathrm{d}x^2} - n^2 \tau M_n \cos nx)^T, \tag{11}$$

where $\tau \in R$, $\omega = (\omega_1, \omega_2)^T \in M$. By the Liapunov-Schmidt reduction principle, (8) is equivalent to

$$P_n F(\tau \varphi_n + \omega,\ \lambda) = 0, \tag{12}$$

$$(I - P_n) F(\tau \varphi_n + \omega, \lambda) = 0. \tag{13}$$

According to the implicit function theorem, from (12) we can get a unique $\omega(\tau, \lambda)$ satisfying $\omega(0,0) = 0$. Substituting $\omega(\tau, \lambda)$ into (13) yields an equivalent equation

$$G(\tau, \lambda) = \langle \psi_n, F(\tau\varphi_n + \omega(\tau, \lambda), \lambda) \rangle = 0, \tag{14}$$

which is called the bifurcation equation of (8). The following formulae can be easily calculated: [3]

$$G_{\tau} = \langle \psi_n, \mathrm{d}F(\varphi_n + \omega_{\tau}) \rangle, \tag{15}$$

$$G_{\tau^2} = \langle \psi_n, \mathrm{d}F(\omega_{\tau^2}) + \mathrm{d}^2 F(\varphi_n + \omega_{\tau}, \varphi_n + \omega_{\tau}) \rangle, \tag{16}$$

$$G_{\tau^3} = \langle \psi_n, \mathrm{d}F(\omega_{\tau^3}) + 3\mathrm{d}^2 F(\varphi_n + \omega_{\tau}, \omega_{\tau^2}) + \mathrm{d}^3 F(\varphi_n + \omega_{\tau}, \varphi_n + \omega_{\tau}, \varphi_n + \omega_{\tau}) \rangle, \tag{17}$$

$$\cdots\ \cdots\ \cdots$$

$$G_{\lambda} = \langle \psi_n, \mathrm{d}F(\omega_{\lambda}) + F_{\lambda} \rangle, \tag{18}$$

$$G_{\tau\lambda} = \langle \psi_n, \mathrm{d}F_{\lambda}(\varphi_n + \omega_{\tau}) + \mathrm{d}F(\omega_{\tau\lambda}) + \mathrm{d}^2 F(\varphi_n + \omega_{\tau}, \omega_{\lambda}) \rangle, \tag{19}$$

$$\cdots\ \cdots\ \cdots$$

where $\omega_{\tau}, \omega_{\tau^2}, \cdots, \omega_{\lambda}, \omega_{\tau\lambda}, \cdots$ are given by Eqs. (20)~(24) derived below.

Differentiating (12) with respect to τ and λ leads to

$$P_n \mathrm{d}F(\varphi_n + \omega_{\tau}) = 0, \tag{20}$$

$$P_n \mathrm{d}^2 F(\varphi_n + \omega_{\tau}, \varphi_n + \omega_{\tau}) + P_n \mathrm{d}F(\omega_{\tau^2}) = 0, \tag{21}$$

$$P_n \mathrm{d}^3 F(\varphi_n + \omega_{\tau}, \varphi_n + \omega_{\tau}, \varphi_n + \omega_{\tau}) + 3P_n \mathrm{d}^2 F(\varphi_n + \omega_{\tau}, \omega_{\tau^2}) + P_n \mathrm{d}F(\omega_{\tau^3}) = 0, \tag{22}$$

$$\cdots\ \cdots\ \cdots$$

$$P_n \mathrm{d}F(\omega_{\lambda}) + P_n F_{\lambda} = 0, \tag{23}$$

$$P_n \mathrm{d}F_{\lambda}(\varphi_n + \omega_{\tau}) + P_n \mathrm{d}F(\omega_{\tau\lambda}) + P_n \mathrm{d}^2 F(\varphi_n + \omega_{\tau}, \omega_{\lambda}) = 0. \tag{24}$$

$$\cdots\ \cdots\ \cdots$$

3 Bifurcation analysis

It is easy to see that at $(\tau, \lambda) = (0,0)$, (20) becomes $P_n L_n(\varphi_n + \omega_{\tau}(0,0)) = 0$. Because $\varphi_n \in L_n$, $\omega_{\tau} \in M$, $P_n L_n = L_n$ and $L_n : M \rightarrow$ range L_n is regular, it follows from $L_n \omega_{\tau}(0,0) = 0$ that

$$\omega_{\tau}(0,0) = 0 \tag{25}$$

and by (15),

$$G_{\tau}(0,0) = \langle \psi_n, (\mathrm{d}F)_{(0,0)}(\varphi_n + \omega_{\tau}) \rangle = \langle \psi_n, L_n(\varphi_n + \omega_{\tau}(0,0)) \rangle = 0. \tag{26}$$

Similarly $\omega_{\lambda}(0,0) = 0$, $G_{\lambda}(0,0) = 0$.

Because

$$(d^2F)_{(0,0)}(\xi_1,\xi_2)=\frac{\partial^2}{\partial t_1\partial t_2}[L(t_1\xi_1+t_2\xi_2)+h(t_1\xi_1+t_2\xi_2)]\,|_{t_1=t_2=0}$$

$$=\left(\frac{2b^2\gamma}{(a+1)^2}[\xi_{11}\xi_{21}-\frac{b}{a+1}(\xi_{11}\xi_{22}+\xi_{12}\xi_{21})+(\frac{b}{a+1})^2\xi_{12}\xi_{22}],\ 2\gamma\xi_{11}\xi_{21}\right)^T, \tag{27}$$

where ξ_{11}, ξ_{12} and ξ_{21}, ξ_{22} are the components of ξ_1 and ξ_2 respectively,

$$(d^2F)_{(0,0)}(\varphi_n,\varphi_n)=\begin{pmatrix}\frac{2b^2\gamma}{(a+1)^4}(a+1-bM_n)^2\cos^2 nx\\ 2\gamma\cos^2 nx\end{pmatrix}=\begin{pmatrix}2\beta\cos^2 nx\\ 2\gamma\cos^2 nx\end{pmatrix}, \tag{28}$$

where $\beta=b^2\gamma\,(a+1-bM_n)^2/(a+1)^3$.

Using (21), (25) and (28), we get $P_n d^2F(\varphi_n,\varphi_n)+P_n dF(\omega_{\tau^2}(0,0))=0$. At $(\tau,\lambda)=(0,0)$, (21) becomes

$$L_n\omega_{\tau^2}(0,0)=-(\beta,\gamma)^T(\cos 2nx+1). \tag{29}$$

To solve (29), it is assumed that

$$\omega_{\tau^2}(0,0)=(c_1\cos 2nx+c_2,\ c_3\cos 2nx+c_4)^T. \tag{30}$$

Substituting (30) into (29) yields

$$c_1=\frac{1}{D}[\beta(\gamma+4n^2d_n)-(\frac{b\gamma}{a+1})^2],$$

$$c_2=\frac{b^2M_n(bM_n-2a-2)}{(a+1)^4},$$

$$c_3=\frac{1}{D}(2\beta\gamma\frac{a+1}{b}+4n^2\gamma+b\gamma^2\frac{a-1}{a+1}),$$

$$c_4=1+\frac{2bM_n}{(a+1)^3}(bM_n-2a-2),$$

where $D=(4n^2+b\gamma\frac{a-1}{a+1})(\gamma+4n^2d_n)+\frac{2b\gamma^2}{a+1}$. By means of $\int_0^\pi\cos^3 nx\,dx=0$ we get

$$G_{\tau^2}(0,0)=\langle\psi_n,L_n\omega_{\tau^2}(0,0)+d^2F_{(0,0)}(\varphi_n,\varphi_n)\rangle=0. \tag{31}$$

Furthermore,

$$(d^3F)_{(0,0)}(\xi_1,\xi_2,\xi_3)=\frac{\partial^3}{\partial t_1\partial t_2\partial t_3}[L(t_1\xi_1+t_2\xi_2+t_3\xi_3)+h(t_1\xi_1+t_2\xi_2+t_3\xi_3)]\,|_{t_1=t_2=t_3=0}$$

$$=(-2b^4\gamma\Phi(\xi_1,\xi_2,\xi_3)/((a+1)^4),\ 0)^T, \tag{32}$$

where $\Phi(\xi_1,\xi_2,\xi_3)=(\xi_{11}\xi_{21}\xi_{32}+\xi_{11}\xi_{22}\xi_{31}+\xi_{12}\xi_{21}\xi_{31})-\frac{2b}{a+1}(\xi_{11}\xi_{22}\xi_{32}+\xi_{12}\xi_{21}\xi_{32}+\xi_{12}\xi_{22}\xi_{31})+3(\frac{b}{a+1})^2\xi_{12}\xi_{22}\xi_{32}$ and ξ_{ij} are the j-th(j=1,2) components of $\xi_i\,(i=1,2,3)$. Therefore

$$(d^3F)_{(0,0)}(\varphi_n,\varphi_n,\varphi_n)=(\alpha\cos^3 nx,\ 0)^T,$$

where $\alpha=-\frac{6b^4\gamma M_n}{(a+1)^4}(\frac{a+1-bM_n}{a+1})^2$. At $(\tau,\lambda)=(0,0)$,

$$G_{\tau^3}(0,0)=\langle\psi_n,L_n\omega_{\tau^3}(0,0)+3d^2F_{(0,0)}(\varphi_n,\omega_{\tau^2})+d^3F_{(0,0)}(\varphi_n,\varphi_n,\varphi_n)\rangle$$

where

$$d^2F_{(0,0)}(\varphi_n,\omega_{\tau^2})=\begin{pmatrix}c_5\cos^3 nx+c_6\cos nx\\ 4\gamma c_1\cos^3 nx+2\gamma(c_2-c_1)\cos nx\end{pmatrix} \tag{33}$$

and

$$c_5=\frac{4b^2\gamma}{(a+1)^2}[c_1-\frac{b}{a+1}(c_3+M_nc_1)+(\frac{b}{a+1})^2M_nc_3],$$

$$c_6 = \frac{2b^2\gamma}{(a+1)^2}[(c_2 - c_1) - \frac{b}{a+1}(c_4 - c_3 + M_n(c_2 - c_1)) + (\frac{b}{a+1})^2 M_n(c_4 - c_3)].$$

Hence

$$G_{\tau^3}(0,0) = \langle \psi_n,\ (\alpha + 3c_5,\ 12\gamma c_1)^T \cos^3 nx + (3c_6,\ 6\gamma(c_2 - c_1))^T \cos nx \rangle$$
$$= 3\pi(\alpha + 3c_5 + 4N_n\gamma c_1 + 4c_6 + 8N_n\gamma c_2)/8. \tag{34}$$

Next let (24) be evaluated at $(\tau, \lambda) = (0,0)$ and $(dF_\lambda)_{(0,0)}\varphi_n = (0, -n^2 M_n \cos nx)^T$. Hence we have

$$L_n\omega_{\tau\lambda}(0,0) = -P_n dF_\lambda(\varphi_n) \tag{35}$$

and where $L_n: M \to$ range L_n is regular. Similarly we can obtain

$$\omega_{\tau\lambda}(0,0) = (\Lambda_1 \cos nx, \Lambda_2 \cos nx)^T, \tag{36}$$

where $\Lambda_1 = \frac{n^2 M_n^2}{(1+N_n^2)(2\gamma\frac{a+1}{b} + \gamma + dn^2)}$, $\Lambda_2 = -\frac{\Lambda_1}{M_n}$. Substituting it into (19) yields

$$G_{\tau\lambda}(0,0) = \int_0^\pi (-n^2 M_n N_n \cos^2 nx)dx = -\frac{\pi}{2}n^2 M_n N_n. \tag{37}$$

Therefore, $G(\tau, \lambda)$ is strongly equivalent [2] to

$$\frac{1}{16}\pi(\alpha + 3c_5 + 4N_n\gamma c_1 + 4c_6 + 8N_n\gamma c_2)\tau^3 - \frac{\pi}{2}n^2 M_n N_n \tau\lambda = 0 \tag{38}$$

and the approximate expression of the nontrivial solution of (8) is

$$y = \tau\varphi_n + \omega = \tau\varphi_n + \frac{\tau^2}{2}\omega_{\tau^2}(0,0) + \tau\lambda\omega_{\tau\lambda}(0,0) + O(\tau^2|\lambda| + \lambda^2|\tau| + |\tau^3|), \tag{39}$$

where $\omega_{\tau^2}(0,0)$ and $\omega_{\tau\lambda}(0,0)$ are respectively given by (30) and (36).

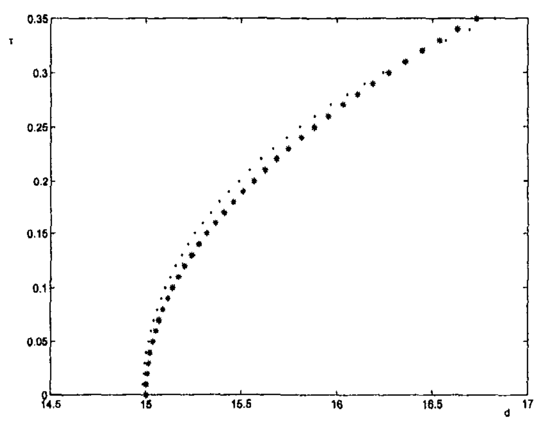

Fig 1　Bifurcation diagram　star points: solutions of (40); dot points: numerical results by difference method

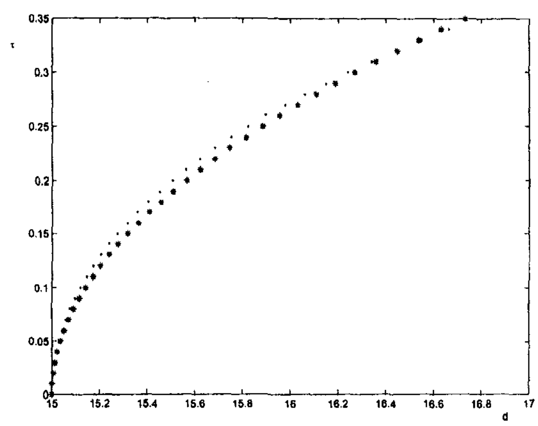

Fig 2　Bifurcation diagram　star points: solutions of (40); dot points: numerical results by pseudo-spectral method

4　Example

Let us take $n = 1$, $a = 0.25$, $b = 1$, and $\gamma = 3$. Then simple calculations show that $u_0 = 1.25$, $v_0 = 1.5625$, $d_1 = 15$, $M_1 = 0.41667$, $N_1 = -0.10667$, $\alpha = -2.048$, $\beta = 0.85333$, $c_1 = 0.31373$, $c_2 = -0.35556$, $c_3 = 0.08497$, $c_4 = 0.11111$, $c_5 = 1.25825$, $c_6 = -1.81509$, $\Lambda_1 = 0.007429$,

$$\Lambda_2 = -0.017829 \text{ and } L_1 = \begin{bmatrix} \frac{d^2}{dx^2} + 1.8 & -1.92 \\ 7.5 & -3 + d_1 \frac{d^2}{dx^2} \end{bmatrix}.$$

The approximate bifurcation equation is

$$G(\tau, \lambda) \doteq 0.06981\, \tau\lambda - 0.98665\, \tau^3 = 0. \quad (40)$$

The approximate expression of the nontrivial solution of (5) is

$$p \doteq \tau\cos x + \frac{\tau^2}{2}(0.31373\cos 2x - 0.35556) + 0.007429\tau\lambda\cos x, \quad (41)$$

$$q \doteq 0.41667\tau\cos x + \frac{\tau^2}{2}(0.08497\cos 2x + 0.11111) - 0.017829\tau\lambda\cos x. \quad (42)$$

Fig1 and Fig2 show the comparison of the approximate solutions of (40) with the numerical results by the difference method and pseudo-spectral method respectively. The fact that the approximate solutions (41), (42) of (5) nearly coincides with the numerical results shows the effectiveness of our analysis.

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生物学中反应扩散方程的分歧分析

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摘 要: 应用 Liapunov-Schmidt 方法研究了一类生物学中的非线性反应扩散方程,在分歧点附近,得到了从平凡解分歧出来的非平凡解的近似解析表达式,并与数值解作了比较,结果表明方法是正确的.

关键词: 反应扩散方程;分歧; Liapunov-Schmidt 约化