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A Generalization of Caristi's Fixed Point Theorem and Its Applications

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Abstract: The paper generalizes the classical Caristi's fixed point theorem. As an application, the classical Ekeland variational principle is generalized. In addition, it is proved that the generalized Caristi's fixed point theorem is equivalent to the generalized Ekeland variational principle.

Key words: fixed point theorem; Zorn's lemma; general principle on ordered set; variational principle.
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1. Introduction

In 1976, Caristi[1] put forward Caristi's fixed point theorem:

Theorem A (Caristi's Fixed Point Theorem) *Assume that X is a complete metric space, $g : X \to X$ is an arbitrary mapping and $\varphi : X \to R_+$ is a lower semi-continuous functional. If*

$$d(x, g(x)) \le \varphi(x) - \varphi(g(x)),\ \forall x \in X,$$

then g has a fixed point, i.e., there exists $x_0 \in X$ such that $g(x_0) = x_0$.

In 1977, Downing and Kirk[2] generalized Theorem A:

Theorem B *Let X and Y be complete metric spaces and $g : X \to X$ an arbitrary mapping, and $f : X \to Y$ a closed mapping (thus for $\{x_n\} \subset X$ the conditions $x_n \to x$ and $f(x_n) \to y$ imply $f(x) = y$). If there exist a lower semi-continuous functional $\varphi : f(X) \to R_+$ and a constant $c > 0$ such that for each $x \in X$,*

$$\begin{cases} d(x, g(x)) \le \varphi(f(x)) - \varphi(f(g(x))), \\ cd(f(x), f(g(x))) \le \varphi(f(x)) - \varphi(f(g(x))), \end{cases} \tag{0}$$

then g has a fixed point.

Note that it is needed in Theorem A and Theorem B that φ is bounded from below, and this condition plays an important role in the proofs. The main results of this paper weaken the condition essentially.

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In 1974, Ekeland[5] put forward a variational principle:

Theorem C (Ekeland Variational Principle) *Let X be a complete metric space, and let $\varphi : X \to R \cup \{+\infty\}$ be bounded from below, lower semi-continuous, and $\not\equiv +\infty$. If there exist $\varepsilon > 0$ and $x \in X$ such that $\varphi(x) \leq \inf\limits_X \varphi + \varepsilon$, then there exists $y \in X$ such that*

$$\varphi(y) \leq \varphi(x); \quad d(x,y) \leq 1; \quad \varphi(z) > \varphi(y) - \varepsilon d(y,z), \quad \forall z \neq y.$$

This variational principle had been applied to many fields, including control theory, optimal theory, geometry in Banach spaces and big area analysis, etc[6,7].

In 1987, Shi[8] proved the equivalence of Theorem A and Theorem C. This makes us want to generalize Theorem C to more general case naturally, and the generalized result is equivalent to the generalized Caristi's fixed point Theorem (Corollary 1 in this paper). We know that the classical general principle on ordered sets[3] can deduce Theorem A. In this paper, we utilize the generalized general principle on ordered sets[4] to give another proof of the main result in this paper.

Lemma 1 (Generalized General Principle on Ordered Sets)[4] *Assume that X is a partial ordered set and a Hausdorff topological space, which satisfies*

(i) $\forall x \in X$, $\{y \in X | y \geq x\}$ *is a sequential closed set;*

(ii) *if* $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$, *then* $\{x_n\}$ *has a convergent subsequence;*

(iii) *there exists* $\psi : X \to R$ *such that*

$$x \in X,\ y \in X,\ x \leq y,\ x \neq y \Rightarrow \psi(x) < \psi(y).$$

Then X has maximal element.

2. Main results

Theorem 1 *Let X and Y be complete metric spaces and $g : X \to X$ an arbitrary mapping. Suppose there exist a closed mapping $f : X \to Y$, a lower semi-continuous functional $\varphi : f(X) \to R$, which is bounded from below on every bounded set, and constants $c > 0$, $a < 0$, $-a < c$ and $x_0 \in X$ such that*

$$\liminf_{d(x,x_0)\to+\infty} \frac{\varphi(f(x))}{d(x,x_0)} > -1, \tag{1}$$

$$\liminf_{d(f(x),f(x_0))\to+\infty} \frac{\varphi(f(x))}{d(f(x),f(x_0))} > a, \tag{2}$$

and for each $x \in X$,

$$\begin{cases} d(x,g(x)) \leq \varphi(f(x)) - \varphi(f(g(x))), \\ cd(f(x),f(g(x))) \leq \varphi(f(x)) - \varphi(f(g(x))). \end{cases}$$

Then g has at least one fixed point.

Proof From (1)(2), it follows that $M > 0$, $\alpha_1 > -1$, $0 \geq \alpha_2 > a$, such that $\varphi(f(x)) > \alpha_1 d(x,x_0)$ when $d(x,x_0) \geq M$, and $\varphi(f(x)) > \alpha_2 d(f(x),f(x_0))$ when $d(f(x),f(x_0)) \geq M$. (We may assume that $\alpha_1 \le 0$, otherwise φ is bounded from below, then the conclusion is proved by Theorem B). Since φ is bounded from below on every bounded set, there exists $\beta_1 \ge 0$ such that $\varphi(f(x)) > -\beta_1$ when $d(x, x_0) \le M$, and $\varphi(f(x)) > -\beta_1$ when $d(f(x), f(x_0)) \le M$. Then we have

$$\varphi(f(x)) > \alpha_1 d(x, x_0) - \beta_1, \forall x \in X;$$

and

$$\varphi(f(x)) > \alpha_2 d(f(x), f(x_0)) - \beta_1, \forall x \in X.$$

Let $\gamma_1 = 1 + \alpha_1$, $\gamma_2 = c + \alpha_2$, then $\gamma_1 > 0$, $\gamma_2 > 0$. For $x, y \in X$, define $x \le y$ provided

$$\begin{cases} \gamma_1 d(x,y) \le \varphi(f(x)) - \alpha_1 d(x, x_0) - [\varphi(f(y)) - \alpha_1 d(y, x_0)], \\ \gamma_2 d(f(x), f(y)) \le \varphi(f(x)) - \alpha_2 d(f(x), f(x_0)) - [\varphi(f(y)) - \alpha_2 d(f(y), f(x_0))]. \end{cases}$$

We first prove "$\le$" defined above satisfies the three axioms of partial order. Indeed, (i) $x \le x$ is obviously; (ii) if $x \le y$, $y \le z$, then

$$\gamma_1 d(x,y) \le \varphi(f(x)) - \alpha_1 d(x, x_0) - [\varphi(f(y)) - \alpha_1 d(y, x_0)],$$

$$\gamma_1 d(y,z) \le \varphi(f(y)) - \alpha_1 d(y, x_0) - [\varphi(f(z)) - \alpha_1 d(z, x_0)];$$

and

$$\gamma_2 d(f(x), f(y)) \le \varphi(f(x)) - \alpha_2 d(f(x), f(x_0)) - [\varphi(f(y)) - \alpha_2 d(f(y), f(x_0))],$$

$$\gamma_2 d(f(y), f(z)) \le \varphi(f(y)) - \alpha_2 d(f(y), f(x_0)) - [\varphi(f(z)) - \alpha_2 d(f(z), f(x_0))].$$

From the above four inequalities, we have

$$\gamma_1 d(y,z) \le \gamma_1 d(x,y) + \gamma_1 d(y,z) \le \varphi(f(x)) - \alpha_1 d(x, x_0) - [\varphi(f(z)) - \alpha_1 d(z, x_0)],$$

and

$$\begin{aligned} \gamma_2 d(f(y), f(z)) &\le \gamma_2 d(f(x), f(y)) + \gamma_2 d(f(y), f(z)) \\ &\le \varphi(f(x)) - \alpha_2 d(f(x), f(x_0)) - [\varphi(f(z)) - \alpha_2 d(f(z), f(x_0))], \end{aligned}$$

i.e., $x \le z$; (iii) if $x \le y$, $y \le x$, then as in the process of (ii), we have

$$\gamma_1 d(x,y) + \gamma_1 d(y,x) \le 0, \quad \gamma_2 d(f(x), f(y)) + \gamma_2 d(f(y), f(x)) \le 0,$$

which yields $x = y$.

Suppose that $M = \{x_\alpha | \alpha \in I\}$ is a totally ordered set in X with $x_\alpha \le x_\beta \Leftrightarrow \alpha \le \beta$. Let $\varphi_1(f(x)) = \varphi(f(x)) - \alpha_1 d(x, x_0)$ and $\varphi_2(f(x)) = \varphi(f(x)) - \alpha_2 d(f(x), f(x_0))$, then

$$\begin{cases} \gamma_1 d(x,y) \le \varphi_1(f(x)) - \varphi_1(f(y)), \\ \gamma_2 d(f(x), f(y)) \le \varphi_2(f(x)) - \varphi_2(f(y)). \end{cases}$$

Hence

$$x_\alpha \le x_\beta \Leftrightarrow \begin{cases} 0 \le \gamma_1 d(x_\alpha, x_\beta) \le \varphi_1(f(x_\alpha)) - \varphi_1(f(x_\beta)), \\ 0 \le \gamma_2 d(f(x_\alpha), f(x_\beta)) \le \varphi_2(f(x_\alpha)) - \varphi_2(f(x_\beta)). \end{cases}$$

It is obvious that $\{\varphi_1(f(x_\alpha))\}_{\alpha\in I}$ and $\{\varphi_2(f(x_\alpha))\}_{\alpha\in I}$ are both monotone decreasing sequences. In addition, $\varphi_1(f(x)) > -\beta_1$, $\forall x\in X$, and $\varphi_2(f(x)) > -\beta_1$, $\forall x\in X$. Then there exist $t_1,\ t_2\in R$ such that $\varphi_1(f(x_\alpha))\to t_1$ and $\varphi_2(f(x_\alpha))\to t_2$. Hence $\forall\varepsilon>0$, $\exists\alpha_0\in I$, such that when $\alpha>\alpha_0$,

$$t_1\le\varphi_1(f(x_\alpha))\le t_1+\varepsilon,\quad t_2\le\varphi_2(f(x_\alpha))\le t_2+\varepsilon.$$

For $\beta\ge\alpha\ge\alpha_0$,

$$\begin{cases}\gamma_1 d(x_\alpha,x_\beta)\le\varphi_1(f(x_\alpha))-\varphi_1(f(x_\beta))\le\varepsilon,\\ \gamma_2 d(f(x_\alpha),f(x_\beta))\le\varphi_2(f(x_\alpha))-\varphi_2(f(x_\beta))\le\varepsilon.\end{cases}$$

Thus $\{x_\alpha\}$ is a Cauchy sequence in X and $\{f(x_\alpha)\}$ is a Cauchy sequence in Y. By completeness of X and Y, there exist $\overline{x}\in X$ and $\overline{y}\in Y$, such that $x_\alpha\to\overline{x}$, $f(x_\alpha)\to\overline{y}$. Since f is a closed mapping, $f(\overline{x})=\overline{y}$. Note that φ is lower semi-continuous, we have

$$\varphi_1(f(\overline{x}))=\varphi(f(\overline{x}))-\alpha_1 d(\overline{x},x_0)\le t_1,$$

$$\varphi_2(f(\overline{x}))=\varphi(f(\overline{x}))-\alpha_2 d(f(\overline{x}),f(x_0))\le t_2.$$

Moreover, if $\alpha,\beta\in I$ with $\beta\ge\alpha$, then

$$\begin{cases}\gamma_1 d(x_\alpha,x_\beta)\le\varphi_1(f(x_\alpha))-\varphi_1(f(x_\beta))\le\varphi_1(f(x_\alpha))-t_1,\\ \gamma_2 d(f(x_\alpha),f(x_\beta))\le\varphi_2(f(x_\alpha))-\varphi_2(f(x_\beta))\le\varphi_2(f(x_\alpha))-t_2.\end{cases}$$

Taking limits with respect to β yields

$$\begin{cases}\gamma_1 d(x_\alpha,\overline{x})\le\varphi_1(f(x_\alpha))-t_1\le\varphi_1(f(x_\alpha))-\varphi_1(f(\overline{x})),\\ \gamma_2 d(f(x_\alpha),f(\overline{x})\le\varphi_2(f(x_\alpha))-t_2\le\varphi_2(f(x_\alpha))-\varphi_2(f(\overline{x})).\end{cases}$$

Thus, $x_\alpha\le\overline{x}$, $\forall\alpha\in I$, i.e., $\overline{x}$ is an upper bound of M. We apply Zorn's Lemma to obtain the maximal element $x^*\in X$, i.e., $\forall x\ne x^*$,

$$\gamma_1 d(x^*,x)>\varphi(f(x^*))-\alpha_1 d(x^*,x_0)-[\varphi(f(x))-\alpha_1 d(x,x_0)],\tag{3}$$

or

$$\gamma_2 d(f(x^*),f(x))>\varphi(f(x^*))-\alpha_2 d(f(x^*),f(x_0))-[\varphi(f(x))-\alpha_2 d(f(x),f(x_0))].\tag{4}$$

From (3), we have

$$\begin{aligned}\varphi(f(x)) &> \varphi(f(x^*))-\alpha_1 d(x^*,x_0)+\alpha_1 d(x,x_0)-\gamma_1 d(x^*,x)\\ &\ge \varphi(f(x^*))+\alpha_1 d(x^*,x)-\gamma_1 d(x^*,x)\\ &= \varphi(f(x^*))-d(x^*,x),\end{aligned}$$

i.e.,

$$\varphi(f(x))>\varphi(f(x^*))-d(x^*,x),\ \forall x\ne x^*.$$

From (4), we have

$$\begin{aligned}\varphi(f(x)) &> \varphi(f(x^*))-\alpha_2 d(f(x^*),f(x_0))+\alpha_2 d(f(x),f(x_0))-\gamma_2 d(f(x^*),f(x))\\ &\ge \varphi(f(x^*))+\alpha_2 d(f(x^*),f(x))-\gamma_2 d(f(x^*),f(x))\\ &= \varphi(f(x^*))-cd(f(x^*),f(x)),\end{aligned}$$

i.e.,

$$\varphi(f(x)) > \varphi(f(x^*)) - cd(f(x^*), f(x)),\ \forall x \neq x^*.$$

If g has no fixed point, then $g(x) \neq x,\ \forall x \in X$, especially, $g(x^*) \neq x^*$, then

$$\varphi(f(g(x^*))) > \varphi(f(x^*)) - d(x^*, g(x^*)),$$

or

$$\varphi(f(g(x^*))) > \varphi(f(x^*)) - cd(f(x^*), f(g(x^*))).$$

The above two inequalities contradict the hypothesis, and hence g has at least one fixed point.

Remark 1 We claim that Theorem 1 generalizes Theorem A and Theorem B essentially. Indeed if φ is bounded from below, we have

$$\liminf_{d(x,x_0)\to+\infty} \frac{\varphi(f(x))}{d(x,x_0)} \geq 0,$$

and

$$\liminf_{d(f(x),f(x_0))\to+\infty} \frac{\varphi(f(x))}{d(f(x),f(x_0))} \geq 0.$$

It is obvious that the two above inequations are stronger then (1) and (2) in Theorem 1 respectively.

In Theorem 1, if $X = Y$ and f is the identity mapping, we have the following conclusion.

Theorem 2 *Let X be a complete metric space and $g : X \to X$ an arbitrary mapping. Suppose that there exists a lower semi-continuous functional $\varphi : X \to R$, which is bounded from below on each bounded set, and $x_0 \in X$ such that*

$$\liminf_{d(x,x_0)\to+\infty} \frac{\varphi(x)}{d(x,x_0)} > -1,$$

and

$$d(x, g(x)) \leq \varphi(x) - \varphi(g(x)),\quad \forall x \in X,$$

then there exists $x^* \in X$ such that $g(x^*) = x^*$.

Remark 2 If f in Theorem 1 is continuous, then Theorem 1 can be proved by Lemma 1.

Remark 3 If X is a finite dimensional space in Theorem 1 or Theorem 2, then we do not need to assume that $\varphi : X \to R$ is bounded from below.

Corollary 1 *In Theorem 2, if we enhance "> -1" in the first inequation as "≥ 0", the conclusion of Theorem 2 is still valid.*

3. Applications

From [8], we know that Theorem A is equivalent to Theorem C, then from Theorem 2, Theorem C can be generalized.

Theorem 3 *Let X be a complete metric space, and let $\varphi : X \to R \cup +\infty$, $\not\equiv +\infty$, lower semi-continuous, and bounded from below on each bounded set. If there exists $x_0 \in X$ such that*

$$\liminf_{d(x,x_0)\to+\infty} \frac{\varphi(x)}{d(x,x_0)} \geq 0,$$

then $\forall \varepsilon > 0$, φ has $\varepsilon-$approximately minimal point, i.e., $\exists x^* \in X$ such that

$$\varphi(x) > \varphi(x^*) - \varepsilon d(x,x^*), \quad \forall x \neq x^*.$$

Proof We assume, by contradiction, that there is $\varepsilon_0 > 0$ such that for all $x \in X$, there exists $y_x \in X, y_x \neq x$, satisfying $\varphi(y_x) \leq \varphi(x) - \varepsilon_0 d(y_x, x)$. We define $g(x) = y_x$, then $g(x) \neq x$, and $\varphi(g(x)) \leq \varphi(x) - \varepsilon_0 d(g(x), x)$. So $g: X \to X$ has no fixed point. On the other hand, from

$$d(g(x), x) \leq [\varphi(x) - \varphi(g(x))]/\varepsilon_0, \quad \forall x \in X,$$

and

$$\liminf_{d(x,x_0)\to+\infty} \frac{\varphi(x)}{d(x,x_0)} \geq 0,$$

we can see that φ/ε_0 satisfies all the conditions of Corollary 1, then g has at least one fixed point. It brings about a contradiction.

Remark 4 In Theorem 3, we can not enlarge "≥ 0" in the inequation as "$> \alpha$" (where $\alpha < 0$). Indeed, let $X = R$, $\varphi(x) = \beta x$ (where $\beta < 0$), then

$$\liminf_{|x|\to+\infty} \frac{\varphi(x)}{|x|} \geq \beta.$$

But for each $x,\ y \in R,\ x > y,\ 0 < \varepsilon_0 < -\beta$, we have

$$\varphi(x) - \varphi(y) = \beta(x-y) < -\varepsilon_0 |x-y|,$$

i.e., φ has no ε_0-approximately minimal point.

Theorem 4 *Theorem 3 is equivalent to Corollary 1.*

Proof Noting the proof of Theorem 3, we only need to prove Theorem 3$\Rightarrow$Corollary 1. Assume that g has no fixed point, i.e., $\forall x \in X,\ g(x) \neq x$. From the hypothesis that φ satisfies the conditions of Theorem 2, it follows that $\forall \varepsilon > 0$, $\exists x_\varepsilon \in X$ such that

$$\varphi(w) > \varphi(x_\varepsilon) - \varepsilon d(w, x_\varepsilon), \quad \forall w \neq x_\varepsilon.$$

In particular, let $\varepsilon = 1$, then

$$\varphi(w) > \varphi(x_1) - d(w, x_1), \quad \forall w \neq x_1.$$

Since $g(x_1) \neq x_1$, we have

$$\varphi(g(x_1)) > \varphi(x_1) - d(g(x_1), x_1),$$

i.e., $d(g(x_1), x_1) > \varphi(x_1) - \varphi(g(x_1))$, which contradicts the hypothesis that $d(x, g(x)) \leq \varphi(x) - \varphi(g(x))$, $\forall x \in X$. Hence g has at least one fixed point.

Remark 5 Theorem 3 develops classical Ekeland variational principle. For example, $f(x) = x^{\frac{1}{3}}, x \in R$, satisfies all the conditions of Theorem 3, however it can not be solved by classical Ekeland variational principle.

Corollary 2 *Assume that X is a reflexive Banach space, and that $\varphi : X \to R \cup +\infty, \not\equiv +\infty$, is weakly lower semi-continuous, and satisfies*

$$\liminf_{||x|| \to \infty} \frac{\varphi(x)}{||x||} \geq 0.$$

Then $\forall \varepsilon > 0$, φ has $\varepsilon-$approximately minimal point.

Proof Since X is reflexive, we have that $\forall M > 0$, $B = \{x \in X \, |\|x\| \leq M\}$ is weakly sequential compact. Note that B is convex and closed, then B is a weakly closed set. Since φ is weakly lower-continuous, we know that φ is bounded from below on weakly sequential compact and weakly closed set B. The conclusion is proved by Theorem 3.

Corollary 3 *Assume that X is a reflexive Banach space, $\varphi : X \to R \cup +\infty, \not\equiv +\infty$, is a convex lower semi-continuous functional, and satisfies*

$$\liminf_{||x|| \to \infty} \frac{\varphi(x)}{||x||} \geq 0.$$

Then $\forall \varepsilon > 0$, φ has $\varepsilon-$approximately minimal point.

Proof From [10], φ is convex and lower semi-continuous $\Rightarrow \varphi$ is weakly lower semi-continuous on weakly convex set $\{x \in X \, |\|x\| \leq M\}, \forall M > 0$. The conclusion is proved by Corollary 2.

Theorem 5 *Let X be a Banach space, and let $\varphi : X \to R \cup +\infty, \not\equiv +\infty$, and have Gâteaux derivative. Suppose φ is bounded from below on each bounded set, and satisfies*

$$\liminf_{||x|| \to \infty} \frac{\varphi(x)}{||x||} \geq 0.$$

Then $\forall \varepsilon > 0$, $\exists x_\varepsilon \in X$ such that

(a) $\varphi(x_\varepsilon) \leq \varphi(x) + \varepsilon||x - x_\varepsilon||$, $\forall x \in X$.

(b) $||\varphi'(x_\varepsilon)|| \leq \varepsilon$.

Proof From Theorem 3, (a) is valid. Hence $\forall y \in X$,

$$\varphi(x_\varepsilon + y) \geq \varphi(x_\varepsilon) - \varepsilon||y||.$$

Since φ has Gâteaux derivative, there exists $\varphi'(x_\varepsilon) \in X^*$ such that

$$\varphi(x_\varepsilon + tu) - \varphi(x_\varepsilon) = \varphi'(x_\varepsilon)(tu) + \circ(||tu||), \quad \forall u \in X.$$

Select u satisfying $||u|| = 1$, then

$$\varphi'(x_\varepsilon)(tu) + \circ(t) = \varphi(x_\varepsilon + tu) - \varphi(x_\varepsilon) \geq -\varepsilon||tu|| = -\varepsilon|t|.$$

When $t > 0$, we have

$$\varphi'(x_\varepsilon)(u) + \frac{\circ(t)}{t} \geq -\varepsilon,$$

i.e., $\varphi'(x_\varepsilon)(u) \geq -\varepsilon$; When $t < 0$, we have

$$\varphi'(x_\varepsilon)(u) + \frac{\circ(t)}{t} \leq \varepsilon,$$

i.e., $\varphi'(x_\varepsilon)(u) \leq \varepsilon$. From the above inequalities, we have $|\varphi'(x_\varepsilon)(u)| \leq \varepsilon$ for $u \in X$ satisfying $||u|| = 1$, then $||\varphi'(x_\varepsilon)|| \leq \varepsilon$, i.e., (b) holds.

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Caristi 不动点定理的推广及应用

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摘要: 本文推广了古典的 Garisti 不动点定理，作为应用，古典的 Ekeland 变分原理得到了推广，并且证明了推广的 Garisti 不动点定理和推广的 Ekeland 变分原理是等价的.

关键词: 不动点定理； Zorn 引理；序集一般原理；变分原理.