第32卷第2期 上海师范大学学报(自然科学版) Vol.32, No.2
2003年6月 Journal of Shanghai Normal University (Natural Sciences) Jun. 2003

The Existence of Drazin Inverses over Integral Domains

YU Yao-ming[1], WANG Guo-rong[2]

(1. College of Education Science, Shanghai Normal University, Shanghai 200234, China;

2. Mathematics and Sciences College, Shanghai Normal University, Shanghai 200234, China

Abstract: MANJUNATHA PRASAD K and others[4] discussed the existence of Drazin inverses over integral domains and gave some sufficient and necessary conditions. In this paper, we further discuss the existence of Drazin inverses over integral domains and gain another sufficient and necessary condition, i. e., $R(A^k) \oplus N(B^k) = R^n$ for some positive integral k.

Key words: Drazin inverses; regular matrices andintegral domains

CLC number: O151.21 **Document code**: A **Article ID**: 1000-5137(2003)02-0012-04

Throughout this paper, R denote an integral domain and $R^{n\times m}$ denotes the set of $n \times m$ matrices over R. By a module we mean a right R-module. Let $A \in R^{n\times m}$. We denote the number of independent columns of A by rank (A) and the maximal order of a nonvanishing minor of A by $\rho(A)$, called the determinantal rank of A over R. Moreover we write module $\{Ax \mid x \in R^n\}$ by $R(A)$ and $\{x \in R^n \mid Ax = 0\}$ by $N(A)$. It is easy to see that $R(A)$ is generated by columns of A.

Let A be an $n \times m$ matrix. If there exists an $m \times n$ matrix G such that

$$AGA = A,$$

then G is called a generalized inverse of A and A is said to be regular.

Let A be an $n \times n$ matrix. If there exist an $n \times n$ matrix G and a positive integer k such that

$$A^k = A^{k+1}G, \quad GAG = G, \quad GA = AG,$$

then G is called the Drazin inverse of A. Especially, when $k = 1$, it is called the group inverse of A.

Let A be an $n \times m$ matrix, and let α be a subset of $\{1,2,\cdots,n\}$ with $|\alpha| = r$, say $\alpha = \{i_1,\cdots,i_r\}$, and β a subset of $\{1,2,\cdots,m\}$ with $|\beta| = r$. Then we denote by A^α_β the submatrix of A determined by rows indexed by α and columns indexed by β, especially we write A^n_β (A^α_m) instead of A^α_β when $|\alpha| = n$ ($|\beta| = m$).

First, we will show the following three lemmas.

Lemma 1 Let $\alpha_1, \alpha_2, \cdots, \alpha_w \in R^n (w \leqslant n)$ and $T = (\alpha_1, \alpha_2, \cdots, \alpha_w)$. Then $\alpha_1, \alpha_2, \cdots, \alpha_w$ are lin-

Received date: 2002-09-01

Foundation item: Supported by NSFC Project (19971057) and Science and Technology Foundation of Shanghai and Shanghai Higher Eduction Project (00JC14057).

Biography: YU Yao-ming (1963-), male, associate professor, College of Education Science, Shanghai Normal University; WANG Guo-rong (1940-), male, professor, Mathematics and Sciences College, Shanghai Normal University.

early dependent over R if and only if $\det(T_w^\alpha) = 0$, where α is any subset of $\{1,2,\cdots,n\}$ with $|\alpha| = w$.

Proof Since $\alpha_1, \alpha_2, \cdots, \alpha_w$ are linearly dependent, there exist $a_1, a_2, \cdots, a_w \in R$, at least one of which is nonzero, such that

$$\sum_{i=1}^{w} \alpha_i a_i = 0.$$

Without loss of generality, we suppose $a_1 \neq 0$. So

$$\begin{aligned}(\det(T_w^\alpha))a_1 &= \det((\alpha_1 a_1, \alpha_2, \cdots, \alpha_w)_w^\alpha) \\ &= \det((\sum_{i=1}^{w} \alpha_i a_i, \alpha_2, \cdots, \alpha_w)_w^\alpha) \\ &= \det((0, \alpha_2, \cdots, \alpha_w)_w^\alpha) = 0,\end{aligned}$$

where $|\alpha| = w$. Since there exist no non-zero divisors of zero in R, $\det(T_w^\alpha) = 0$. Conversely, assume that $\alpha_1, \alpha_2, \cdots, \alpha_w \in R^n (w \leq n)$ are linearly independent over R. Let F be the field of quotient of R. Suppose there exist w elements $a_i \in F$, $i = 1,2,\cdots,w$, such that

$$\sum_{k=1}^{w} \alpha_i a_i = 0.$$

Since $a_i \in F$, there exist $b_i, d_i \in R$ with $b_i \neq 0$ such that $a_i = d_i b_i^{-1}$. Thus

$$\sum_{k=1}^{w} \alpha_i d_i b_i^{-1} = 0,$$

and so

$$\sum_{k=1}^{w} \alpha_i p_i = 0,$$

where $p_i = d_i \prod_{\substack{j=1 \\ j \neq i}}^{w} b_j \in R$, $i = 1,2,\cdots,w$. By the assumption, we have $p_i = 0$, $i = 1,2,\cdots,w$. Hence $d_i = 0$ and so $a_i = 0$, $i = 1,2,\cdots,w$. We obtain that $\alpha_1, \alpha_2, \cdots, \alpha_w \in R^n (w \leq n)$ are linearly independent over F.

Without loss of generality, there exists a nonsingular matrix $Z \in F^{w \times w}$ such that

$$T = \begin{pmatrix} B \\ C \end{pmatrix} Z,$$

where $B \in F^{w \times w}$ is a lower triangular matrix with $\det(B) \neq 0$ and $C \in F^{(n-w) \times w}$. Thus when $\alpha = \{1,2,\cdots,w\}$, by Cauchy – Binet formula, $\det(T_w^\alpha) = \det(B)\det(Z) \neq 0$, over F, and so $\det(T_w^\alpha) \neq 0$ over R.

□

It is clear from Lemma 1 that we have the following result.

Corollary 1 Let R be an integral domain and $A \in R^{n \times m}$. Then $\rho(A) = \mathrm{rank}(A)$.

Lemma 2 Let $A = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ be an $n \times m$ matrix of determinantal rank $\rho(A) = r$ such that

$$R(A) \oplus N(A) = R^n.$$

Then there exist finite elements of $N(A)$ such that these elements and the columns of A generate R^n. Moreover, any $n - r + 1$ elements of $N(A)$ are linearly dependent.

Proof If X is a subset of $N(A)$ and generates $N(A)$, then $X \cup Y$ generates R^n where Y consists of the columns of A. Because R^n is finite free module, it is easy to show that there exists a finite subset of X such that it generates $N(A)$. Hence we have the result of the first part.

By Corollary 1, there exist r columns of A, say $\alpha_1, \alpha_2, \cdots, \alpha_r$, which are linearly independent.

We will show that any $n - r + 1$ elements are linearly dependent in $N(A)$. Let $\beta_1, \beta_2, \cdots, \beta_{n-r+1}$ be any

$n - r + 1$ elements in $N(A)$.

Over the field of quotients F of R, we know, $\alpha_1, \alpha_2, \cdots, \alpha_r, \beta_1, \beta_2, \cdots, \beta_{n-r+1}$ are linear dependent in F^n and there exist $n + 1$ elements $a_i \in F(i = 1,2, \cdots, n + 1)$, at least one of which is nonzero, such that

$$\sum_{i=1}^{r} a_i\alpha_i + \sum_{i=r+1}^{n+1} a_i\beta_{i-r} = 0.$$

We know that $a_i = d_i b_i^{-1} (i = 1,2, \cdots, n + 1)$ where $b_i, d_i \in R$ and $b_i \neq 0$. Hence we have

$$\sum_{i=1}^{r} p_i\alpha_i + \sum_{i=r+1}^{n+1} p_i\beta_{i-r} = 0,$$

where $p_i = d_i \prod_{\substack{j=1 \\ j\neq i}}^{n+1} b_j (i = 1,2, \cdots, n+1)$. Since $p_i \in R$ $(i = 1, \cdots, n+1)$, at least one of which is nonzero, we obtain that $\alpha_1, \alpha_2, \cdots \alpha_r, \beta_1, \beta_2, \cdots, \beta_{n-r+1}$ are linearly dependent in R^n . By the condition, we can obtain

$$\sum_{i=1}^{r} p_i\alpha_i = - \sum_{i=r+1}^{n+1} p_i\beta_{i-r} \in R(A) \cap N(A) = \{0\}.$$

Since $\alpha_1, \alpha_2, \cdots, \alpha_r$ are linearly independent, we have $p_i = 0$ $(i = 1, \cdots, r)$, and so at least one element of $\{p_{r+1}, p_{r+2}, \cdots, p_{n-r+1}\}$ is nonzero. Hence $\beta_1, \beta_2, \cdots, \beta_{n-r+1}$ are linearly dependent.

Lemma 3 Let A be an $n \times n$ matrix of determinantal rank $\rho(A) = r \leqslant n$ over R, and suppose that P is an $n \times s$ $(s \geqslant n - r)$ matrix with determinantal rank $\rho(P) \leqslant n - r$ such that $T = (A, P)$ has a right inverse. Then A is regular.

Proof Suppose T has a right inverse X . Then $TX = I$, and so there exists a linear combination of all the $n \times n$ minors of T

$$\sum_{\alpha} (\det(T_{\alpha}^{n}))c^{\alpha} = 1$$

where $\alpha \subset \{1,2,\cdots,n+s\}$ with $|\alpha| = n$ and $c^{\alpha} \in R$ by [3, Theorem 1 and Corollary 2]. By Laplace expansion, we have

$$\det(T_{\alpha}^{n}) = \sum_{\gamma} \mathrm{sgn}(\gamma)\det(A_{\beta}^{\gamma})\det(P_{\beta^c}^{\gamma^c})$$

where $\gamma \subset \{1,2,\cdots, n\}$, $\gamma^c = \{1,2,\cdots, n\} - \gamma$ and $\beta \subset \alpha$, $\beta^c = \alpha - \beta$.

Because $\rho(A) = r$ and rank $(P) \leqslant n - r$, by Lemma 1, we have $\det(T_{\alpha}^{n}) = 0$ when $|\beta| \neq r$.

Therefore, we have

$$\sum_{\alpha}\sum_{\gamma} \mathrm{sgn}(\gamma)\det(A_{\beta}^{\gamma})\det(P_{\beta^c}^{\gamma^c})c^{\alpha} = 1$$

where $|\gamma| = |\beta| = r$. So

$$\sum_{\beta,\gamma} (\mathrm{sgn}(\gamma)\det(P_{\beta^c}^{\gamma^c})c^{\alpha})\det(A_{\beta}^{\gamma}) = 1,$$

that is, a linear combination of all the $r \times r$ minors of A equal to 1. A is regular by [3, Theorem 8].

Now we have the following main theorem.

Theorem 1 Let A be an $n \times n$ matrix. Then A has the Drazin inverse if and only if

$$R(A^k) \oplus N(A^k) = R^n$$

for some positive integer k.

Proof Let G be the Drazin inverse of A. Then $GAG = G$. Thus GA is idempotent linear map on $R^n \to R^n$. So, by [1, Lemma 5.6]

$$R(GA) \oplus N(GA) = R^n.$$

By using $A^k = A^{k+1}G$ and $GAG = G$, it is easy to see that $R(GA) = R(A^k)$ and $N(GA) = N(A^k)$. Hence $R(A^k) \oplus N(A^k) = R^n$.

By Lemma 1, we can denote $T = (A^k, P)$ where $P \in R^{n \times s}$ and $R(P) = N(A^k)$ such that the columns of T generate R^n. Since R^n is a free module with a finite basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ where $\mathbf{e}_i (i = 1, 2, \cdots, n)$ has zero coordinates except the ith coordinate being 1 and $(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = I \in R^{n \times n}$, there exists an $s \times n$ matrix X such that $I = TX$, i e, $I = (A^k, P)X$. So

$$A^k = A^k(A^k, P)X = (A^{2k}, 0)X = A^{k+1}(A^{k-1}, 0)X.$$

Thus $R(A^k) = R(A^{k+1})$. Immediately, $\rho(A^k) = \rho(A^{k+1})$ and $R(A^k) = R(A^{k+l})$ for any natural number l. So

$$R(A^{2k+1}) \oplus N(A^k) = R^n.$$

By Lemma 2, (A^{2k+1}, P) has right inverse. Hence A^{2k+1} is regular by Lemma 3. Thereby, A has theDrazin inverse by [4, Theorem 9]. □

Remark Since the real number field is an especial integral domain, the theorem extends the result in [5, Lemma 2.1].

For the group inverse, we have the following corollary immediately.

Corollary 2 Let A be an $n \times n$ matrix. Then A has the group inverse if and only if

$$R(A) \oplus N(A) = R^n.$$

Remark It is obvious that the result in [2, P162, Theorem 1] is an especial case of Corollary 2.

References:

[1] FRANK W Anderson, KENT R Full. Rings and Categories of Modules[M]. New York-San Francisco-London: Springerverlag, 1973.

[2] BEN-ISRSEL A, GREVILLE T N E. Generalized Inverses: Theory and Applications[M]. New York, John Wiley, 1974.

[3] BHASKARA K R S Rao. On generalized inverses of matrices over integral domains[J]. Linear Algebra Appl, 1983, 49: 179-189.

[4] MANJUNATHA K Prasad, BHASKARA K P S Rao, BAPAT R B. Generalized inverses over integral domains II, Group inversed and Drazin inverses[J]. Linear Algebra Appl, 1991, 146: 31-47.

[5] WEI Y M. Index splitting for Drazin inverse and the sigular linear system[J]. Appl Math Comput, 1998, 95: 115-124.

整环上 Drazin 逆的存在性

俞耀明[1], 王国荣[2]

(1. 上海师范大学 教育科学学院, 上海 200234; 2. 上海师范大学 数理信息学院, 上海 200234)

摘要: 讨论了整环上 Drazin 逆的存在性,并且给出了一些充分必要条件. 进一步讨论整环上 Drazin 逆的存在性,并且给出了另一个充分必要条件,即对某个正整数 k, $R(A^k) \oplus N(B^k) = R^n$.

关键词: Drazin 逆;正则矩阵;整环