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Sufficient Conditions for Heegaard Splittings with Disjoint Curve Property

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Abstract In the paper, we give two conditions that the Heegaard splitting admits the disjoint curve property. The main result is that for a genus g ($g \geq 2$) strongly irreducible Heegaard splitting $(C_1, C_2; F)$, let D_i be an essential disk in C_i, $i = 1, 2$, satisfying (1) at least one of ∂D_1 and ∂D_2 is separating in F and $|\partial D_1 \cap \partial D_2| \leq 2g - 1$; or (2) both ∂D_1 and ∂D_2 are non-separating in F and $|\partial D_1 \cap \partial D_2| \leq 2g - 2$, then $(C_1, C_2; F)$ has the disjoint curve property.

Keywords Heegaard splitting; disjoint curve property.

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1. Introduction

Let M denote a compact orientable 3-manifold and $(C_1, C_2; F)$ a Heegaard splitting of M. We say that $(C_1, C_2; F)$ is reducible (weakly reducible, resp.) if there are essential disks $D_1 \subset C_1$ and $D_2 \subset C_2$ with $\partial D_1 = \partial D_2$ ($\partial D_1 \cap \partial D_2 = \emptyset$, resp.). $(C_1, C_2; F)$ is irreducible (strongly irreducible, resp.) if it is not reducible (weakly reducible, resp.). Heegaard splitting was first introduced in [1,4]. We say that $(C_1, C_2; F)$ has the disjoint curve property, or simply, DCP, if there exist essential simple closed curves c ,a, b on F such that c is disjoint from a and b, a bounds a disk in C_1, and b bounds a disk in C_2.

Clearly, a reducible Heegaard splitting is weakly reducible, and a weakly reducible Heegaard splitting has DCP. The inversions are not true in general. Many related results are given in [1,2,5,8].

Thompson[5] gave a sufficient condition for a Heegaard splitting of genus 2 to have DCP as follows:

Theorem 1.1 *Let $(C_1, C_2; F)$ be a genus two Heegaard splitting of a 3-manifold M. If there are essential disks D_1 in C_1 and D_2 in C_2 such that $|\partial D_1 \cap \partial D_2| \leq 3$, then $(C_1, C_2; F)$ has the disjoint curve property.*

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In the present paper, we give a sufficient condition for a Heegaard splitting of genus ≥ 3 to have DCP. The statement and proof of the theorem are included in Section 2. As a corollary, we also describe an alternative proof of Theorem 1.1.

All surfaces and 3-manifolds are assumed to be orientable throughout the paper. Notations and terminology not defined in the paper are standard, see for example[2,3]. More results about disjoint curve property can be found in [6]–[8].

2. A sufficient condition for DCP Heegaard splittings

A 3-manifold M is a compression body if there is a compact connected surface F such that M is obtained from $F \times [0,1]$ by attaching 2-handles along mutually disjoint simple loops in $F \times 1$ and capping off the resulting 2-sphere boundary components by 3-handles. Then $\partial_+ C$ denotes the component of ∂C corresponding to $F \times 0$, and $\partial_- C$ denotes $\partial C \backslash \partial_+ C$. If $\partial_- C = \emptyset$, then C is called a handlebody. The genus of the surface F is called the genus of the Heegaard splitting.

Let $(C_1, C_2; F)$ be an Heegaard splitting of a 3-manifold M. The splitting is *stabilized* if there exist essential disks D_i in $C_i (i = 1, 2)$ respectively such that $|\partial D_1 \cap \partial D_2| = 1$. It is well known that a stabilized splitting of genus at least 2 is reducible.

For a finite set A, we use $|A|$ to denote the number of the elements in A.

First, we show two lemmas.

Lemma 2.1 *Let F be a once-punctured orientable surface of genus g. Let $\mathcal{A}$ be a union of pairwise disjoint simple arcs properly embedded in F. If $\mathcal{A}$ cuts F into a union of disks, then $\mathcal{A}$ contains at least $2g$ arcs.*

Proof We induct on g to finish the proof. Clearly, the conclusion holds for $g = 0$. Suppose it holds for all once-punctured surfaces with genus less than $g > 0$. Let F be a once-punctured surface of genus $g = n$ and $\mathcal{A}$ a union of pairwise disjoint simple arcs properly embedded in F which cuts F into a union of disks. Then there exists an arc α in $\mathcal{A}$ which is essential in F. There are two possibilities:

(1) α is non-separating in F. Then the surface F' obtained by cutting F along α is a twice-punctured surface with genus $n-1$. Since $\mathcal{A}$ cuts F into a union of disks, there exists arc $\beta \in \mathcal{A}$ such that $\beta \neq \alpha$ and β connects the distinct boundary components of F'. Let F'' be the surface obtained by cutting F' open along β. Then F'' is a once-punctured surface of genus $n-1$, and $\mathcal{A} - \{\alpha, \beta\}$ cuts F'' into a union of disks. By induction, $|\mathcal{A}| - 2 \geq 2(n-1)$, therefore $|\mathcal{A}| \geq 2n$.

(2) α is separating in F. Then α cuts F into two once-punctured surfaces F_1 and F_2 with $g_1 = g(F_1) > 0$, $g_2 = g(F_2) > 0$, and $g_1 + g_2 = n$. Let $\mathcal{A}_1 = \mathcal{A} \cap F_1 - \{\alpha\}$ and $\mathcal{A}_2 = \mathcal{A} \cap F_2 - \{\alpha\}$. Then $\mathcal{A}_i$ cuts F_i into a union of disks, therefore, by induction, $|\mathcal{A}_i| \geq 2g_i$, $i = 1, 2$. Thus $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| + 1 \geq 2(g_1 + g_2) + 1 > 2n$.

A direct consequence of Lemma 2.1 is

Lemma 2.2 *Let F be a twice-punctured surface genus g, and $\mathcal{A}$ a union of pairwise disjoint arcs properly embedded in F such that $\mathcal{A}$ cuts F into a union of disks. Then $|\mathcal{A}| \geq 2g + 1$.*

We now come to

Theorem 2.3 *Let $(C_1, C_2; F)$ be a strongly irreducible Heegaard splitting of a genus g ($g \geq 2$) for 3-manifold M. Let D_i be an essential disk in C_i, $i = 1, 2$. Suppose one of following conditions is satisfied:*

(1) At least one of ∂D_1 and ∂D_2 is separating in F and $|\partial D_1 \cap \partial D_2| \leq 2g - 1$; or

(2) Both ∂D_1 and ∂D_2 are non-separating in F and $|\partial D_1 \cap \partial D_2| \leq 2g - 2$.

Then $(C_1, C_2; F)$ has the disjoint curve property.

Proof Denote $n = |\partial D_1 \cap \partial D_2|$. If $n = 1$, the Heegaard splitting is stabilized, therefore is reducible. Next we assume $n \geq 2$.

Assume that ∂D_1 is separating in F. ∂D_1 cuts F into two once-punctured surfaces F_1 and F_2. Then $g_1 = g(F_1) > 0$, $g_2 = g(F_2) > 0$, and $g_1 + g_2 = g$. In this case, n is even. Let $\mathcal{A}_i = F_i \cap \partial D_2$, $i = 1, 2$. Then both $\mathcal{A}_1$ and $\mathcal{A}_2$ contain $\frac{n}{2}$ arcs. If for $i = 1, 2$, the surface F_i' obtained by cutting F_i open along $\mathcal{A}_i$ totally consists of disks, then by Lemma 2.2, $\frac{n}{2} \geq 2g_i$. So $n \geq (g_1 + g_2) = 2g$, contradicting the assumption $n \leq 2g - 1$. Thus some F_i' has a component which is not a disk. Let α be an essential simple closed curve in F_i'. Then α is essential in F and is disjoint from $\partial D_1 \cup \partial D_2$. Therefore, $(C_1, C_2; F)$ has the disjoint curve property.

Now assume that ∂D_1 is non-separating in F. Let F' be the surface obtained by cutting F open along ∂D_1. Then F' is a twice-punctured surface of genus $g - 1$, and $\mathcal{A} = \partial D_2 \cap F'$ is a union of n arcs properly embedded in F'. By assumption, $n = |\partial D_1 \cap \partial D_2| \leq 2g - 2$, thus Lemma 2.1 implies that at least one of the component F'' of the surface obtained by cutting F' open along $\mathcal{A}$ is not a disk. As above, this shows that $(C_1, C_2; F)$ has the disjoint curve property.

As a corollary, we describe an alternative proof of Theorem 1.1:

Proof of Theorem 1.1 Let $(C_1, C_2; F)$ be a genus 2 Heegaard splitting for 3-manifold M. Suppose there are essential disks D_1 in C_1 and D_2 in C_2 such that $|\partial D_1 \cap \partial D_2| \leq 3$. Clearly, if $|\partial D_1 \cap \partial D_2| \leq 2$, the conclusion holds. Next we consider the case $|\partial D_1 \cap \partial D_2| = 3$.

If one of ∂D_1 and ∂D_2 is separating in F, the conclusion follows from Theorem 2.3(1). Now we assume that both ∂D_1 and ∂D_2 are non-separating in F. Let F' be the surface obtained by cutting F open along ∂D_1. Then F' is a twice-punctured torus with two boundary components α and β, and $\mathcal{A} = \partial D_2 \cap F'$ is a union of 3 arcs properly embedded in F'. Either each of the 3 arcs connects α and β, or only one of them connects α and β. In the first case it is easy to check that at least two arcs in $\mathcal{A}$ are parallel on F', which implies that the surface obtained by cutting F' open along $\mathcal{A}$ has a non-disk component. In the second case the two components of the surface obtained by cutting F' open along $\mathcal{A}$ also contain one non-disk component. The conclusion follows.

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