第26卷第1期 数 学 研 究 与 评 论 Vol.26, No.1
2006年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2006

Article ID: 1000-341X(2006)01-0040-03 **Document code**: A

Weak Global Dimension of Smash Products of Hopf Algebras

WANG Zhi-xi, ZHAO Hui

(Dept. of Math., Capital Normal University, Beijing 100037, China)

(E-mail: wangzhx@mail.cnu.edu.cn)

Abstract: Let H be a finite dimensional Hopf algebra and A a commutative H-module algebra. We prove that the smash product $A\#H$ is of the same weak global homological dimension as A, provided that H^* is unimodular and there is a trace one element in A.

Key words: weak global dimension; module algebra; smash product.
MSC(2000): 16W30, 16S40
CLC number: O153

In this paper, all modules concerned are left modules unless otherwise specified. For a ring R, by w.dim(R) we denote the weak global dimension of R and fdim(M) the flat dimension of the R-module M. Let H be a finite dimensional Hopf algebra over a field of characteristic zero. We fix $0 \neq t \in \int_H$, the space of left integrals of H. For an H-module algebra A, by $A\#H$ we denote the associated smash product. It is known that M is an $A\#H$-module if and only if M is both an A-module and an H-module such that $h\cdot(am) = \sum(h_1\cdot a)(h_2\cdot m)$. In order to reach our main result, we need two lemmas about injectivity and flatness for modules over smash products.

Lemma 1 *Let H be a finite dimensional Hopf algebra such that H^* is unimodular, and A a left H-module algebra. Assume that $t\cdot c = 1$ for some $c \in Z(A)$, the center of A. If an $A\#H$-module Q is injective as an A-module, then Q is also injective as an $A\#H$-module.*

Proof By the assumptions one sees that $S^2(t) = t$ ([5, Corollary 5]) and $x = S(t) \in \int_H^r$.

Suppose that the $A\#H$-module Q is injective as an A-module. To see that Q is an injective $A\#H$-module, let $i : {}_{A\#H}M \to {}_{A\#H}N$ be a monomorphism and $\psi : M \to Q$ an $A\#H$-homomorphism. If Q is injective as an A-module, then there exists an A-homomorphism $\lambda : N \to Q$ such that $\lambda \circ i = \psi$. Define $\bar{\lambda} : N \to Q$ by

$$\bar{\lambda}(n) = \sum [c\lambda(x_2 n)]S(x_1).$$

By [2, Proposition 2], it is easy to see that $\bar{\lambda}$ is an $A\#H$-homomorphism. To see $\bar{\lambda}\circ i = \psi$, by [4, 10.3.12] and the assumption, one can obtain

$$\sum x_1 \otimes x_2 \otimes S^{-1}(x_3) = \sum x_2 \otimes x_3 \otimes S(x_1).$$

Received date: 2004-04-02
Foundation item: the National Natural Science Foundation of China (10271081) and the Natural Science Foundation of Beijing (1042004)

Hence $S(x)\otimes 1=\sum S(x_2)\otimes S(x_1)x_3$. For $m\in M$, we have

$$\begin{aligned}\bar{\lambda}\circ i(m)&=\sum S(x_1)[c\lambda(x_2 i(m))]=\sum[S(x_2)\cdot c][(S(x_1)x_3)\psi(m)]\\&=(S(x)\cdot c)\psi(m)=\psi(m).\end{aligned}$$

Hence Q is an injective $A\#H$-module.

Lemma 2 *Let H be a finite dimensional Hopf algebra such that H^* is unimodular, and A a left H-module algebra. Assume that $t\cdot c=1$ for some $c\in Z(A)$, the center of A and M a right $A\#H$-module. Then M is a flat right $A\#H$-module if and only if M is flat as a right A-module.*

Proof Let M be flat as a right $A\#H$-module. Considering an exact sequence of right $A\#H$-modules

$$0\to K\to F\to M\to 0,$$

where F is right $A\#H$-free (hence A-free since $A\#H_A$ is free). One gets easily from [6, Theorem 3.57] that M is right A-flat.

Conversely, assume that M is flat as a right A-module. We regard $\mathbf{Q}$ (the field of rational numbers) and $\mathbf{Z}$ (the ring of integers) as $\mathbf{Z}$-modules. Then $M^*=\mathrm{Hom}_{\mathbf{Z}}(M,\mathbf{Q}/\mathbf{Z})$ defined in the natural way is a left $A\#H$-module. Clearly, it is also a left A-module, which is just the one induced by the right A-module M. Thus, M^* is injective as a left A-module. By Lemma 1 M^* is also injective as a left $A\#H$-module. It follows that M is flat as a right $A\#H$-module.

Theorem 1 *Let H be a finite dimensional Hopf algebra and A a left H-module algebra. Assume that H^* is unimodular and A is commutative such that $t\cdot c=1$ for some $c\in A$. Then* w.dim(A) = w.dim(A#H).

Proof By Lemma 2, we have that w.dim(A) $\geq$ w.dim(A#H). So we only need to show that w.dim(A) $\leq$ w.dim(A#H).

Let N be an arbitrary A-module. We observe that $A\#H$ is an (A, A)-bimodule and A is an (A, A)-bimodule direct summand of $A\#H$. The (A, A)-bimodule structure of $A\#H$ defined by $(a\#h)\leftarrow b=(ba)\#h$ and $b\cdot(a\#h)=ba\#h$ as A is commutative. The (A, A)-bimodule structure of A is left and right multiplications.

By assumption we have that A is a projective left $A\#H$-module. This means that $A\#H\to A$ defined by $\varepsilon_l: a\#h\mapsto\varepsilon(h)a$ is a split $A\#H$-epimorphism ([7, Lemma 1.2]) and there exists an $A\#H$-module homomorphism $\delta: A\to A\#H$ such that $\varepsilon_l\circ\delta=id$. The homomorphism δ is determined by $\delta(1)$ which is in tA. That is, there is a $c\in A$ with $\delta(1)=tc$. So $\varepsilon_l(\delta(1))=t\cdot c=1$.

It remains to show that both ε_l and δ are (A, A)-bimodule homomorphisms. This follows from

$$\varepsilon_l(b(a\#h)\leftarrow c)=\varepsilon(h)abc=b\varepsilon_l(a\#h)\leftarrow c.$$

Similarly, δ is also an (A, A)-bimodule homomorphism. So we have that A is an (A, A)-bimodule direct summand of $A\#H$. Thus, every A-module M is a direct summand of the A-module

$(A\#H) \otimes_A M$ (the action via the left component). Hence

$$\mathrm{fdim}_A(M) \leq \mathrm{fdim}_A((A\#H) \otimes_A M),$$

and hence w.dim(A) $\leq$ w.dim(A#H). The required equality follows.

Corollary 1 *Let H be a finite dimensional cosemisimple Hopf algebra and A be a commutative $H-$module algebra. Then $A\#H$ is von Neumann regular if and only if so is A and $t \cdot c = 1$ for some $c \in A$.*

Proof Clearly, H^* is unimodular. If $t \cdot c = 1$ for some $c \in A$, then w.dim(A) = w.dim(A#H) by Theorem 1. Thus, if A is von Neumann regular, so is $A\#H$.

Conversely, if $A\#H$ is von Neumann regular, then $t \cdot c = 1$ by [3, 2.6]. By Theorem 1, w.dim(A) = w.dim(A#H) = 0. So A is von Neumann regular.

Remark 1 If A is not commutative, Corollary 1 is not true even in group algebra case (see [1, Remark 2.5]).

References:

[1] ALFARO R, ARA P, DELRÍO A. *Regular skew group rings* [J]. J. Australian Math. Soc., Ser. A, 1995, **58**: 167–182.
[2] COHEN M, FISCHMAN D. *Hopf algebra actions* [J]. J. Algebra, 1986, **100**: 363–379.
[3] COHEN M, FISCHMAN D, MONTGOMERY S. *Hopf Galois extensions, smash products, and Morita equivalence* [J]. J. Algebra, 1990, **133**: 351–372.
[4] MONTGOMERY S. *Hopf Algebras and Their Actions on Rings* [M]. CBMS, Lecture in Math. 82, AMS, Providence, RI, 1993.
[5] RADFORD D E. *The order of the antipode of a finite dimensional Hopf algebras is finite* [J]. Amer. J. Math., 1976, **98**: 333–355.
[6] ROTMAN J J. *An Introduction to Homological Algebra* [M]. Academic Press. Inc., New York, 1979.
[7] YANG S, CHEN J. *Hopf actions on factor rings and induced Hopf actions* [J]. Comm. Algebra, 1997, **25**(4): 1103–1110.

Hopf 代数的冲积的弱整体维数

王志玺， 赵 慧
(首都师范大学数学系，北京 100037)

摘要: 设 H 是有限维 Hopf 代数， A 是交换的 H- 模代数. 当 H^* 是幺模且 A 中存在迹为 1 的元素时，本文证明冲积 $A\#H$ 与代数 A 的弱整体维数相等.

关键词: 弱整体维数；模代数；冲积.