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The Aysmptotic Behaviour of Theoretical and Numerical Solutions for Nonlinear Differential System with Several Delay Terms

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Abstract The purpose of this paper is to study the asymptotic behaviour of the theoretical and numerical solution of the nonlinear differential system with several delay terms. Under the suitable assumptions for right hand function, we prove that, the theoretical solutions of the nonlinear differential system with several delay terms are asymptotically stable. The analogous behaviour of the numerical solutions of implicit Euler method are also investigated.

Keywords nonlinear differential system; asymptotic stability; numerical solution; theoretical solution

1 Introduction

First of all, we consider the following linear system of delay differential equation:

$$y'(t) = Ay(t) + B_1y(t-\tau_1) + B_2y(t-\tau_2) + \ldots + B_my(t-\tau_m), t > 0, \quad (1.1)$$

$$y(t) = \varphi(t), \qquad t \leq 0, \quad (1.2)$$

where

$$y(t) = (y_1(t), y_2(t), \ldots, y_N(t))^T \in C^N,$$

$$y(t-\tau_j) = (y_1(t-\tau_j), y_2(t-\tau_j), \ldots, y_N(t-\tau_j))^T \in C^N, \qquad 1 \leq j \leq m.$$

$A = (a_{ij})_{N\times N}, B = (b_{ij})_{N\times N}$ are constant matrices and $0 < \tau_j \leq \tau, 1 \leq j \leq m$.

Theorem 1.1 [7] For any positive integer $m \geqslant 1$ if

(i) $\eta(A) = 1/2\lambda_{\max}(A + A^*) < 0$,

(ii) $\sum_{j=1}^m \| B_j \| < -\eta(A)$,

then every solution $y(t)$ of (1.1) satisfies

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$$\lim_{t\to\infty} y(t) = 0.$$

Where $\lambda_{\max}(A) = \max\{\lambda:\lambda \in \sigma(A)\}$, $\| B \| = \sup_{\|\xi\|=1} \| B\xi \|$, and $\| \xi \|^2 = < \xi,\xi > = \sum_{j=1}^{N} \xi_j\bar{\xi}_j, \xi \in C^N$.

As a special example of (1. 1), it is

$$y'(t) = ay(t) + \sum_{j=1}^{m} b_j y(t-\tau_j), \qquad t > 0, \tag{1.3}$$

$$y(t) = \varphi(t), \qquad t \leq 0. \tag{1.4}$$

Corollary 1. 2 [7] If

(1) $\mathrm{Re}(a) < 0$,

(2) $\sum_{j=1}^{m} |b_j| < -\mathrm{Re}(a)$,

then every solution $y(t)$ of (1. 3) satisfies

$$\lim_{t\to\infty} y(t) = 0. \tag{1.5}$$

Concerning numerical solution of (1. 3) and (1. 4), we have the following concepts.

Definition 1. 1 [7] A numerical method for DDEs (1. 3) and (1. 4) is called P_m-stable if, under conditions (1) and (2), $y_n \to 0$, as $n \to \infty$, for every stepsize h such that $h = \tau_j m_j$, where $m_j, 1 \leq j \leq m$, are positive integers.

Definition 1. 2 [7] A numerical method for DDEs (1. 3) and (1. 4) is called GP_m-stable if, under condition (1) and (2), $y_n \to 0$, as $n \to \infty$ for every stepsize $h > 0$.

In Torelli's paper [10], the numerical solution for the nonlinear system of delay differential equations was studied and the conditions for theoretical and numerical solutions which continuously depended on the initial value were obtained.

The purpose of this paper is to show that, under similar assumption of right hand function with several delay terms as that of paper [7], its theoretical solution is asymptotically stable and that the numerical solution has same property for implicit Euler method.

2 Asymptotic stability of the theoretical solution for nonlinear system with several delay terms

We consider the following nonlinear system with two delay terms

$$y'(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2)), \qquad t > 0, \tag{2,1}$$

$$y(t) = \varphi(t), \qquad t \leq 0, \tag{2.2}$$

and

$$z'(t) = f(t, z(t), z(t-\tau_1), z(t-\tau_2)), \qquad t > 0, \tag{2.3}$$

$$z(t) = \psi(t), \qquad t \leq 0, \tag{2.4}$$

where $f:[0,\infty] \times C^s \times C^s \times C^s \to C^s$, $y(t)$, $z(t)$ $:R \to C^s, \tau_j > 0, i = 1,2$.

Definition 2. 1 The nonlinear delay system (2. 1) with any continuous initial function $\varphi(t)$ is called asymptotically stable if

$$\| y(t) - z(t) \| \to 0, \text{ as } t \to \infty,$$

where $z(t)$ is the solution of (2. 3) with any continuous initial function $\psi(t)$, and $\|x\|^2 = < x, x >$

is the inner product on C^s.

Theorem 2.1 Assume that $\varphi(t)$ and $\psi(t)$ are continuous and

$$\mathrm{Re} < f(t,y_1,u,v) - f(t,y_2,u,v), y_1 - y_2 > \leq \sigma\|y_1 - y_2\|^2, \quad \forall t \in \bar{R}, \forall u,v,y_1,y_2 \in C^s. \tag{2.5}$$

$$\|f(t,y,u_1,v) - f(t,y,u_2,v)\| \leq \gamma_1\|u_1 - u_2\|, \quad \forall t \in \bar{R}, \forall y,v,u_1,u_2 \in C^s, \tag{2.6}$$

$$\|f(t,y,u,v_1) - f(t,y,u,v_2)\| \leq \gamma_2\|v_1 - v_2\|, \ \forall t \in \bar{R}, \quad \forall y,u,v_1,v_2 \in C^s \tag{2.7}$$

$$\gamma_1, \gamma_2 > 0, \sigma < 0 \text{ are constants and } \gamma_1 + \gamma_2 < -\sigma. \tag{2.8}$$

where $\bar{R} = [0,\infty)$. If the solutions of (2.1)—(2.2) and (2.3)—(2.4) exist uniquely, then

$$\lim_{t\to\infty}\|y(t) - z(t)\| = 0. \tag{2.9}$$

Proof According to the definition of the norm on C^s, we have

$$\begin{aligned}
1/2\,(d/dt)(\|y(t) - z(t)\|^2) &= \mathrm{Re} < y'(t) - z'(t), y(t) - z(t) > \\
&= \mathrm{Re} < f(t,y(t),y(t-\tau_1),y(t-\tau_2)) - f(t,z(t),z(t-\tau_1),z(t-\tau_2)), y(t) - z(t) > \\
&= \mathrm{Re} < f(t,y(t),y(t-\tau_1),y(t-\tau_2)) - f(t,z(t),y(t-\tau_1),y(t-\tau_2)), y(t) - z(t) > \\
&+ \mathrm{Re} < f(t,z(t),y(t-\tau_1),y(t-\tau_2)) - f(t,z(t),z(t-\tau_1),y(t-\tau_2)), y(t) - z(t) > \\
&+ \mathrm{Re} < f(t,z(t),z(t-\tau_1),y(t-\tau_2)) - f(t,z(t),z(t-\tau_1),z(t-\tau_2)), y(t) - z(t) >.
\end{aligned} \tag{2.10}$$

An application of Schwartz's inequality, yields

$$\begin{aligned}
1/2(d/dt)(\|y(t) - z(t)\|^2) \leq \sigma\|y(t) - z(t)\|^2 &+ \gamma_1\|y(t) - z(t)\|\|y(t-\tau_1) - z(t-\tau_1)\| \\
&+ \gamma_2\|y(t) - z(t)\|\|y(t-\tau_2) - z(t-\tau_2)\|.
\end{aligned}$$

Let $Y(t) = \|y(t) - z(t)\|$. Then

$$1/2(d/dt)(Y(t)^2) \leq \sigma Y(t)^2 + \gamma_1 Y(t)Y(t-\tau_1) + \gamma_2 Y(t)Y(t-\tau_2),$$

or

$$Y(t)Y'(t) \leq \sigma Y(t)^2 + \gamma_1 Y(t-\tau_1)Y(t) + \gamma_2 Y(t)Y(t-\tau_2).$$

Notice that the solutions of (2.1)—(2.2) and (2.3)—(2.4) exist uniquely. Then

$$Y'(t) \leq \sigma Y(t) + \gamma_1 Y(t-\tau_1) + \gamma_2 Y(t-\tau_2), \qquad t \geqslant 0, \tag{2.11}$$

$$Y(t) = \|\varphi(t) - \psi(t)\| = \Phi(t). \qquad t \leq 0. \tag{2.12}$$

It can be written as

$$Y'(t) = \sigma Y(t) + \gamma_1 Y(t-\tau_1) + \gamma_2 Y(t-\tau_2) - \alpha(t), \qquad t \geqslant 0, \tag{2.13}$$

$$Y(t) = \Phi(t), \qquad t \leq 0, \tag{2.14}$$

where $\alpha(t) \geqslant 0$ is continuous bit by bit.

Let $\tau = \min\{\tau_1, \tau_2\}$ and $t \in [0,\tau]$. Then the formula (2.13) may be written in the form

$$Y'(t) = \sigma Y(t) + \gamma_1\Phi(t-\tau_1) + \gamma_2\Phi(t-\tau_2) - \alpha(t), \qquad 0 \leq t \leq \tau, \tag{2.15}$$

$$Y(t) = \Phi(t), \qquad t \leq 0. \tag{2.16}$$

Set $W(t) = y(t) - Z(t)$, where the $Z(t)$ fulfills the following differential equation and initial value

$$Z'(t) = \sigma Z(t) + \gamma_1 Z(t-\tau_1) + \gamma_2 Z(t-\tau_2), \qquad 0 \leq t \leq \tau, \tag{2.17}$$

$$Z(t) = \Phi(t), \qquad t \leq 0. \tag{2.18}$$

Thus

$$W'(t) = \sigma W(t) - \alpha(t), \qquad 0 \leq t \leq \tau, \tag{2.19}$$

$$W(t) = 0, \qquad t \leq 0. \tag{2.20}$$

The solution of the above differential equation (2.19) with initial value (2.20) is

$$W(t) = e^{\Sigma(t)}[0 + \int_0^t e^{-\Sigma(t)}(-\alpha(t))dt], \qquad 0 \leq t \leq \tau, \tag{2.21}$$

and

$$W(t) = 0, \qquad t \leq 0.$$

where

$$\Sigma(t) = \int_0^t \sigma(t)dt,$$

From (2.21) and $\alpha(t) \geqslant 0$, it follows

$$W(t) \leq 0, \qquad 0 \leq t \leq \tau,$$

namely

$$Y(t) \leq Z(t), \qquad 0 \leq t \leq \tau. \tag{2.22}$$

When $t \in [0, 2\tau]$, we consider the following equations

$$\begin{cases} Y'(t) = \sigma Y(t) + \gamma_1 Y(t - \tau_1) + \gamma_2 Y(t - \tau_2) - \alpha(t), & 0 \leq t \leq 2\tau, \\ Y(t) = \Phi(t), & t \leq 0, \end{cases}$$

and

$$\begin{cases} Z'(t) = \sigma Z(t) + \gamma_1 Z(t - \tau_1) + \gamma_2 Z(t - \gamma_2), & 0 \leq t \leq 2\tau, \\ Z(t) = \Phi(t), & t \leq 0. \end{cases}$$

Thus

$$\begin{cases} W'(t) = \sigma W(t) + \gamma_1 W(t - \tau_1) + \gamma_2 W(t - \tau_2) - \alpha(t), & 0 \leq t \leq 2\tau, \\ W(t) = 0, & t \leq 0. \end{cases} \tag{2.23}$$

From formula (2.22), we obtain

$$Y(t - \tau_1) - Z(t - \tau_1) \leq 0, \qquad t \in [0, 2\tau],$$

and

$$Y(t - \tau_2) - Z(t - \tau_2) \leq 0, \qquad t \in [0, 2\tau].$$

Then (2.23) may be rewritten in the form

$$\begin{cases} W'(t) = \sigma W(t) - \bar{\alpha}(t), & 0 \leq t \leq 2\tau, \\ W(t) = 0, & t \leq 0, \end{cases} \tag{2.24}$$

here $\bar{\alpha}(t) = \alpha(t) - \gamma_1[Y(t - \tau_1) - Z(t - \tau_1)] - \gamma_2[Y(t - \tau_2) - Z(t - \tau_2)] \geqslant 0.$

Therefore

$$W(t) \leq 0, \qquad 0 \leq t \leq 2\tau,$$

that is

$$Y(t) \leq Z(t), \qquad 0 \leq t \leq 2\tau.$$

Continue this process, we arrive at

$$Y(t) \leq Z(t), \qquad t \geqslant 0.$$

and

$$\begin{cases} Z'(t) = \sigma Z(t) + \gamma_1 Z(t-\tau_1) + \gamma_2 Z(t-\tau_2), & t \geqslant 0, \\ Z(t) = \Phi(t), & t \leq 0. \end{cases}$$

From the condition (2. 8) and corollary 1. 2, it follows

$$\lim_{t\to\infty} Z(t) = 0.$$

Thus

$$0 \leq Y(t) \leq Z(t) \to 0, \text{ as } t \to \infty.$$

This completes the proof of this theorem.

The result can be generalized to the nonlinear systems with m delay terms.

Theorem 2. 2 For the systems

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau_1), \ldots, y(t-\tau_m)), & t \geqslant 0, & (2.25) \\ y(t) = \varphi(t), & t \leq 0. & (2.26) \end{cases}$$

and

$$\begin{cases} Z'(t) = f(t, z(t), z(t-\tau_1), \ldots, z(t-\tau_m)), & t \geqslant 0, & (2.27) \\ z(t) = \psi(t), & t \leq 0. & (2.28) \end{cases}$$

Assume that the $\varphi(t)$ and $\psi(t)$ are continuous and

$$\mathrm{Re} < f(t, y, u_1, \ldots, u_m) - f(t, \bar{y}, u_1, \ldots, u_m), y - \bar{y} > \leq \sigma \|y - \bar{y}\|^2,$$
$$\forall\, t \in \bar{R}, \forall\, y, \bar{y}, u_1, \ldots, u_m \in C^s, \qquad (2.29)$$

$$\|f(t, y, u_1, \ldots, u_i, \ldots, u_m) - f(t, y, u_1, \ldots, \bar{u}_i, \ldots, u_m)\| \leq \gamma_i \|u_i - \bar{u}_i\|,$$
$$\forall\, t \in \bar{R}, \forall\, y, u_1, \ldots, u_m, \bar{u}_i \in C^s, 1 \leq i \leq m, \qquad (2.30)$$

$$\sigma < 0, \sum_{j=1}^{m} \gamma_j < -\sigma. \qquad (2.31)$$

If the solutions of (2. 25)—(2. 26) and (2. 27)—(2. 28) exist uniquely, then

$$\lim_{t\to\infty} \|y(t) - z(t)\| = 0. \qquad (2.31)$$

The proof of theorem 2. 2 is analogous to the proof of theorem 2. 1.

3 The asymptotic stability of implicit Euler method

From Theorem 2. 1, if the function f satisfies the conditions (2. 5) — (2. 8), then the solution of (2. 1)—(2. 2) is asymptotically stable. So requiring the numerical process having similar property is reasonable.

In this section, assume that the function f is real and $\tau_2 \geqslant \tau_1 > 0$, $\sigma < 0$, and $\gamma_1 + \gamma_2 < -\sigma$. Denote $\bar{\sigma} = h\sigma, \bar{\gamma}_i = h\gamma_i (i = 1, 2)$ and $h = \tau_1/m_1 = \tau_2/m_2$ the stepsize, where m_1, m_2 are positive integer.

Definition 3. 1 A numerical method for DDEs is called asymptotically stable at $(\bar{\sigma}, \bar{\gamma}_1, \bar{\gamma}_2)$ if, under conditions (2. 5) — (2. 8), the numerical solutions y_n and z_n at the mesh points of (2. 1)—(2. 2) and (2. 3)—(2. 4) respectively, satisfy the condition

$$\|y_n - z_n\| \to 0, \text{ as } n \to \infty,$$

for every stepsize h such that $h = \tau_1/m_2 = \tau_2/m_2$.

By applying the implicit Euler method to solve the equation (2. 1)—(2. 2), we arrive at

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}, y_{n-m_1+1}, y_{n-m_2+1}), \qquad (3.1)$$

where $t_n = nh, h = \tau_1/m_1 = \tau_2/m_2, y_n \sim y(t_n)$, and

$$z_{n+1} = z_n + hf(t_{n+1}, z_{n+1}, z_{n-m_1+1}, z_{n-m_2+1}). \tag{3.2}$$

Let $V_n = y_n - z_n$. From (3.1) and (3.2), it follows

$$\begin{aligned} V_{n+1} &= V_n + h[f(t_{n+1}, y_{n+1}, y_{n-m_1+1}, y_{n-m_2+1}) - f(t_{n+1}, z_{n+1}, z_{n-m_1+1}, z_{n-m_2+1})] \\ &= V_n + h[f(t_{n+1}, y_{n+1}, y_{n+1-m_1}, y_{n-m_2+1}) - f(t_{n+1}, z_{n+1}, y_{n+1-m_1}, y_{n+1-m_2})] \\ &+ h[f(t_{n+1}, z_{n+1}, y_{n+1-m_1}, y_{n+1-m_2}) - f(t_{n+1}, z_{n+1}, z_{n+1-m_1}, y_{n+1-m_2})] \\ &+ h[f(t_{n+1}, z_{n+1}, z_{n+1-m_1}, y_{n+1-m_2}) - f(t_{n+1}, z_{n+1}, z_{n+1-m_1}, z_{n+1-m_2})]. \end{aligned} \tag{3.3}$$

Take the inner product with V_{n+1} on both sides of (3.3) and use condition (2.5), (2.6) and (2.7) to obtain

$$\|V_{n+1}\|^2 \le \|V_n\|\|V_{n+1}\| + h\sigma\|V_{n+1}\|^2 + h\gamma_1\|V_{n+1-m_1}\|\|V_{n+1}\| + h\gamma_2\|V_{n+1-m_2}\|\|V_{n+1}\|.$$

Therefore

$$\|V_{n+1}\| \le \|V_n\| + h\sigma\|V_{n+1}\| + h\gamma_1\|V_{n+1-m_1}\| + h\gamma_2\|V_{n+1-m_2}\|, \tag{3.4}$$

and

$$\|V_{n+1}\| \le (\|V_n\| + h\gamma_1\|V_{n+1-m_1}\| + h\gamma_2\|V_{n+1-m_2}\|)/(1 - h\sigma). \tag{3.5}$$

Consider the following difference equation

$$U_{n+1} = \{U_n + h\gamma_1 U_{n+1-m_1} + h\gamma_2 U_{n+1-m_2}]/(1 - h\sigma), \tag{3.6}$$

with initial value

$$U_{-i} = \|V_{-i}\|, \qquad i = 0, 1, 2, \ldots, m_2 - 1. \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we can obtain recursively

$$\|V_n\| \le U_n, \qquad n = 1, 2, \cdots.$$

To the recurrence relation (3.6) we adjoin the characteristic polynomial where

$$P_M(z) = z^{m_2} - (1 - h\sigma)^{-1}[z^{m_2-1} - h\gamma_1 z^{m_2-m_1} - h\gamma_2].$$

The polynomial $P_M(z)$ is a Schur polynomial if (see [7])

(a) $|1/(1 - h\sigma)| < 1$,

(b) $|h\gamma_1/(1 - h\sigma)| < |z - 1/(1 - h\sigma)|, \qquad |z| = 1$,

(c) $|h\gamma_2/(1 - h\sigma)| < |z - 1/(1 - h\sigma)| - |h\gamma_1/(1 - h\sigma)|, \quad |z| = 1$.

Since $\sigma < 0$, the condition (a) is trivial. For condition (b), noticing

$$\min\{|z - 1/(1 - h\sigma)| : |z| = 1\} = -h\sigma/(1 - h\sigma)$$

and $\gamma_1 < -\sigma$, we can conclude that the (b) is held. The condition (c) is equivalent to

$$h\gamma_1/(1 - h\sigma) + h\gamma_2/(1 - h\sigma) < -h\sigma/(1 - h\sigma).$$

Since $\gamma_1 + \gamma_2 < -\sigma$, it follows that the (c) is true. Therefore $P_M(z)$ is a Schur polynomial. Let $\xi_1, \xi_2, \ldots, \xi_{m_2}$ are the zeros of $P_M(z)$. Then $|\xi_j| < 1$. The solution of difference equation (3.7) with initial condition is

$$U_n = c_1\xi_1^n + c_2\xi_2^n + \ldots + c_{m_2}\xi_{m_2}^n, \text{ and } \lim_{n\to\infty} U_n = 0. \tag{3.8}$$

Thus

$$\|V_n\| \le U_n \to 0, \ n \to \infty.$$

Then we establish the Theorem 3.1

Theorem 3.1 For any range $(\bar{\sigma},\bar{\gamma}_1,\bar{\gamma}_2) \in S$, the implicit Euler method is asymptotically stable. Where $S = \{(\bar{\sigma},\bar{\gamma}_1,\bar{\gamma}_2); \bar{\sigma} < 0,\ \bar{\gamma}_1 + \bar{\gamma}_2 < -\bar{\sigma}\}$.

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非线性多滞时微分系统理论及数值解

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提　要　本文旨在研究非线性多滞时微分系统的理论解和数值解的渐近性态．可以证明，在对右端函数给出适当条件下，非线性多滞时微分系统的理论解是渐近稳定的．并且隐式欧拉公式得到的数值解具有相同性态．

关键词　非线性微分系统；渐近稳定性，数值解；理论解

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