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The representation of the W-weighted Drazin inverse $(A \otimes B)_{d,w}$ and its applications

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Abstract: The representation of the W-weighted Drazin inverse $(A \otimes B)_{d,w}$ of the Kronecker product $A \otimes B$ of two matrices A and B is given. The relation between the Kronecker product of the projectors is established. Using the above results and the Cramer rule, the unique W-weighted Drazin inverse solution $x \in R(((A \otimes B)(W_1 \otimes W_2))^{k_1})$ of a special kind of restricted linear equations is found.

Key words: Kronecker product; W-weighted Drazin inverse; index; projector; Cramer rule

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1 Introduction

Let $A = (a_{ij}) \in C^{m \times n}, B = (b_{ij}) \in C^{p \times q}$. The Kronecker product $A \otimes B$ of the two matrices A and B is the $mp \times nq$ matrix expressible in the partitioned form.

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}. \tag{1.1}$$

The properties of this product can be found in [1].

Lemma 1.1 Let A, B, A_1, B_1, A_2, B_2 and O are matrices of proper sizes.

(1) $O \otimes A = A \otimes O = O$;

(2) $(A_1 + A_2) \otimes B = (A_1 \otimes B) + (A_2 \otimes B)$;

(3) $A \otimes (B_1 + B_2) = (A \otimes B_1) + (A \otimes B_2)$;

(4) $(A_1 A_2) \otimes (B_1 B_2) = (A_1 \otimes B_1)(A_2 \otimes B_2)$;

(5) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;

(6) $(A \otimes B)^{+} = A^{+} \otimes B^{+}$;

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(7) $r(A \otimes B)^l = r(A^l)\, r(B^l)$,

where $r(A)$ denotes the rank of A .

The representations of the generalized inverse $(A \otimes B)^{+}_{MN}$, $(A \otimes B)_d$ and $(A \otimes B)_g$ are given in [3]. In[2], Wang Guorong showed that for the generalized inverses $(A \otimes B)^{(1,2)}_{T,S}$ and $(A \otimes B)^{(2)}_{T,S}$, there exist two representations.

$$(A \otimes B)^{(1,2)}_{T,S} = A^{(1,2)}_{T_1,S_1} \otimes B^{(1,2)}_{T_2,S_2}, \tag{1.2}$$

where $T = T_1 \otimes T_2$, $S = S_1 \otimes S_2$, and

$$(A \otimes B)^{(2)}_{T,S} = A^{(2)}_{T_1,S_1} \otimes B^{(2)}_{T_2,S_2}, \tag{1.3}$$

where $T = T_1 \otimes T_2$, $S = S_1 \otimes S_2$.

If $A \in C^{m\times n}, W \in C^{n\times m}$, then $X = [(AW)_d]^2 A$ is the unique solution to the following equations.

$$(AW)^{k+1}XW = (AW)^k,\ XWAWX = X,\ AWX = XWA, \tag{1.4}$$

where $k = \mathrm{Ind}(AW)$, the index of AW , is the smallest nonnegative interger for which $r[(AW)^k] = r[(AW)^{k+1}]$. The matrix X is called the W-weighted Drazin inverse of A and is written as $X = A_{d,w}$ [4].

Lemma 1.2[4] Let $A \in C^{m\times n}, W \in C^{n\times m}$ with $\mathrm{Ind}(AW) = k_1$ and $\mathrm{Ind}(WA) = k_2$, we have:

(a) $A_{d,W} = A[(WA)_d]^2 = [(AW)_d]^2 A$;

(b) $A_{d,W}W = (AW)_d, WA_{d,W} = (WA)_d$;

(c) $A_{d,W}WAW = (AW)_d AW = P_{R[(AW)^{k_1}],N[(AW)^{k_1}]} = P_{R[(AW)^k],N[(AW)^k]}$, where $k \geq k_1$.

2 Results

First, we give the representation of $(A \otimes B)_{d,w}$.

Theorem 2.1 Let $A \in C^{m\times n}$, $B \in C^{p\times q}$, $W_1 \in C^{n\times m}$, and $W_2 \in C^{q\times p}$, with $\mathrm{Ind}(AW_1) = k_1$, $\mathrm{Ind}(BW_2) = k_2$, and $k = \max(k_1, k_2)$. Then

$$(A \otimes B)_{d,W} = A_{d,W_1} \otimes B_{d,W_2}, \tag{2.1}$$

and

$$\mathrm{Ind}(AW_1 \otimes BW_2) = k, \tag{2.2}$$

where $W = W_1 \otimes W_2$.

Proof From the properties of the Kronecker product, we have

$$(AW_1 \otimes BW_2)^l = (AW_1)^l \otimes (BW_2)^l, \tag{2.3}$$

$$r(AW_1 \otimes BW_2)^l = r(AW_1)^l r(BW_2)^l, \tag{2.4}$$

$$r(AW_1 \otimes BW_2)^{l+1} = r(AW_1)^{l+1} r(BW_2)^{l+1}. \tag{2.5}$$

By the assumptions, we have

$$r(AW_1)^{k_1} = r(AW_1)^{k_1+1}, \tag{2.6}$$

and

$$r(AW_1)^{k_2} = r(AW_1)^{k_2+1}. \tag{2.7}$$

It is obvious that the smallest nonegative integer l such that $r(AW_1 \otimes BW_2)^{l+1} = r(AW_1 \otimes BW_2)^l$ is k . Hence (2.2) is true.

From the properties of the Kronecker product and Lemma1.1[4], we have

$$[(A \otimes B)(W_1 \otimes W_2)]^{k+1}(A_{d,W_1} \otimes A_{d,W_2})(W_1 \otimes W_2) = [(A \otimes B)(W_1 \otimes W_2)]^k,$$

$$(A_{d,W_1} \otimes A_{d,W_2})(W_1 \otimes W_2)(A \otimes B)(W_1 \otimes W_2)(A_{d,W_1} \otimes A_{d,W_2}) = A_{d,W_1} \otimes A_{d,W_2},$$

$$(A\otimes B)(W_1\otimes W_2)(A_{d,W_1}\otimes A_{d,W_2})=(A_{d,W_1}\otimes A_{d,W_2})(W_1\otimes W_2)(A\otimes B).$$

We can obtain (2.1) immediately. □

Corollary 2.1[3] Let $A\in C^{m\times m}$, $B\in C^{n\times n}$, $\mathrm{Ind}(A)=k_1$, $\mathrm{Ind}(B)=k_2$, $k=\max(k_1,k_2)$. Then

$$(A\otimes B)_d=A_d\otimes B_d, \tag{2.8}$$

and

$$\mathrm{Ind}(A\otimes B)=k. \tag{2.9}$$

Theorem2.2 Let $A\in C^{m\times n}$, $B\in C^{p\times q}$, $W_1\in C^{n\times m}$, and $W_2\in C^{q\times p}$, with $\mathrm{Ind}(AW_1)=k_1$, $\mathrm{Ind}(BW_2)=k_2$, and $k=\max(k_1,k_2)$. Then

$$P_{R[(A\otimes B)(W_1\otimes W_2)]^k,N[(A\otimes B)(W_1\otimes W_2)]^k}=P_{R[(AW_1)^k],N[(AW_1)^k]}\otimes P_{R[(BW_2)^k],N[(BW_2)^k]}. \tag{2.10}$$

Proof It follows from Lemma1.2(c)

$$A_{d,W_1}W_1AW_1=(AW_1)_d(AW_1)=P_{R[(AW_1)^k],N[(AW_1)^k]},$$

and

$$B_{d,W_2}W_2BW_2=(BW_2)_d(BW_2)=P_{R[(BW_2)^k],N[(BW_2)^k]}.$$

So we have

$$\begin{aligned}&P_{R[(AW_1)^k],N[(AW_1)^k]}\otimes P_{R[(BW_2)^k],N[(BW_2)^k]}\\&=(A_{d,W_1}W_1AW_1)\otimes(B_{d,W_2}W_2BW_2)\\&=(A\otimes B)_{d,(W_1\otimes W_2)}(W_1\otimes W_2)(A\otimes B)(W_1\otimes W_2)\\&=P_{R[(A\otimes B)(W_1\otimes W_2)]^k,N[(A\otimes B)(W_1\otimes W_2)]^k}.\end{aligned}$$

This completes the proof. □

Corollary 2.2[2] Let the assumptions be the same as those in Corollary 2.1. Then

$$P_{R(A^k\otimes B^k),N(A^k\otimes B^k)}=P_{R(A^k),N(A^k)}\otimes P_{R(B^k),N(B^k)}. \tag{2.11}$$

Theorem 2.3 Let $A\in C^{m\times n}$, $W\in C^{n\times m}$ with $\mathrm{Ind}(AW)=k_1$, $\mathrm{Ind}(WA)=k_2$. Then

$$\mathrm{r}[(AW)^{k_1}]=\mathrm{r}[(WA)^{k_2}]. \tag{2.12}$$

Proof Suppose that $k_1\geqslant k_2$. By hypothesis, we deduce $k_1+1>k_2$. From the properties of the index of WA, we have

$$\mathrm{r}[(WA)^{k_2}]=\mathrm{r}[(WA)^{k_1+1}]. \tag{2.13}$$

Since $(WA)^{k_1+1}=W(AW)^{k_1}A$, it follows that

$$\mathrm{r}[(WA)^{k_1+1}]\leqslant\mathrm{r}[(AW)^{k_1}]. \tag{2.14}$$

From (2.13) and (2.14), it holds that

$$\mathrm{r}[(WA)^{k_2}]\leqslant\mathrm{r}[(AW)^{k_1}]. \tag{2.15}$$

Since $(AW)^{k_1+1}=(AW)^{k_2+1}(AW)^{k_1-k_2}=A(WA)^{k_2}W(AW)^{k_1-k_2}$, it follows that

$$\mathrm{r}[(AW)^{k_1+1}]\leqslant\mathrm{r}[(WA)^{k_2}]. \tag{2.16}$$

From the fact $\mathrm{r}[(AW)^{k_1+1}]=\mathrm{r}[(AW)^{k_1}]$, combined with (2.16), we have

$$\mathrm{r}[(AW)^{k_1}]\leqslant\mathrm{r}[(WA)^{k_2}]. \tag{2.17}$$

By (2.15) and (2.17), we can obtain (2.12). □

3 Applications

For a nonsingular matrix A, for any b, the solution of the linear equation

$$Ax=b \tag{3.1}$$

is given by the classical Cramer rule. (For an elegant proof see [5]).

In[6], there is a Cramer rule for the unique Drazin inverse solution, A_db, of the restricted linear equation

$$Ax = b, x \in R(A^k), \tag{3.2}$$

where $A \in C^{n\times n}$ with Ind $(A) = k$ and $b \in R(A^k)$.

In[7], Y. Wei showed a Cramer rule for the W-weighted Drazin inverse solution, $A_{d,w}b$, of a general restricted linear equation

$$WAWx = b, x \in R[(AW)^{k_1}], \tag{3.3}$$

where $A \in C^{m\times n}$, $W \in C^{n\times m}$, Ind $(AW) = k_1$, and Ind $(WA) = k_2$ with $b \in R[(WA)^{k_2}]$.

In this section, we will consider the W-weighted Drazin inverse solution, $(A\otimes B)_{d,W_1\otimes W_2}(b_1\otimes b_2)$, of a general restricted linear equation

$$(W_1\otimes W_2)(A\otimes B)(W_1\otimes W_2)x = b_1\otimes b_2, x \in R(((A\otimes B)(W_1\otimes W_2))^{k_1}), \tag{3.4}$$

where $A \in C^{m\times n}$, $B \in C^{p\times q}$, $W_1 \in C^{n\times m}$, $W_2 \in C^{q\times p}$, $k_3 = \text{Ind}(AW_1)$, $k_4 = \text{Ind}(BW_2)$, $k_5 = \text{Ind}(W_1A)$, $k_6 = \text{Ind}(W_2B)$, $k_1 = \text{Ind}((A\otimes B)(W_1\otimes W_2)) = \max(k_3, k_4)$, $k_2 = \text{Ind}((W_1\otimes W_2)(A\otimes B)) = \max(k_5, k_6)$, $b_1 \in R[(W_1A)^{k_5}]$, and $b_2 \in R[(W_2B)^{k_6}]$.

Lemma 3.1[7] Let $A \in C^{m\times n}$, $W_1 \in C^{n\times m}$ with $k_3 = \text{Ind}(AW_1)$, $k_5 = \text{Ind}(W_1A)$ and $r[(AW_1)^{k_3}] = r[(WA_1)^{k_5}] = r_1$. Suppose that $U_1 \in C^{n\times(n-r_1)}_{n-r_1}$, $V_1^* \in C^{m\times(m-r_1)}_{m-r_1}$ be matrices whose columns form bases for $N[(W_1A)^{k_5}]$ and $N[(AW_1)^{k_3^*}]$, respectively and $U_2^* \in C^{n\times(n-r_1)}_{n-r_1}$, $V_2 \in C^{m\times(m-r_1)}_{m-r_1}$ be matrices whose columns form bases for $N[(W_1A)^{k_5^*}]$ and $N[(AW_1)^{k_3}]$, respectively. Then

$$D_1 = \begin{pmatrix} W_1AW_1 & U_1 \\ V_1 & 0 \end{pmatrix} \tag{3.5}$$

is nonsingular and its (regular) inverse is

$$D_1^{-1} = \begin{pmatrix} A_{d,W_1} & V_2(V_1V_2) \\ (U_2U_1)^{-1}U_2 & -(U_2U_1)^{-1}U_2W_1AW_1V_2(V_1V_2)^{-1} \end{pmatrix}. \tag{3.6}$$

Lemma 3.2[7] Let $A, W_1, U_1, V_1^*, U_2^*$ and V_2 be the same as in lemma3.1. Let $b_1 \in R[(W_1A)^{k_5}]$, and $r[(AW_1)^{k_3}] = r[(W_1A)^{k_5}] = r_1$. Then the unique W-weighted Drazin inverse solution $x = (x_1, x_2, \cdots, x_m)^T$ of (3.3) satisfies

$$x_j = \frac{\det\begin{pmatrix} W_1AW_1(j\to b_1) & U_1 \\ V_1(j\to 0) & 0 \end{pmatrix}}{\det\begin{pmatrix} W_1AW_1 & U_1 \\ V_1 & 0 \end{pmatrix}}, \tag{3.7}$$

where j=1,2, ⋯ ,m.

Theorem 3.1 Let A, W_1, U_1, V_1^*, U_2^* and V_2 be the same as those in lemma 3.1. Let $B \in C^{p\times q}$, $W_2 \in C^{q\times p}$, $U_3 \in C^{q\times(q-r_2)}_{q-r_2}$, $V_3^* \in C^{p\times(p-r_2)}_{p-r_2}$ be matrices whose columns form bases for $N[(W_2B)^{k_6}]$ and $N[(BW_2)^{k_4^*}]$, respectively and $U_4^* \in C^{q\times(q-r_2)}_{q-r_2}$, $V_4 \in C^{p\times(p-r_2)}_{p-r_2}$ be matrices whose columns form bases for $N[(W_2B)^{k_6^*}]$ and $N[(BW_2)^{k_4}]$, respectively. Let $b_1 \in R[(W_1A)^{k_5}]$, $b_2 \in R[(W_2B)^{k_6}]$ and $r[(AW_1)^{k_3}] = r[(W_1A)^{k_5}] = r_1$, $r[(BW_2)^{k_4}] = r[(W_2B)^{k_6}] = r_2$. Then the unique W-weighted Drazin inverse solution $x = (x_1, x_2, \cdots, x_{mp})^T$ of (3.4) satisfies

$$x_s = \frac{\det\begin{pmatrix} W_1AW_1((\lfloor\frac{s}{p}\rfloor+1)\to b_1) & U_1 \\ V_1((\lfloor\frac{s}{p}\rfloor+1)\to 0) & 0 \end{pmatrix} \det\begin{pmatrix} W_2BW_2((s-p\lfloor\frac{s}{p}\rfloor)\to b_2) & U_3 \\ V_3((s-p\lfloor\frac{s}{p}\rfloor)\to 0) & 0 \end{pmatrix}}{\det\begin{pmatrix} W_1AW_1 & U_1 \\ V_1 & 0 \end{pmatrix} \det\begin{pmatrix} W_2BW_2 & U_3 \\ V_3 & 0 \end{pmatrix}}, \tag{3.8}$$

where s=1,2, ⋯ , mp.

Proof Let $x=(A\otimes B)_{d,(W_1\otimes W_2)}(b_1\otimes b_2)$. It is obvious that x is the solution of (3.4) from [7]. By Theorem 2.1, we arrive at

$$x=(A\otimes B)_{d,(W_1\otimes W_2)}(b_1\otimes b_2)=(A_{d,W_1}\otimes B_{d,W_1})(b_1\otimes b_2)=A_{d,W_1}b_1\otimes A_{d,W_2}b_2. \quad (3.9)$$

Let $x_1=A_{d,W_1}b_1$ and $x_2=A_{d,W_2}b_2$, we have

$$x=x_1\otimes x_2. \quad (3.10)$$

Thus, (3.4) is equivalent to the following equations(3.10)~(3.12):

$$W_1AW_1x_1=b_1, x_1\in R[(AW_1)^{k_3}], \quad (3.11)$$

where $A\in C^{m\times n}$, $W_1\in C^{n\times m}$, Ind$(AW_1)=k_3$ and Ind$(W_1A)=k_5$ with $b_1\in R[(W_1A)^{k_5}]$.

$$W_2BW_2x_2=b_2, x_2\in R[(BW_2)^{k_4}], \quad (3.12)$$

where $B\in C^{p\times q}$, $W_2\in C^{q\times p}$, Ind$(BW_2)=k_4$ and Ind$(W_2B)=k_6$ with $b_2\in R[(W_2B)^{k_6}]$. By hypothesis and Lemma 3.1, we have

$$D_1=\begin{pmatrix}W_1AW_1 & U_1\\ V_1 & 0\end{pmatrix},$$

$$D_1^{-1}=\begin{pmatrix}A_{d,W_1} & V_2(V_1V_2)\\ (U_2U_1)^{-1}U_2 & -(U_2U_1)^{-1}U_2W_1AW_1V_2(V_1V_2)^{-1}\end{pmatrix},$$

and

$$D_2=\begin{pmatrix}W_2BW_2 & U_3\\ V_3 & 0\end{pmatrix}, \quad (3.13)$$

$$D_2^{-1}=\begin{pmatrix}B_{d,W_2} & V_4(V_3V_4)\\ (U_4U_3)^{-1}U_4 & -(U_4U_3)^{-1}U_4W_2BW_2V_4(V_3V_4)^{-1}\end{pmatrix}. \quad (3.14)$$

By lemma 3.2, we obtain immediately

$$x_{1j}=\frac{\det\begin{pmatrix}W_1AW_1(j\to b_1) & U_1\\ V_1(j\to 0) & 0\end{pmatrix}}{\det\begin{pmatrix}W_1AW_1 & U_1\\ V_1 & 0\end{pmatrix}} \quad (3.15)$$

where j=1,2, ⋯ , m, and

$$x_{2t}=\frac{\det\begin{pmatrix}W_2BW_2(t\to b_2) & U_3\\ V_3(t\to 0) & 0\end{pmatrix}}{\det\begin{pmatrix}W_2BW_2 & U_3\\ V_3 & 0\end{pmatrix}}, \quad (3.16)$$

where t=1,2, ⋯ , p.

Let $x_1=(x_{11},x_{12},\cdots,x_{1m})^T$ and $x_2=(x_{21},x_{22},\cdots,x_{2p})^T$, from (3.10), we have

$$x=(x_{11}(x_{21},x_{22},\cdots,x_{2p}),x_{12}(x_{21},x_{22},\cdots,x_{2p}),\cdots,x_{1m}(x_{21},x_{22},\cdots,x_{2p}))^T. \quad (3.17)$$

It is easy to verify that

$$x_s=x_{1,(\lfloor\frac{s}{p}\rfloor+1)}*x_{2,(s-p\lfloor\frac{s}{p}\rfloor)}, \quad (3.18)$$

where $s=1,2,\cdots,mp$.

From (3.15), (3.16) and (3.18), we know that (3.8) is true. □

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加 W 权 Drazin 逆 $(A \otimes B)_{d,w}$ 的表示及应用

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摘要:讨论了 Kronecker 积 $A \otimes B$ 的加 W 权 Drazin 逆 $(A \otimes B)_{d,w}$ 的表示式,并建立投影算子的 Kronecker 积之间的关系。最后,运用上面的结果和 Cramer 法则,得到了一类约束线性方程的加权 Drazin 逆解 $x \in R(((A \otimes B)(W_1 \otimes W_2))^{k_1})$.

关键词:Kronecker 积; Drazin 逆; 指标; 投影算子; Cramer 法则